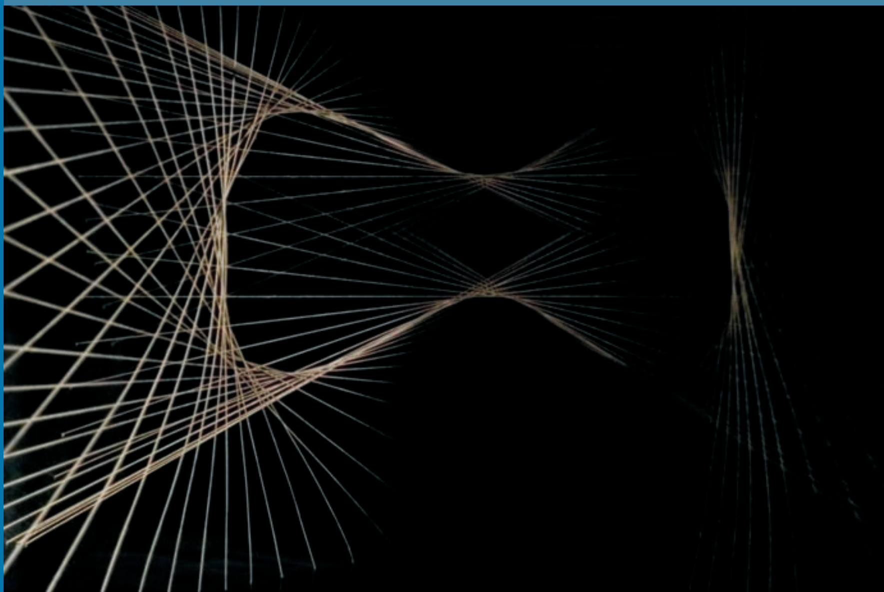
Andrey Tyurin

Vector Bundles

edited by Fedor Bogomolov,
Alexey Gorodentsev, Victor Pidstrigach,
Miles Reid and Nikolay Tyurin



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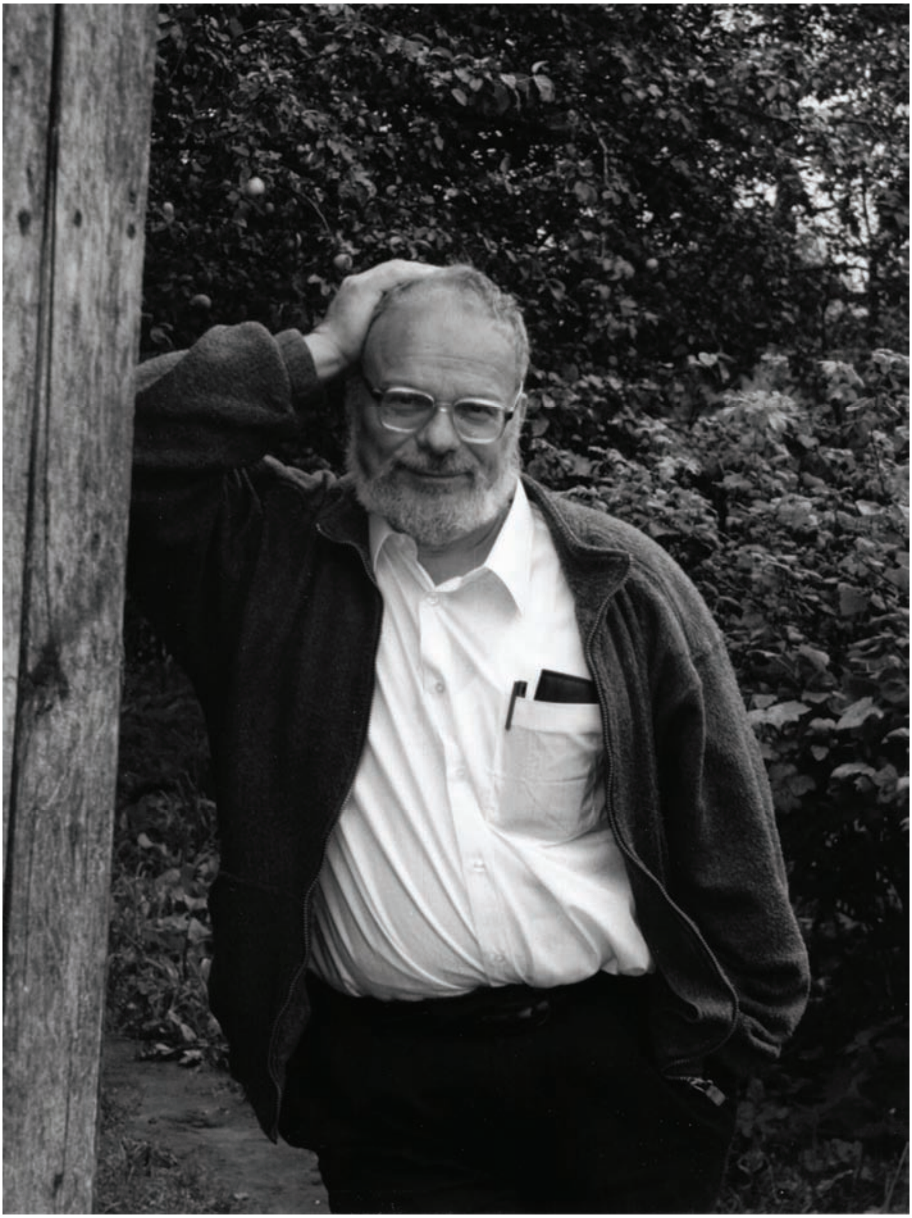

Andrey Tyurin (24.02.1940 - 27.10.2002)

ANNOTATION

This is the first volume of a three volume collection of Andrey Nikolaevich Tyurin's Selected Works. It includes his most interesting articles in the field of classical algebraic geometry, written during his whole career from the 1960s; most of these papers treat different problems of the theory of vector bundles on curves and higher dimensional algebraic varieties. This theory is central to algebraic geometry and most of its applications. The spectrum of the problems considered is very broad, ranging from the geometry of stable vector bundles on algebraic curves to the description of symplectic structures and metrics on the moduli varieties of vector bundles on surfaces, from the method of superposition in the theory of mathematical instantons to the application of classical enumerative geometry to the description of differentiable structures on four manifolds, from the theory of theta functions and Lagrangian geometry to the construction of Delzant models in Quantum Conformal Field Theory.

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Introduction of the Chief-Editor

This is Volume 1 of a three volume collection of selected works of Andrey Nikolaevich Tyurin. This publication is not an complete one, and we omit a number of his papers, although the substance of his work is well reflected in the articles included here.

This publication serves a two-fold purpose. On the one hand it is our homage to the memory of a close friend and colleague, whose work in the field of algebraic geometry was most productive and influential over many years. On the other hand, Andrey Tyurin played a very essential role in the creation of the modern theory of vector bundles, and in advancing the methods of algebraic geometry within topology and theoretical physics. His articles collected together provide a bright and dramatic picture of the development of this area of algebraic geometry over the last forty years. We hope this publication will serve future generations of scientists as a good introduction to a number of very interesting problems in algebraic geometry. We would like to provide an introduction to a broad circle of ideas and projects of Andrey Tyurin, some of which were left unfinished due to his sudden and untimely death. Several generations of mathematician were fortunate enough to work and develop in the atmosphere of creative quest that his energy and talent fostered. We hope that this collection will at least in part communicate a colourful image of Andrey Tyurin's personality to future generations.

F. Bogomolov

Andrey Nikolaevich Tyurin

Andrey Tyurin chose his mathematical theme very early in his career, already as an undergraduate, and he was to remain faithful to it in essence for his whole life. This chosen theme, the theory of vector bundles and their moduli spaces, turned out to be very brilliant: alongside its own internal beauty, the subject has many diverse connections with other branches of geometry and mathematical physics. One aspect of Tyurin's own personality is already apparent from this: his ability to spot these connections as soon as they appear, to find them for himself and to take part in their development.

The theory of vector bundles was just beginning to take shape as an area of mathematics when Tyurin started to work on it in the early 1960s. A prewar paper of André Weil served as a foundation for it, using the language of "matrix-valued divisors" to describe what we now called vector bundles on algebraic curves. This theory already contained one of the foundational points for the development of the subject, the idea that it can be understood as a "non-Abelian analog" of the classical theory of divisors, in which we replace numbers (that is, 1×1 matrices) by matrices of any rank. This idea then took hold of Tyurin, and formed the driving force for his work over many years. Also available at the time were lecture notes of Weil, in which he showed how the notions of the theory of vector bundles, currently popular in topology and differential geometry, could be formulated in algebraic geometry. Serre's lecture at the Bourbaki seminar stimulated assimilation of these ideas. Among the more concrete results known at the time were the classification of vector bundles on curves of genus 0 (Grothendieck) and genus 1 (Atiyah).

This was the starting point for Tyurin's own research. In a paper written while still an undergraduate, he found all rank 2 vector bundles on curves of genus 2 having determinant 0, and showed that they are described by points of the variety $\mathbb{P}^3 \setminus V$, where $V \subset \mathbb{P}^3$ is the Kummer surface. His subsequent studies were concerned with the analogous problem over curves of any genus, and later also its generalisation to the case of dimension ≥ 2.

Here he ran into the fact that posing the problem is itself a difficulty. As he soon discovered, this relates to nonrepresentability of certain functor under discussion. In order to make it representable, one introduces a certain extra rigidity, from which point onwards the question about the moduli space makes sense, and can be studied. A natural point of view on this type of problem is given by the notion of stability, which originates with Hilbert, and was revived in more recent times by Mumford, in connection with the theory of moduli of algebraic curves. Tyurin soon mastered this circle of ideas. Quite generally, I believe it is fair to say that our Moscow circle of algebraic geometers was able to assimilate the "stability philosophy" due in large part to Tyurin's influence.

In application to the theory of vector bundles over a curve, the conclusion from this philosophy is that the natural object to study is the moduli space of stable bundles (or its completion, that includes the semistable ones). In this direction, Tyurin was responsible for a fundamental "noncommutative" analog of the classical Torelli theorem. Namely, he proved that, for bundles over a curve of fixed rank and fixed degree of their determinant, if the rank and the degree are coprime, the moduli space of these bundles determines the original curve uniquely. Slightly after this (and entirely independently – obviously, in view of the complete lack of connections between the Soviet Union and the rest of the world), Mumford, Newstead, Ramanan and Seshadri obtained similar results. They obtained another striking result, one that can be stated as the "absence of a Schottky problem" in the theory of higher rank vector bundles. This period represents a second flight of the theory of vector bundles over algebraic curves, following Weil's initial work. Tyurin's survey, setting out precisely everything known at this time, was to stimulate the subsequent development of the subject.

Tyurin was swift to adapt to the new ideas in algebraic geometry, reworking them enthusiastically from his "philosophy of vector bundles" point of view. For example, Hartshorne drew attention to certain properties of *extensions* of a variety, when a projective variety X can be realized as a hyperplane section of a variety X_1; (this is always possible if X is a hypersurface or a complete intersection). Varieties for which this extension can be carried out sufficiently far (so that one obtains a "variety of small codimension") have a number of remarkable properties. Tyurin formulated these ideas as a certain theory of "infinite dimensional manifolds" (that is, he assumed the possibility of extending the manifold indefinitely). In this formulation, Hartshorne's results mean that an infinite dimensional manifold is a complete intersection in an "infinite dimensional projective space". Naturally enough, he related these ideas at once to the theory of vector bundles. Here we consider an *extension of pairs* (X, E) where E is a vector bundle on X, and we demand not just that X should be realized as a hyperplane section of a variety X_1, but also that E should be the restriction to X of a bundle E_1 on X_1. An infinite sequence of pairs is called a vector bundle (of finite rank) on an infinite dimensional algebraic manifold. One of the results then says that any such bundle is a direct sum of line bundles. This result (and his other results in this paper) also admits a finite statement, when the infinite dimensionality of the manifold is replaced by a sufficiently long series of extensions.

Needless to say, Tyurin's interests were not limited to the theory of vector bundles (although certain analogies with vector bundles played a role in most cases). Thus, he proved two completely different "Torelli-type theorems". The first of these starts off from the classical problem of classifying linear transformations (or classifying $n \times n$ matrices up to conjugacy $A \mapsto C^{-1}AC$). The characteristic polynomial $\det(A - tE)$ is obvious an invariant, and with it the "spectrum" of the matrix – its set of roots. A fundamental classical result asserts that in the case of a "simple" spectrum, the spectrum is a complete system of invariants. In the 19th century, mathematicians preferred to speak of a *pencil of matrices* $\lambda A + \mu B$, and to consider the discriminant form $\det(\alpha A + \mu B) = \varphi(\lambda, \mu)$. The assumption of "simple spectrum" corresponds to the case when the discriminant does not have multiple factors. Here the natural transformations are of the form $(A, B) \mapsto (CAD, CBD)$. If we are discussing symmetric matrices (that is, quadratic forms), we set $D = C^*$, with $\det C \neq 0$. An analog of the theorem mentioned above is the statement that in the case of a simple spectrum, a pair of quadratic forms can simultaneously be reduced to a sum of squares. Going further, the next question is a *net of matrices* $\lambda A + \mu B + \nu C$, or the intersections of the three corresponding quadrics. The analog of the spectrum is now the plane curve $\det(\lambda A + \mu B + \nu C) = 0$, and the "simple spectrum" assumption corresponds to the case that this curve has no singular points. The theorem Tyurin proved is that in this case the "spectrum" (that is, the plane discriminant curve) again determines the net of quadrics, but only up to a finite number of possibilities. These additional invariants are determined by a certain double covering of the curve $\det(\lambda A + \mu B + \nu C) = 0$; he indicated exactly which type of covering curves can occur, so that as a result one obtains a one-to-one correspondence (in the "simple spectrum" case) between intersections of three quadrics and their nonsingular discriminant curves of degree n in the projective plane plus a certain double cover, where both are considered up to projective transformation.

While Tyurin was writing this paper, he was unaware that his main result had been obtained by Dixon in the early 20th century (this is not the Leonard Dickson who is well-known for his study of finite fields and algebraic groups over them, but another person, Arthur Dixon, whose surname is even spelt differently). However, this phenomenon is inevitable. Over a few decades, mathematicians start to think in different terms, and cease to understand their predecessors. Their results are forgotten, except for some really famous ones. For this reason, reestablishing these results in a modern setting is no less of an achievement than proving new ones.

Tyurin's activity with nets of quadrics led him to a construction that I am convinced has yet to play its full role in geometry; it relates to Riemannian geometry (or pseudo-Riemannian geometry). For every point $x \in X$ of a (pseudo-)Riemannian 4-manifold X, the exterior square $\bigwedge^2 T_x$ of the tangent space has three intrinsically defined quadratic forms. The first of these is the exterior product (since $\bigwedge^4 T_x$ is a line). The second is defined by the (pseudo-)Riemannian metric on X. The third is given by the curvature form. In our case rank $\bigwedge^2 T_x = 6$, and the projectivization of the whole construction defines three quadrics in $\mathbb{P}^5$. The intersection of these quadrics (assuming that it is transversal) is a K3 surface, or in general a certain degeneration of one. Thus, in a quite remarkable way, every point of a Riemannian 4-manifold X corresponds to a certain K3 surface (possibly degenerate). Thus there is defined a map of X into the moduli space of K3 surfaces (compactified in a suitable way). The K3 surfaces corresponding to points of X are by no means arbitrary K3s. As well as being realized as surfaces of degree 8 in $\mathbb{P}^5$, they have the special property that their lattice of integral 2-dimensional homology classes contains a sublattice of algebraic cycles of rank 9, with completely determined intersection properties. However, it is known that these special K3 surfaces also have a moduli space, which is where the 4-manifold X gets mapped to. Moreover, the 3 quadratic forms we have constructed are real, so that we are in the domain of K3 surfaces over $\mathbb{R}$, that Nikulin has already studied; these constructions further stimulated Nikulin's work. This seems to be a very promising direction of study, with many interesting questions arising.

Another Torelli-type theorem proved by Tyurin concerns a classical object of study in algebraic geometry – the nonsingular cubic hypersurface in $\mathbb{P}^4$. It has been known for a long time that a hypersurface of this type contains infinitely many lines, which are parametrized by a surface, the *Fano surface* of the cubic. Whether the cubic is determined by its Fano surface was also a long-standing open question. It was Tyurin who gave the positive solution to this question; at the same time, he showed that the Fano surface of the cubic satisfies the analog of the Torelli theorem. This paper belongs to a period of intense activity on 3-folds in Moscow, in connection with Manin and Iskovskikh's negative solution of the Lüroth problem, and the study of Fano 3-folds by Iskovskikh and his school. At around the same time, Tyurin also wrote his survey on 3-folds, stimulated by the work of Griffiths and Clemens on the irrationality of the cubic 3-fold; the survey in its published form is entitled "Five lectures on threefolds", and these five lectures were really given at the algebraic geometry seminar in Moscow. It was at these lectures that the majority of the seminar participants first became acquainted with intermediate Jacobians and Prym varieties and their application to the proof of the irrationality of certain algebraic 3-folds.

Tyurin always assimilated new ideas appearing in algebraic geometry in a creative way, especially those that he was able to connect with the theory of vector bundles. Thus, starting off from an observation of Gunning, according to which the classical theory of uniformization of Riemann surface can be viewed as specifying a "special geometric structure" whose "transition functions" are fractional-linear transformations, Tyurin related this to theory of quadratic differentials that had been developed a long time ago (since Poincaré), and to the theory of rank 2 vector bundles (or in the current application, their projectivization) on Riemann surfaces. In the spirit of the general conception of vector bundles as a non-Abelian analog of the theory of divisors, he proved the non-Abelian analog of the main theorems of the theory of differentials of the third kind, determining how they are specified by their periods and principal parts in a neighborhood of the poles.

It became clear during the 1980s that the theory of vector bundles can be fruitfully applied in the geometry of manifolds of dimension greater than 1;

Tyurin played a large role in this. Thus he showed how the theory of vector bundles can be applied to the study of zero-cycles on algebraic surfaces, and he obtained especially vivid results in the case of K3 surfaces. These articles contain in particular the extension to the 2-dimensional case of Brill–Noether theory of "special divisors" on algebraic curves. The construction discovered by Tyurin led to a new direction of research: how to use a complex symplectic structure on an algebraic surface S to construct a structure of the same type on its moduli space of vector bundles.

At the same time, during the 1980s, relations developed between the theory of vector bundles and their moduli spaces with questions of mathematical physics (quantum field theory). This direction of study occupied Tyurin during his last two decades. However, classical constructions from the theory of vector bundles were frequently visible under the cloak of physical terminology. This applies right up to the final publication of Tyurin's life, where the terminology of Delzant models — however exotic it may sound to the traditional mathematician — corresponds to the moduli space of vector bundles on an algebraic curve. In particular, this paper involves a role for the space $\mathbb{P}^3 \setminus V$, where V is the Kummer surface, that appeared in Tyurin's very first paper.

There are many of his ideas that Tyurin did not himself carry to a conclusion in a published paper, sharing them instead with the young mathematicians who constantly surrounded him. For example, he drew attention to the paper of Drézet and Le Potier, in which certain at first sight strange fractions appeared as the "slopes" of exceptional bundles on $\mathbb{P}^2$. He succeeded in interesting young mathematicians working at the time in A.N. Rudakov's seminar in these questions. They succeeded in linking them with Markov numbers, braid groups, derived categories,.... This led to a beautiful theory, in which the impact of Tyurin's influence will probably be felt for a long time to come. The same applies to many of the scientific impulses originating with him.

As an eye-witness of the whole of Tyurin's scientific life, from his first steps onwards, I was constantly amazed by the extent of his emotional involvement. This moved Tyurin himself, and attracted many young mathematicians to work with him. It seems to me that his scientific work can best be characterized in the words of our common beloved Aristophanes:

> ...like a torrent of glory rushing across the plain, uprooting oak, plane tree and rivals and bearing them pell-mell in its wake.
>
> Aristophanes' *Hippeis* (Knights), 526

(Here "rivals" stands for scientific and logical difficulties, of course.)

I.R. Shafarevich

The geometry of moduli of vector bundles

The article originates from a short course of lectures given in I.R. Shafarevich's seminar at the Moscow State University, in September – Oktober 1973. The theme is the geometry of moduli of many-dimensional vector bundles over an algebraic curve.

Russian Math. Surveys. 1974. 29:6.

Preface.

An algebraic curve is both a one-dimensional subscheme of projective space and a Riemann surface. Hence the theory of the Jacobian of a curve has two aspects: the geometric and the analytic. The geometric theory of the Jacobian is the description of its properties as a projective variety, above all, of those properties that are preserved under variations, that is, the geometry of Abelian varieties.

The analytic theory of the Jacobian is the theory of θ–functions. From the geometric point of view the Jacobian = the Picard variety = the variety of moduli of one-dimensional bundles of degree 0. From the analytic point of view it is the manifold of one-dimensional unitary representations of the fundamental group of the curve.

The variety S of moduli of many-dimensional vector bundles can also be regarded both as an algebraic variety and as a set of unitary representations of the fundamental group of the curve. In the present article we are concerned exclusively with the geometric aspect of the theory of moduli of bundles.

The most striking result of this theory is: *every "Abelian" variety* (that is, a variation of S) *is the "Jacobian" of a curve*, (that is, the variety of moduli of bundles over some curve) (Chapter II, § 2). The geometrical ideas operating here were suggested by Newstead, Narasmhan and Ramanan (see [8], [10] and [13]). So that these ideas do not become lost in technicalities, we include in this article our own proofs of the basic results of the theory. As it is self-contained, it should allow the non-specialist to master its contents quickly.

CHAPTER 1
Introduction

§ 1 Various interpretations of *vector bundle* concept.

A vector bundle of dimension n on a manifold X is an element of $H^1(X, \mathbf{GL}(n))$, where $\mathbf{GL}(n)$ is the sheaf of germs of mappings of X into the full linear group $GL(n)$.

Each cocycle $h \in H^1(X, \mathbf{GL}(n))$ associates with an affine covering $\{U_i\}$ of X a set $\{h_{ij}\}$ of matrix functions such that h_{ij} is regular and regularly invertible on $U_i \cap U_j$ and $h_{ij}h_{jk}h_{ki} = 1$ on $U_i \cap U_j \cap U_k$. If V_0 is an n-dimensional vector space, then the matrix functions h_{ij} enable us to glue together the $V_0 \times U_i$ into a single variety V with projection $V \xrightarrow{\pi} X$ $\pi^{-1}(U_i) = V_0 \times U_i$. This triple (V, π, X) is again called a *vector bundle.*

In algebraic geometry vector bundle have four different interpretations.

I. The sheaf interpretation. A vector bundle is a locally free sheaf on X.

More precisely, with a bundle V we can associate its sheaf $\mathbf{V}$ of germs of sections. This is a locally free sheaf. Conversely, each locally free sheaf is the sheaf of germs of sections of a unique bundle V. The rank of the locally free sheaf is the dimension of the bundle.

This association has become so habitual that V and $\mathbf{V}$ are not even distinguished notationally.

II. The geometric interpretation. Over the complete variety X a vector bundle V as a variety is neither affine nor complete. There is a simple method of turning it into a complete (compact)variety, the operation of projectivization. Let $P(V_0)$ be projective space corresponding to the vector space V_0. Then $P(V_0) \times U_i$ can be glued by the same matrices h_{ij} into projective variety $P(V)$ with projection $\pi : P(V) \longrightarrow X$. This is now a complete variety, a geometrical object. It is easy to see that $P(V) = P(V')$ if and only if there is a one-dimensional bundle L such that $(V') = V \otimes L$.

The bundle $\pi^*(V)$ on $P(V)$ contains a one-dimensional "tautological" subbundle L whose fibre L_p at the point $p \in P(V)$ is the same one-dimensional subspace that defines the point p of the projectivization $P(V)$. The bundle $\tau = L^*$ is called the **anti-tautological** bundle on $P(V)$.

The pair $(P(V), \tau)$ uniquely determines the bundle V on X (in fact, constructively, by $V = (R^0\pi(\tau))^*$).

Projectivization is the simplest but not the only method of associating with a vector bundle a bundle with a complete fibre. Let k be an integer with $0 < k < \dim V$. Then $G_k(V) \xrightarrow{\pi} X$is a bundle on the Grassmann manifolds of k-subspaces of fibers. It is a compact manifold. In $\pi^*(V)$ there is a sub-bundle E of dimension k, whose fiber over a point of the Grassmannian is the subspace corresponding to that point. The bundle $\tau = E^*$ is called the *anti-tautological* bundle over $G_k(V)$.

The pair $(G_k(V), \tau)$ uniquely determines V on X: constructively, by

$$V = (R^0\pi(\tau))^* .$$

It is clear that $P(V) = G_1(V)$.

III. The arithmetic interpretation. A vector bundle is a class of *matrix divisors*.

Let X be a curve. A matrix bundle on X associates with each point $x \in X$ a functional matrix M_x in such a way that there are only finitely many points $x \in X$ at which M_x is not regular and regularly invertible at x. The assignments M_x and M'_x are equivalent if $M_x^{-1}M'_x$ is regular and regularly invertible at x for every x.

A class of matrix assignments is called a *matrix divisor*. The matrix divisors M_x and M'_x are equivalent if $M'_x \cdot M_x^{-1} = G$ is a matrix of rational functions on X not depending on the point $x \in X$.

The concept of a matrix divisor is analogous to that of a divisor, and the connection between matrix divisors and vector bundles is the same as that between divisors and one-dimensional (linear) bundles.

IV. The analytic interpretation. Among the vector bundles there are those obtained from a representation ρ of the fundamental group $\pi_1(X)$ in the full linear group $GL(n)$. These are the so-called *flat* bundles. Their precise construction is the following: let $\rho : \pi_1(X) \longrightarrow GL(n)$ be a representation of the fundamental group, U the universal covering manifold of X, and V_0 an n-dimensional vector space. Then $\pi_1(X)$ acts diagonally on $U \times V_0$, that is, $g(u, v) = (g(u), \rho_g(v))$ and $U \times V_0/\pi_1(X) = V$ is a bundle on X.

The condition for a bundle to be flat is purely algebraic [2] and for curves is very simple.

These four interpretations of a vector bundle can be well illustrated in the one-dimensional case.

EXAMPLE. *One-dimensional bundles.* This example is well-known to everybody. We only recall that the equivalence of a divisor class with a locally free sheaf of rank 1 provides the divisors with higher cohomology; in fact, cohomology first entered into arithmetic through this equivalence.

The geometric interpretation gives nothing for one-dimensional bundles, since $P(V) = X$ and $\tau = V^*$.

A one-dimensional bundle can be obtained from a representation of the fundamental group if and only if its Chern class is 0. Hence we get the analytic construction of the Picard variety: $\operatorname{Pic} X = \{\text{set of unitary characters of } \pi_1(X)\}$ [14].

The one-dimensional case is the intuitive mental baggage on which we depend in this article.

We recall some techniques of the geometric interpretation in the many-dimensional case, which are tautologous or lacking in the one-dimensional case.

These simple methods play such an important role in what follows that we have decided to devote a separate section to them.

§ 2 Exact triples of Grassmannizations.

Let V_0 be an n-dimensional vector space and $G_k(V_0)$ the Grassmann manifold of k-dimensional subspaces in V_0.

Let $\mathbf{V}_0$ denote the trivial bundle on $G_k(V_0)$ with fibre V_0. Then $\mathbf{V}_0$ contains a sub-bundle $E \subset \mathbf{V}_0$that is tautological, that is, the subspace $h \subset V_0$ is itself the fibre E_h at the point $h \in G_k(V_0)$. The bundle $E^* = \tau_{\mathbf{V}_0}$ is called *anti-tautological.* We have the embedding

$$0 \longrightarrow \tau^*_{V_0} \longrightarrow \mathbf{V}_0 .$$

To describe the factor bundle, we note that $G_k(V_0) = G_{n-k}(V_0^*)$ on $G_k(V_0)$ is the second antitautological bundle $\tau_{V_0^*}$, and is also the factor bundle:

$$0 \longrightarrow \tau^*_{V_0} \longrightarrow \mathbf{V}_0 \longrightarrow \tau_{V_0^*} \longrightarrow 0. \tag{1}$$

The dual triple

$$0 \longrightarrow \tau^*_{V_0^*} \longrightarrow \mathbf{V}_0^* \longrightarrow \tau_{V_0} \longrightarrow 0$$

corresponds to the right-hand side of the equation

$$G_k(V_0) = G_{n-k}(V_0^*) \ .$$

The bundles τ_{V_0} and $\tau^*_{V_0}$ have very simple cohomology.

Proposition 1.

1) $H^i(G_k(V_0), \tau^*_{V_0}) = 0$ *for all* i;

2) $H^0(G_k(V_0), \tau_{V_0}) = \mathbf{V}_0^*$;

3) $H^i(G_k(V_0), \tau_{V_0}) = 0$ *for all* $i \neq 0$;

4) $H^0(G_k(V_0), \operatorname{End} \tau_{V_0}) = \mathbf{C}$;

5) $H^i(G_k(V_0), \operatorname{End} \tau_{V_0}) = 0, i > 0$.

This will be proved later.

It is easy to show (and even easier to recall), that the tangent bundle to the Grassmannian coincides with $\mathrm{Hom}(\tau^*_{V_0}, \tau_{V_0^*})$:

$$\Theta(G_k(V_0)) = \tau_{V_0} \otimes \tau_{V_0^*}. \tag{2}$$

Now let V be a vector bundle on X and $\pi : G_k(V) \longrightarrow X$ its Grassmannization. Then $\pi^*(V)$ Contains the tautological bundle $E = \tau^*_V$ and τ_V is the anti-tautological bundle. Since $G_k(V) = G_{n-k}(V^*)$, we have, analogously to (1):

$$0 \longrightarrow \tau^*_V \longrightarrow \pi^*V \longrightarrow \tau_{V^*} \longrightarrow 0 \quad \text{(I.G.e.t.)}.$$

Definition 1. This exact triple of bundles on $G_k(V)$ is called the first Grassmannization exact triple (or I.G.e.t., for short).

Thus, if $x \in X$ is a point and V_x the fibre of V over x, then the Grassmannian $G_k(V_x)$ is the fibre of $G_k(V)$ over x, $\tau_V \mid_{G_k(V_x)} = \tau_{V_x}$ and the restriction of I.G.e.t. to $G_k(V_x)$ gives (1).

Let $G_k(V) \xrightarrow{\pi} X$ be the projection of the bundle. The symbol $R^i\pi$, as usual, denotes the i-th direct image of a sheaf.

Proposition 2.

1) $R^i\pi\tau^*_V = 0$ *for any* i;

2) $R^0\pi\tau_V = V^*$;

3) $R^i\pi\tau_V = 0$, $i > 0$;

4) $R^0\pi\mathrm{End}\tau_V = \mathcal{O}_X$;

5) $R^i\pi\mathrm{End}\tau_V = 0$, $i > 0$.

PROOF. Applying the see-saw principle [7], we find that assertion j) of Proposition 2 follows from the corresponding assertion of Proposition 1 when j is odd.

We consider the triple on $G_k(V)$ dual to I.G.e.t.:

$$0 \longrightarrow \tau^*_{V^*} \longrightarrow \pi^*V^* \longrightarrow \tau_V \longrightarrow 0,$$

and the direct image functor

$$0 \longrightarrow R^0\pi\tau^*_{V^*} \longrightarrow R^0\pi(\pi^*V^*) \longrightarrow R^0\pi(\tau_V) \longrightarrow R^1\pi\tau^*_{V^*}.$$

But $R^0\pi\tau^*_{V^*} = R^1\pi\tau^*_{V^*}$ by 1). In addition,

$$R^0\pi(\pi^*V^*) = V^* \otimes R^0\pi\mathcal{O}_{G_k(V)} = V^* \otimes \mathcal{O}_X = V^*.$$

Hence we get the isomorphism 2).

Let E be any bundle. Then $\mathrm{End}E$ splits into the direct sum $\mathrm{End}E = I \oplus \mathrm{ad}\ E$, where I is the trivial one-dimensional bundle and $\mathrm{ad}\ E$ is the bundle on the endomorphisms with trace zero.

Definition 2. The bundle ad E is called the *adjoint* bundle of E.

Clearly, if L is a one-dimensional bundle, then

$$\mathrm{ad}\,(E \otimes L) = \mathrm{ad}\ E, \quad \mathrm{ad}\ E^* = \mathrm{ad}\ E, \quad (\mathrm{ad}\ E)^* = \mathrm{ad}\ E.$$

Thus, $\mathrm{End}\tau_V = I \oplus \mathrm{ad}\ \tau_V$ and, According to 4) of Proposition 1, $R^0\pi\mathrm{End}\tau_V = R^0\pi(\mathcal{O}_{G_k(V)}) = \mathcal{O}_X$. This proves Proposition 1.

The projection $G_k(V) \xrightarrow{\pi} X$ is nowhere degenerate, and we have the epimorphism of tangent bundles

$$\Theta G_k(V) \xrightarrow{d\pi} \pi^*\Theta X \longrightarrow 0;$$

the kernel of $d\pi$ is called the *relative* tangent bundle and is denoted by Θ_π.

It is easy to obtain a formula analogous to (2): $\Theta_\pi = \tau_V \otimes \tau_{V^*}$. Multiplying I.G.e.t. by τ_V, we get the triple

$$0 \longrightarrow \mathrm{End}\tau_V \longrightarrow \pi^*(V) \otimes \tau_V \longrightarrow \Theta_\pi \longrightarrow 0 \quad \text{(II.G.e.t.)},$$

which is called the *second Grassmannization exact triple* (II.G.e.t.).

The bundle $\Theta^*_\pi = \Omega_\pi$ is called the *relative cotangent bundle* or the *bundle on relative differentials.*

Proposition 3.

1) $R^0\pi\Theta_\pi = \mathrm{ad}\ V$;

2) $R^i\pi\Theta_\pi = 0$, $i > 0$;

3) $R^1\pi\Omega_\pi = \mathcal{O}_X$;

4) $R^i\pi\Omega_\pi = 0$, $i \neq 1$.

PROOF. We apply the direct image functor to II.G.e.t.:

$$0 \longrightarrow R^0\pi\mathrm{End}\tau_V \longrightarrow R^0\pi\tau_V \otimes \pi^*(V) \longrightarrow$$
$$\longrightarrow R^0\pi\Theta_\pi \longrightarrow R^1\pi\mathrm{End}\tau_V \longrightarrow \cdots$$

But $R^i\pi\tau_V \otimes \pi^*(V) = V \otimes R^i\pi\tau_V$ and, According to 2) and 3) of Proposition 2, $R^0\pi\tau_V \otimes \pi^*V = V \otimes V^* = \mathrm{End}V$ $R^0\pi\tau_V \otimes \pi^*V = 0$, $i > 0$. Identifying the extreme terms of the resulting sequence by 4) and 5) of Proposition 2, we get

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathrm{End}V \longrightarrow R^0\pi\Theta_\pi \longrightarrow 0$$

and it is clear that the embedding $\mathcal{O}_X \longrightarrow \mathrm{End}V$ corresponds to the endomorphisms of multiplication by a constant. Note that, by definition $R^0\pi\Theta_\pi = \mathrm{ad}\ V$. Assertion 2) follows immediately from 5) of Proposition 2. We invert II.G.e.t.:

$$0 \longrightarrow \Omega_\pi \longrightarrow \pi^*(V^*) \otimes \tau^*_V \longrightarrow \mathrm{End}\tau_V \longrightarrow 0$$

and apply the direct image functor:

$$0 \longrightarrow R^0\pi\Omega_\pi \longrightarrow V^* \otimes R^0\pi\tau_V^* \longrightarrow \mathcal{O}_X \longrightarrow R^1\pi\Omega_\pi \longrightarrow$$
$$\longrightarrow \pi^*(V^*) \otimes R^1\pi\tau_V^* \longrightarrow \cdots$$

But by 1) Proposition 2 $R^i\pi\tau_V^* = 0$, and by 5), $R^i\pi\mathrm{End}\tau_V = 0$. Hence all that remains of the sequence is the isomorphism

$$0 \longrightarrow \mathcal{O}_X \longrightarrow R^1\pi\Omega_\pi \longrightarrow 0,$$

as required.

Now let $\mathcal{F}$ be an arbitrary sheaf on $G_k(V)$. The projection

$$G_k(V) \xrightarrow{\pi} X$$

enables us to decompose the cohomology of $\mathcal{F}$ on $G_k(V)$ into the cohomology of the direct image of $\mathcal{F}$ in X:

$$H^i(X, R^j\pi\mathcal{F}) \Rightarrow H^*(G_k(V), \mathcal{F}).$$

Applying this spectral sequence to the bundles τ, Θ_π and Ω_π we obtain the next result.

Proposition 4. *For any i:*

1) $H^i(G_k(V), \tau_V) = H^i(X, V^*)$;

2) $H^i(G_k(V), \Theta_\pi) = H^i(X, \mathrm{ad}\, V)$;

3) $H^{i+1}(G_k(V), \Omega_\pi) = H^i(X, \mathcal{O}_X)$.

Note that in the last case $H^{i+1}(G_k(V), \Omega_\pi) = H^i(X, R^1\pi\Omega_\pi)$; because of this the dimensions are not the same.

So we see that *the cohomology of Ω_π on $G_k(V)$ does not depend on the bundle V.*

In the special case $k = 1$, Grassmann manifold is projective space $G_1(V) = P(V)$ and Grassmannization is called projectivization. In this case τ_V is a one-dimensional bundle and I.G.e.t. is obtained from II.G.e.t. by multiplication by τ_V^*. In the many-dimensional case we cannot do this and are forced to have two exact triples.

Note now that for $k = 1$, that is, for projective space, Proposition 1 is well known: τ_{V_0} is the hyperplane bundle and $\mathrm{End}\tau_{V_0} = \mathcal{O}_{P_{n-1}}$.

We now prove Proposition 1 for any k.

PROOF OF PROPOSITION 1. We consider the projectivization of the bundle τ on $G_k(V_0)$:

$$P(\tau) \xrightarrow{\pi_1} G_k(V_0)\ .$$

Let τ_τ be the one-dimensional anti-tautological bundle of this projectivization. Then $H^0(P(\tau), \tau_\tau) = V_0^*$ and we get a mapping $\pi_2 : P(\tau) \longrightarrow P(V_0)$, given by the linear kernel $\mid \tau_\tau \mid$. If h is the one-dimensional hyperplane bundle on the projective space $P(V_0)$, then

(3) $$\tau_\tau = \pi_2^*(h).$$

But geometrically π_2 can be split into the embedding $P(\tau) \hookrightarrow P(V_0) \times X$ and the projection onto $P(V_0)$. Thus, the fibre $\pi_2^{-1}(p)$ is the manifold of k-subspaces Passing through the point p in $P(V_0)$, that is $\pi_2^{-1}(p) = G_{k-1}(V_0/p)$. Hence $\pi_2 : P(\tau) \longrightarrow P(V_0)$ is a locally trivial bundle on the Grassmannians. (More precisely, $P(\tau) \xrightarrow{\pi_2} P(V_0)$ is $G_{k-1}(\Theta P(V_0))$, the Grassmannization of the tangent bundle to the projective space.) The spectral sequence

$$H^i(P(V_0), R^i\pi_2(\tau)) \Rightarrow H^i(P(\tau), \tau) = H^i(G_k(V_0), \tau_{V_0})$$

degenerates into the isomorphism $H^i(P(\tau), \tau_\tau) = H^i(P(V_0), h)$, since by (3)

$$R^i\pi_2(\tau_\tau) = h \oplus R^i\pi_2(\mathcal{O}_{P(\tau)}) = \begin{cases} 0, & \text{for } i > 0 \\ 1, & \text{for } i = 0\ , \end{cases}$$

since a Grassmann manifold is regular.

1), 2) and 3) of Proposition 1 follow from this, as do 1), 2) and 3) Proposition 2.

Assertions 4) and 5) of Proposition 1 can be restated as follows:

$$H^i(G_k(V_0), \mathrm{ad}\ \tau_{V_0} = 0 \qquad \forall\ i$$

that is

(4) $$H^i(P(\tau), \Theta_{\pi_1}) = 0.$$

(Here we have used Proposition 3 for *projectivization.*)

The following argument is the exact prototype of the basic core of the whole article (see Chapter V, § 1, diagram (27)).

I. We have two projections of the variety $P(\tau)$:

$$\begin{tikzcd} & P(\tau) \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ G_k(V_0) & & P(V_0) \end{tikzcd}$$

Here $P(\tau) = G_1(\tau)$, and it is easy to see that relative to the second projection

$$P(\tau) = G_{k-1}(\Theta P(V_0))\ .$$

II. It is easy to see that $\Theta_{\pi_1} = \tau^*_{\Theta P(V_0)}$; in words: *the relative tangent bundle of the first projection is the tautological bundle of the Grassmannization of the tangent bundle for the second projection.*

From this (4) follows automatically:

$$H^i(P(\tau), \Theta_{\pi_1}) = H^i(G_{k-1}(\Theta P(V_0)), \tau^*_{\Theta P(V_0)}) = 0$$

by 1) of Proposition 2, which has already been proved. This completes the proof.

§ 3 Special properties of bundles on curves.

Let V be a bundle on a manifold X of dimension n. Then we can apply to V the sheaf operations of exterior powers $\Lambda^i V \quad (i = 1, \cdots)$. For $i > n$, $\Lambda^i V = 0$, and $\Lambda^n V = \det V$ is the one-dimensional bundle called the *determinant* of V.

In this article we consider bundles over an algebraic curve X. Naturally, they have special properties, which simplify their investigation.

In this section we list these properties.

In the first place, every one-dimensional bundle L on X determines an integer, its degree $\deg L$, the degree of the divisor to which it corresponds.

For a many-dimensional bundle V

$$\deg V = \deg \det V.$$

This integer not only splits the variety of classes of bundles into components, but also enables us to distinguish the components of highest dimension, the *stable* bundles.

Definition 3. A bundle V on a curve X is called *stable*, if for any proper sub-bundle $M \subset V$

$$\frac{\deg M}{\dim M} < \frac{\deg V}{\dim V}. \tag{5}$$

Multiplication of V by a one-dimensional bundle L adds $\deg L$ to each side of the inequality (5), and so does not alter it. Hence stability is preserved under multiplication of V by a one-dimensional bundle.

Next, for V^* this inequality is multiplied by -1, but M^* becomes a factor-bundle of V^*, and the inequality is preserved for the kernel. Thus, stability is preserved under inversion.

Intuitively, this numerical concept of stability means nothing to people who do not work with bundles. Hence, for the time being, stable bundles may be thought of as bundles forming the component of maximum dimension in the set of classes, and having no non-trivial endomorphisms.

As we have already noted in § 2, the bundle $\mathrm{End}V$ is the direct sum of the trivial bundle I and ad V. Therefore, also for sections we have: $H^0(X, \mathrm{End}V) = I \oplus H^0(X, \mathrm{ad}\ V)$. The sections of the sheaf ad V are those endomorphisms of V for which the image is a proper subbundle in V; such endomorphisms are also called non-trivial.

If $e \in H^0(X, \text{ad } V)$, then E can be represented as an extension

$$0 \longrightarrow \ker e \longrightarrow E \longrightarrow \mathrm{Im} e \longrightarrow 0;$$

here $\deg E = \deg \ker e + \deg \mathrm{Im} e$ and $\dim E = \dim \ker e + \dim \mathrm{Im} e$. Then the fraction $\frac{\deg E}{\dim E}$ lies *between* $\frac{\deg \ker e}{\dim \ker e}$ and $\frac{\deg \mathrm{Im} e}{\dim \mathrm{Im} e}$ and is less than one of them, violating the inequality (5). Hence *stable* bundles have no non-trivial endomorphisms.

On a curve X every point is a divisor. This leads very easily to the first property:

PROPERTY I. *As an element of the K-functor, every bundle on a curve is a sum of one-dimensional bundles.*

More simply, this means that every bundle can be represented as a chain of extensions of one-dimensional bundles.

PROPERTY II. *If V is generated by global sections, then it has an $(n-1)$-dimensional trivial subbundle, that is, V can be represented in the form of an extension*

$$0 \longrightarrow I_{n-1} \longrightarrow V \longrightarrow \det V \longrightarrow 0. \tag{6}$$

Let $\underline{H^0(X, V)} = H^0(X, V) \times X$ be the trivial bundle on X with fibre $H^0(X, V)$. The mapping $\underline{H^0(X, V)} \xrightarrow{\varphi} E$, $\varphi(s, x) = s(x)$, is an epimorphism if and only if vector of the fibre is generated by global section. Consider $\ker \varphi$ and the projection of $\ker \varphi$ into $H^0(X, V)$. This projection is layer-wise linear and we can projectivize it: $f : P(\ker \varphi) \longrightarrow P(H^0(X, V)) = P$, $\mathrm{codim}_P f(P(\ker \varphi)) \geqslant (n-1)$. But $f(P(\ker \varphi)) \subset P$ is a manifold of sections having zeros. Thus, any subspace not intersecting $f(P(\ker \varphi))$, consists of non-vanishing sections and so defines a trivial sub-bundle on V. Hence it follows that a subbundle I_{n-1} not only exists in V, but can be chosen so that it passes through an arbitrary point of V.

PROPERTY III. *Let V be an arbitrary stable bundle of degree d on X. Then there exists an integer $N(d)$ such that if L is any one-dimensional bundle of degree $\geqslant N(d)$, then V can be represented in the form of an extension*

$$0 \longrightarrow L^* \otimes I_{n-1} \longrightarrow V \longrightarrow \det V \otimes L^{n-1} \longrightarrow 0. \tag{7}$$

The previous property reduces this assertion to the following.

There exists an absolute constant N_0 such that any stable bundle of degree $\geqslant N_0$ is generated by global sections (is very ample).

Let $V(-x) = V \otimes L^*(x)$, where $L(x)$ is the bundle corresponding to the divisor x.

Suppose that the fibre V_x over $x \in X$ is not generated by global sections. This means that the increase of the sections $\dim H^0(X, V) - \dim H^0(X, V(-x))$ is less than the increase in the Euler characteristics $\chi(V) - \chi(V(-x)) = \dim V = n$, but this is equivalent to $H^1(X, V(-x)) \neq 0$. Then, by Serre duality, $H^0(X, \Omega \otimes V^*(x)) \neq 0$, where Ω is the one-dimensional canonical bundle on X. A section of $\Omega \otimes V^* \otimes L(x)$ determines a subbundle M of non-negative degree, but $\Omega \otimes L(x) \otimes V^*$ is stable, so that

$$0 \leqslant \deg M < -\deg V + n(2g-1).$$

It follows that if $\deg V \geqslant n(2g-1)$ and V is stable, then V is generated by global sections.

§ 4 Variations of bundles.

The example of one-dimensional bundles on a curve already shows that bundles have moduli, that they depend on algebraic parameters and can be varied.

We consider first the theory of *local* variation of bundles.

We note at once that the association of the one-dimensional bundle $\det V$ with V reduces the classification of bundles to the description of bundles with a fixed $\det V$ and the description of classes of one-dimensional bundles, which is already known.

Here it does not matter what value the determinant takes, since for a one-dimensional bundle L

$$\det(V \otimes L) = \det V \otimes L^n,$$

and by such a multiplication we can reduce $\det V$ to an arbitrary bundle of the same degree.

In addition, the classification of bundles V with fixed determinant coincides locally with the classification of the projectivization $P(V)$, since there are only *finitely* many bundles V with the same determinant and the same projectivization.

But $P(V)$ is a compact non-singular variety, and we can apply to it the theory of local variation of Kodaira and Spencer [4].

According to this theory, if M is a piece of the variety of moduli around a point of $P(V)$ and $H^2(P(V), \Theta P(V)) = 0$, then

$$\Theta M_{P(V)} = H^1(P(V), \Theta P(V)),$$

that is, the tangent space to the set of variations at a point of $P(V)$ is a one-dimensional cocycle with coefficients in the sheaf of germs of vector fields.

Let us investigate the corresponding cohomology spaces. The variety

$$P(V) \xrightarrow{\pi} X$$

is fibered by the projection π on X, from which we get an exact triple on $P(V)$:

$$(8) \qquad 0 \longrightarrow \Theta_\pi \longrightarrow \Theta P(V) \xrightarrow{d\pi} \pi^*(\Theta X) \longrightarrow 0.$$

We now recall equation 2) of Proposition 3 in § 2:

$$H^i(P(V), \Theta_\pi) = H^i(X, \text{ad } V) .$$

Also $H^i(P(V), \pi^*(\Theta X)) = H^i(X, \Theta X)$. But X is a curve, therefore, $H^i(X, *) = 0$ for $i > 1$, hence of the whole cohomology sequence of the triple (7) there remains the exact triple

$$(9) \qquad 0 \longrightarrow H^1(X, \text{ad } V) \longrightarrow \Theta M_{P(V)} \longrightarrow H^1(X, \Theta X) \longrightarrow 0.$$

and $H^2(P(V), \Theta P(V)) = 0$. This means that *locally* the variation of a projective bundle decomposes into the variation of the base X ($H^1(X, \Theta X)$ is tangent to the variety of moduli of the curves at the point of X), and the variation of the bundle with fixed base.

Here we must pause. The inertia of the classificatory idea urges us to use this decomposition, to forget about the variation of the curve, and to study only the variations of the bundles with fixed base. But if we return to the example — fundamental to our intuitive baggage — of a one-dimensional bundle, we see that such a decomposition should be avoided at all costs.

In fact, the theory of variations of a one-dimensional bundle, that is the theory of the Jacobian variety of a curve, originated in the middle of the last century as the theory of theta-functions, which combined both the variations of the divisor and those of the curve. The unity is not broken, and the price of this is known to any geometer who has ever translated properties of the singular points of the Poincaré divisor of the Jacobian of a curve into the language of moduli of the curve itself.

The theory of the Jacobian of a curve gave rich information about the variety of moduli of curves, and we shall try to present here the theory of variation of many-dimensional bundles so that this information is increased. The étale theorem, the symmetry theorem and Torelli's theorem to be given below show that the many-dimensional theory reflects the properties of curves more precisely, and it is possible that with its help we may succeed in getting decisive information on the geometry of varieties of moduli of curves.

This remark touches on the general direction of our thought, but meanwhile even the local variation, the triple (8), enables us to count the number of moduli of bundles on a fixed curve with a fixed determinant. This number is equal to $\dim H^1(X, \text{ad } V)$, and for a "general" stable bundle it is equal, by the Riemann–Roch theorem, to $\chi(\text{ad } V)$, since $H^0(X, \text{ad } V) = 0$. Now $\deg \text{ad } V = 0$, hence $\chi(\text{ad } V) = \dim \text{ad } V \cdot (g-1) = (n^2-1)(g-1)$, where n is the dimension of the bundle.

We now turn to the global variation, that is, to the global variety of moduli.

The only correctly stated problem on the global variety of moduli is the "universal problem" of Grothendieck. For bundles with fixed base and fixed determinant it runs as follows.

I. *There exists a quasi-projective variety* $S_{n,d}$ *and* n*-dimensional bundle* $U_{n,d}$ *on the direct product* $X \times S_{n,d}$ *such that*:

a) *for any stable bundle* V *on* X *with* $\dim V = n, \deg V = d$, *and with fixed determinant there is a unique point* s *on* $S_{n,d}$ *such that* $V \cong U\,|_{(X\times s)}$;

b) *for any bundle* V *on* $X \times B$, *where all bundles* $V\,|_{(X\times b)}$ *are stable and have the same determinant, there exists a regular morphism* $\varphi : B \longrightarrow S_{n,d}$ *and a one-dimensional bundle* L *on* $S_{n,d}$ *such that* $V = (1 \times \varphi)^*(U \otimes \mathrm{pr}_S^* L)$.

We recall once more that by multiplying a bundle by a one-dimensional bundle we can make its determinant arbitrary, and its degree has the same residue modulo n.

Thus, the possible suffixes n and d of the series of varieties $S_{n,d}$ are arranged as follows: n runs through all integers and d runs through all residues modulo n.

Proposition 5. I. *For any given* (n, d) *there exists a quasi-projective variety* $S_{n,d}$, *parametrizing the classes of stable bundles of dimension* n *with fixed determinant of degree* d.

II. *If* n *and* d *are coprime, then on* $X \times S_{n,d}$ *there exists a universal family* $U_{n,d}$ *with the properties* a) *and* b)*. In this case* $S_{n,d}$ *is a non-singular compact variety.*

I is a simple special case of Mumford's general theory [5]. II is in [14].

Note that if n and d are not coprime, then no universal family exists on $X \times S_{n,d}$ ([13] and [11]).

However, for projective bundles there always exists a universal family that is not the projectivization of a vector bundle and is locally isotrivial, but not trivial [5].

In what follows, we do not need the suffixes n and d and we shall consider a non-singular compact variety S and a universal bundle U on $X \times S$, that is, the case when $(n, d) = 1$, And we shall specially mention all the results that hold for arbitrary n and d. As a simple consequence of the universal problem we have:

Proposition 6. *S is unirational.*

PROOF. For any stable bundle V we have the representation (7)

$$0 \longrightarrow I_{n-1} \otimes L^* \longrightarrow V \longrightarrow \det V \otimes L^{n-1} \longrightarrow 0,$$

where L is a one-dimensional bundle of sufficiently high degree.

The set of such extensions is given by the following projective space: $P = P(H^1(X, \det(V^* \times L^*) \otimes I_n))$ [2]. We can collect all the extensions into a family with base P, and an open set P_0 in P is formed by the stable bundles. (We shall meet this in Chapter III, § 2). Then we have a mapping $\varphi : P_0 \longrightarrow S$ into the set of classes. This is a rational uniformization, which proves the theorem.

It is not much harder to prove the rationality of S (Newstead [12]).

CHAPTER 2
The Poincaré bundle

§ 1 The adjoint Poincaré bundles.

Thus, let us consider a universal family, that is, a bundle U on $X \otimes S$. We can regard it as family of bundles on X and as a family of n-dimensional bundles on S parametrizing the curve X.

Definition 4. The bundle $U_x = U \mid_{(x \times S)}$ on S is called the *Poincaré bundle.*

We note, first of all, that this definition depends on choice of the universal family U on $X \otimes S$. It follows from property b) of the universal family (see § 4 Chapter I) that, for any one-dimensional bundle L on S,

$$U' = U \otimes \mathrm{pr}_S^* L$$

is also universal, and any universal family is of this form.

Hence the Poincaré bundle U_x is only determined up to multiplication by a one-dimensional bundle, that is, only the projectivization $P(U_x)$ is uniquely determined.

This concept is entirely vacuous in the one-dimensional case. To define a Poincaré *vector* bundle uniquely we need the *normalization* of the universal bundle on $X \otimes S$.

The Riemann normalization. In the one-dimensional case $S = \mathrm{Pic}\, X$, the universal bundle U on $X \times \mathrm{Pic}\ X$ corresponds to a divisor D on $X \times \mathrm{Pic}\ X$. Suppose that D is effective and that $H^0(X \times \mathrm{Pic}\ X, D) = C$. Then D is uniquely determined, and $D_x = \Theta \subset \mathrm{Pic}\ X$ is called the *Poincaré divisor* ([7]).

Normalization in the many-dimensional case. As will be proved a little later (the theorem on $h^{*,1}$), $\mathrm{Pic}\ S = \mathbf{Z}$. Hence the bundle on S also have degree $\deg \det V$. The Poincaré vector bundle U_x on S is uniquely determined by the inequalities $0 \leqslant \deg U_x < n$.

Definition 5. The bundle $\mathrm{ad}\ U_x$ is called the *adjoint Poincaré bundle* on S.

The adjoint Poincaré bundle is completely unique because $\mathrm{ad}\,(U_x \otimes L) = \mathrm{ad}\ U_x$.

Hence, there are two families of bundles on S: U_x and $\mathrm{ad}\ U_x$, parametrized by the curve X.

THEOREM 1. *The curve X is the variety of moduli for the Poincaré bundle.*

Let $S(V)$ be the connected component of the variety of moduli of bundles on a manifold M containing V. Then $S(U_x) = X$ for any bundle U_x on S.

If, as in the one-dimensional case, we could fix a "polarisation" on S, the Poincaré bundle U_x, then the construction of Torelli's theorem would be quite simple: $S(U_{x_0}) = X$.

Note that in the one-dimensional case there is no such simple reconstruction of the curve X from the Poincaré divisor, because in this case $S(\Theta) = \text{Pic } X$.

The proof of Theorem 1 is divided into two parts.

THEOREM 2. *All the Poincaré bundles are distinct, that is, $U_x \not\cong U_{x'}$, $x \neq x'$.*

It follows that $X \xrightarrow{\varphi} S(U_x)$ is an embedding. For X to coincide with a component of $S(U_x)$, it suffices to prove that $S(U_x)$ is one-dimensional, or, what comes to the same thing, that the tangent space to $S(U_x)$ is one-dimensional. According to the isomorphism for local variation, $\Theta S(U_x)_{U_x} = H^1(S, \text{ad } U_x)$. Hence the second step of the proof is purely cohomological.

THE NARASIMHAN–RAMANAN THEOREM. *For any $x \in X$*

$$H^i(S, \text{ad } U_x) = H^{i-1}(S, \mathcal{O}_S)$$

for $i \leqslant N_g(n)$, where $N_g(n)$ is a constant depending only on the genus g of the curve and the dimension n of the bundle. A crude estimate is: $N_g(n) \leqslant \frac{n^2}{2}(g-3)$(see Chapter V, § 1).

This theorem has no analogue in the one-dimensional case. The proof of Theorem 2 is an analogue of the theory classically known as the "inversion problem". To make the analogy clear we recall the geometrical part of the "inversion problem" in the one-dimensional case.

I. An immersion $X \overset{\varphi}{\hookrightarrow} \text{Pic } X = S$ is constructed such that the restrictions of the Poincaré divisor Θ_0 to all the *variations of* φ give all the classes of one-dimensional bundles. On the other hand, the restrictions of the divisors Θ_x to $\varphi(X)$ give different divisors on X. Hence it follows finally that $\Theta_x \not\cong \Theta_{x'}$, $x \neq x'$ on Pic X.

II. The immersion $X \overset{\varphi}{\hookrightarrow} \text{Pic } X$ has the simple topological property:

$$\varphi_* : H_1(X, \mathbf{Z}) \longrightarrow H_1(\text{Pic } X, \mathbf{Z})$$

is an isomorphism. In fact, it is defined by this property. The isomorphism φ_* induces an isomorphism of the Albanese variety $A(X) = \text{Pic } X$.

In the many-dimensional case an analogous mapping $\varphi : X \longrightarrow S$ can be constructed having the "inversion" property. Also we can construct a mapping of the direct product $P^1 \times X \xrightarrow{\varphi} S$ such that for each point of the projective line $p \in P^1$ the mapping $\varphi\,|_{(p \times X)}$ is an "inversion" mapping and

$$\varphi_* : H_3(X \times P^1, \mathbf{Z}) \longrightarrow H_3(S, \mathbf{Z}) \tag{10}$$

is an isomorphism. It induces an isomorphism $J(X) \cong J^3(S)$ between the Jacobian of the curve X and the third intermediate Jacobian of S (for definition see [17]).

In Chapter IV we shall construct φ, establish the isomorphism (9), and show that

$$U_x \mid_{\varphi(X\times P^1)} \not\cong U_{x'} \mid_{\varphi(X\times P^1)}$$

for $x \neq x'$, from which the assertion of Theorem 2 follows. Chapter V is devoted to the proof of the Narasimhan-Ramanan theorem, but now we obtain all the geometrical consequences of it and Theorem 2.

§ 2 Tensors.

Let U be the universal bundle on $X \times S$. Then $R^1\mathrm{pr}_S\,\mathrm{ad}\,U = \Theta S$ is the tangent bundle to S. This statement globalizes the local statement $\Theta S_s = H^1(X, \mathrm{ad}\,U_s)$, where $U_s = U\mid_{(X,s)}$, and s is a point of S.

THE SYMMETRY THEOREM. $R^1\mathrm{pr}_S\,\mathrm{ad}\,U = \Theta X$.

PROOF. By the Narasimhan–Ramanan theorem,

$$H^2(S, \mathrm{ad}\,U_x) = 0 \quad \text{and} \quad H^1(X, \mathrm{ad}\,U_x) = 1$$

do not depend on x. By the see-saw theorem ([7]), $R^1\mathrm{pr}_X\,\mathrm{ad}\,U$ is a locally free sheaf of rank 1. The family of bundles $\{U_x\}$, $x \in X$, determines a homomorphism of the local variation

$$\Theta X \longrightarrow R^1\mathrm{pr}_X\,\mathrm{ad}\,U\ .$$

Since by Theorem 2 the variation is non-trivial, this homomorphism is an embedding and the equality of dimensions shows that it is an isomorphism.

Two Leray spectral sequences converge to the space of sheaf cohomology of $\mathrm{ad}\,U$ on $X \times S$:

$$\begin{tikzcd} & H^*(X\times S, \mathrm{ad}\,U) & \\ H^i(X, R^j\mathrm{pr}_x\,\mathrm{ad}\,U) \arrow[ur] & & H^i(S, R^j\mathrm{pr}_s\,\mathrm{ad}\,U) \arrow[ul] \end{tikzcd}$$

But $R^j\mathrm{pr}_x\,\mathrm{ad}\,U = 0$, $j < N_g$, $j \neq 1$, by the Narasimhan–Ramanan theorem, and $R^j\mathrm{pr}_s\,\mathrm{ad}\,U = 0$, $j \geqslant 2$, since the fibre of the bundle pr_s is one-dimensional, and $R^0\mathrm{pr}_s\,\mathrm{ad}\,U = 0$, since $H^0(X, \mathrm{ad}\,U) = 0$, because U_s is a stable bundle on a curve (see Chapter I, § 3). Thus, in dimensions $\leqslant N_g$ the spectral sequence reduces to the isomorphisms:

(11)
$$\begin{tikzcd} & H^i(S\times X, \operatorname{ad} U) \arrow[dl, "\simeq"'] \arrow[dr, "\simeq"] & \\ H^{i-1}(X, R^1\mathrm{pr}_x \operatorname{ad} U) \arrow[rr, equal] & & H^{i-1}(S, R^1\mathrm{pr}_s \operatorname{ad} U) \end{tikzcd}$$

As a corollary we obtain some important theorems.

THE AUTOMORPHISMS THEOREM. *The group of biregular automorphisms of S is finite.*

PROOF. $H^0(S,\Theta S) = H^0(X,\Theta X) = 0$ hence, the group of automorphisms is discrete. Also, the anticanonical bundle of S is ample. Finiteness now follows.

THE ÉTALE THEOREM. $H^1(X,\Theta X) \cong H^1(S,\Theta S)$.

This isomorphism has the following interpretation.

Let M_c be the variety of moduli of curves, and M_S the variety of moduli of S (that is, of variations of S as a variety). Then the correspondence $X \vdash\!\sim\!\longrightarrow S$ induces a morphism $M_c \xrightarrow{f} M_S$. At $X \in M_c$ the tangent space to M_c is identified with $H^1(X,\Theta X)$, and at $S = f(X)$ with $H^1(S,\Theta S)$. The isomorphism of the theorem can be interpreted as df, the differential of f, and so f is étale, that is, the local Torelli theorem holds for $X \vdash\!\sim\!\longrightarrow S$, furthermore, it follows from the global Torelli theorem (see [15] or Chapter III) that f is an isomorphism:

The variety of moduli of S coincides with the variety of moduli of curves.

This many-dimensional effect is absent in the one-dimensional case, since then $M_S = M_a$ is the variety of moduli of Abelian varieties. The mapping $M_c \overset{f}{\vdash\!\sim\!\longrightarrow} M_a$ is an embedding (Torelli's theorem), but the image is not the whole of M_a and has a complicated description by Shottky relations (see also [1]).

THE THEOREM ON $h^{*,1}$. $H^i(S,\Omega S) \cong H^{i-1}(X,\mathcal{O}_X)$ *for* $i \leqslant N_g(n)$

PROOF. By Serre duality,

$$(R^1\mathrm{pr}_S \operatorname{ad} U)^* = R^0\mathrm{pr}_S(\mathrm{pr}_X^*\Omega X \otimes \operatorname{ad} U).$$

Again we use two spectral sequences:

$$\begin{tikzcd} & H^*(S\times X, \mathrm{pr}_X\Omega X\otimes \operatorname{ad} U) & \\ H^i(S, R^j\mathrm{pr}_S(\mathrm{pr}_X^*\Omega X \otimes \operatorname{ad} U)) \arrow[ur] & & H^i(X, \Omega X\otimes R^j\mathrm{pr}_X \operatorname{ad} U) \arrow[ul] \end{tikzcd}$$

It follows from the stability that for $j \neq 0$

$$R^j\mathrm{pr}_S(\mathrm{pr}_X^*\Omega X \otimes \operatorname{ad} U) = 0 \quad \text{and} \quad R^0\mathrm{pr}_{S'}(\mathrm{pr}_X^*\Omega X \otimes \operatorname{ad} U) = \Omega S.$$

By the Narasimhan-Ramanan theorem, $R^j\mathrm{pr}_X \operatorname{ad} U = 0$, $j \neq 1$ (within the necessary limits), and $R^1\mathrm{pr}_X \operatorname{ad} U = TX$ by the symmetry theorem. Hence we obtain isomorphisms

$$
(12)\qquad
\begin{tikzcd}
 & H^i(S \times X, \mathrm{pr}_X^* \Omega X \otimes \mathrm{ad}\, U) \arrow[dl, "\sim"'] \arrow[dr, "\sim"] & \\
H^i(S, \Omega S) \arrow[rr, equal, "\sim"] & & H^{i-1}(X, \mathcal{O}_X)
\end{tikzcd}
$$

and tensor invariants

$$h^{0,1}(S) = 0, \quad h^{1,1}(S) = 1, \quad h^{2,1}(S) = g.$$

$\square$

COROLLARY. Pic $(S) = \mathbf{Z}$.
REMARK. For $i = 2$ the isomorphism (12)

$$H^{2,1}(S) \cong H^{1,0}(X)$$

can be interpreted as the differential of the isomorphism

$$J^3(S) \cong J(X),$$

which will be constructed in Chapter IV.

§ 3 Problems and conjectures.

The problems and conjectures scattered throughout the article and collected in this section indicate certain directions of thought. We do not insist by any means on their competitive value. Many of them are obviously not difficult and are simple exercises, but they either fill out the general picture or underline the analogy with the classical theory ($n = 1$).

First of all, if n and d are not coprime, then S is not complete, but it can be completed by a standard method (Seshadri [14]) and the completion desingularised. Let $\tilde{S}$ be the resulting variety. As already mentioned, there is no Poincaré bundle U_x on $\tilde{S}$, but there is its adjoint ad U_x.

Problem I. *Do the theorems of the previous section hold for $\tilde{S}$?*

On the curve X there exists a series of bundle *canonically* connected with the curve:

$$E_n^k(X) = R^k \mathrm{pr}_X \underbrace{\mathrm{ad}\ \mathrm{ad}\ \cdots\ \mathrm{ad}}_{n \text{ times}} U$$

We know that $E_1^0 = 0$ and $E_1^1 = \Theta X$. There is the same series on S:

$$E_n^k(S) = R^k \mathrm{pr}_S\ \mathrm{ad}\ \mathrm{ad}\ \cdots\ \mathrm{ad}\ U,$$

$$E_1^0 = 0, \quad E_1^1 = \Theta S\,.$$

PROBLEM II. *What are the bundles $E_n^k(X)$ and $E_n^k(S)$?*

Apparently they are bundles on jets.

By the theorem on automorphisms, the group Aut$S$ is finite.

Let $J_n(X)$ be the group of points of order n on J(X), that is, the group of divisor classes of order n on X. This group acts in the following way on S:

$$\sigma \in J(X), \quad E^\sigma = E \otimes L(\sigma).$$

It is clear that $\det E^\sigma = \det E$ and that stability is preserved.

CONJECTURE. There is a central extension

$$1 \longrightarrow J_n(X) \longrightarrow \mathrm{Aut}S \longrightarrow \mathrm{Aut}X \longrightarrow 1$$

where Aut$X$ is the automorphism group of the curve X.

This conjecture has been proved by Newstead [10] in the case $n = 2$, $g = 2$. It is needed for the construction of an analogue to the theory of theta-constants of Mumford's method [6].

Finally: PROBLEM III. *Calculate all the $h^{p,q}(S)$.*

CONJECTURE. *$H^{i+q}(X, \Omega^q) \cong H^{i+p}(S, \Omega^p)$ for any independent q and p.*

CHAPTER 3
Elementary operations and their variations

§ 1 Elementary operations.

I. An elementary operation on the divisor of a curve X is the addition of a point $x \in X$ to the divisor or the subtraction of a point $\xi - x$; in the language of sheaves this means:

$$0 \longrightarrow \mathrm{elm}_x L(\xi) \longrightarrow L(\xi) \xrightarrow{\alpha} \mathcal{O}_x \longrightarrow 0,$$

where α is any non-zero homomorphism into the skyscraper $\mathcal{O}_x$ over $x \in X$.

In the many-dimensional case we consider a k-dimensional subspace $g \in V_x$ in the fibre of a bundle Y over the point $x \in X$. This uniquely determines the epimorphism

$$V \xrightarrow{\alpha(g)} \mathcal{O}_x^{n-k} \longrightarrow 0,$$

where $\mathcal{O}_x^{n-k}$ is the skyscraper of dimension $x \in X$ over $n-k$, where $n = \dim V$. In fact, as a homomorphism into a skyscraper, $\alpha(g)$ is uniquely determined by its restriction $\alpha(g)_x$ to the fibre over x, and $\alpha(g)_x$ is uniquely determined by the condition $\ker \alpha(g)_x = g$. The kernel of $\alpha(g)$ is a locally free sheaf.

Definition 6.

1) The kernel of $\alpha(g)$ is denoted by the symbol $\mathrm{elm}_x^k(g)(V)$.

2) The operation $V \rightsquigarrow \mathrm{elm}^k(g)(V)$ is called an *elementary operation of degree k*.

3) The exact triple

$$0 \longrightarrow \mathrm{elm}_x^k(g)(V) \xrightarrow{i} V \xrightarrow{\alpha(g)} \mathcal{O}_x^{n-k} \longrightarrow 0 \quad \text{(a.e.t.)}$$

is called *adjoint exact triple* (a.e.t. for short).

It is clear that $\dim \operatorname{elm}_x^k(g)(V) = \dim V$. The restriction of a.e.t. to the fibre at a point x gives a four-term sequence:

$$0 \longrightarrow \ker i_x \longrightarrow (\operatorname{elm}_x^k(g)(V))_x \xrightarrow{i_x} V_x \xrightarrow{\alpha(g)_x} \mathcal{O}_x^{n-k} \longrightarrow 0.$$

Thus, the fibre over x of the bundle $\operatorname{elm}_x^k(g)(V)$ contains the $(n-k)$-dimensional subspace $\ker i_x = g'$. It is easy to see that

$$V \otimes L^*(x) = \operatorname{elm}_x^{n-k}(\ker i_x)(\operatorname{elm}_x^k(g)(V)). \tag{13}$$

From a.e.t. it follows immediately that

$$\det \operatorname{elm}_x^k(g)(V) = \det V - (n-k)x. \tag{14}$$

Proposition 7 (ITERATION OF OPERATIONS). *Suppose that V_x contains two subspaces g_1 and g_2, with $g_1 \subset g_2$, $\dim g_1 = k_1$, $\dim g_2 = k_1 + k$. Then $(\operatorname{elm}_x^{k_1+k}(g_2)(V))_x$ contains the subspace $i_x^{-1}(g_1)$ and*

$$\operatorname{elm}_x^{k_1}(g_1)(V) = \operatorname{elm}_x^{n-k}(i_x^{-1}(g_1))(\operatorname{elm}_x^{k_1+k}(g_2)(V)). \tag{15}$$

The proof follows at once from a.e.t.

This proposition shows that among the elm's there is a "most" elementary one, and an elementary transformation of any stage can be obtained from it by superposition. This $\operatorname{elm}_x^{n-1}(g)(V)$, which we shall simply denote by $\operatorname{elm}_x(g)(V)$.

The next propositions follow at once from a.e.t.

Proposition 8.

1) $\operatorname{elm}_x^k(g)(V \otimes L) = L \otimes \operatorname{elm}_x^k(g)(V)$, *that is, an elementary operation commutes with multiplication by a one-dimensional bundle*;

2) $(\operatorname{elm}_x^k(g)(V))^* = \operatorname{elm}_x^{n-k}(g^*)(V^*)$.

From the first statement it follows that an elementary operation can be described in terms of the projective variety $P(V)$.

However, before this description we investigate the relation between the elm$'$'s and stability.

Clearly, the elm of a stable bundle need not be stable.

Definition 7. A bundle V is called *superstable*, if every elementary operation of any degree from V is a stable bundle.

Proposition 9 (NUMERICAL CRITERION FOR SUPERSTABILITY). *A bundle V is superstable if and only if for any proper subbundle $M \subset V$*

$$\frac{\deg M}{\dim M} < \frac{\deg V}{\dim V} - \frac{\dim V - \dim M}{\dim V}.$$

PROOF. We have the inclusion

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{elm}_x^k(g)(V) \arrow[r, "i"] & V \arrow[r, "\alpha"] & \mathcal{O}_x^{n-k} \arrow[r] & 0, \\
0 \arrow[r] & M' \arrow[r] \arrow[u, hook] & M \arrow[r] \arrow[u, hook] & \mathcal{O}_x^l \arrow[r] \arrow[u, hook] & 0,
\end{tikzcd}$$

where $n = \dim V$, $n_1 = \dim M$, $e = \dim(M_x \cap \ker i_x)$. Thus

$$M' = \mathrm{elm}_x^{n_1-l} \times (M_x \cap \ker\, i_x)(M)$$

and $e \geqslant \max(0, n_1 - k)$.

We need the inequality

$$\frac{\deg M - l}{n_1} < \frac{\deg V - (n-k)}{n} \tag{16}$$

which is equivalent to

$$\frac{\deg M}{n_1} < \frac{\deg V}{n} - \frac{n - n_1}{n} + \left(\frac{e}{n_1} - \frac{n_1 - k}{n}\right),$$

that is, (16) + $\left(\frac{e}{n_1} - \frac{n_1-k}{n}\right)$. But the last difference is always positive. This proves Proposition 9.

Let $S_0 \subset S$ be a submanifold of the classes of stable, but not superstable bundles. It is easy to see that S_0 is a closed submanifold.

Proposition 10. $\mathrm{codim}_S S_0 > \frac{n^2}{4}(g-2)$.

PROOF. Let us count constants. Let $M_1 \subset V$ be a proper subbundle with maximal ratio $\deg M_1/\dim M_1$. Then V can be represented as an extension

(17)
$$0 \longrightarrow M_1 \longrightarrow V \longrightarrow M_2 \longrightarrow 0\,,$$

$$\deg M_i = d_i\,, \qquad \dim M_i = n_i \quad (i = 1, 2)\,,$$
$$\deg V = d = d_1 + d_2\,, \qquad \dim V = n = n_1 + n_2$$

If M_1 is a maximal subbundle, then

1) M_1 and M_2 are stable;

2) $H^0(\mathrm{Hom}(M_2, M_1)) = 0$.

Assertion 1) is obvious, for if M_1 is maximal in V, then M_2 is maximal in V^*.

If $s : M_2 \longrightarrow M_1$ is a homomorphism of stable bundles, then $\frac{d_1}{n_1} > \frac{d_2}{n_2}$, hence $\frac{d_1}{n_1} > \frac{d_1+d_2}{n_1+n_2} > \frac{d_2}{n_2}$.

The dimension of the variety of all extensions (17) is equal to

$$\dim S_{n_1,d_1} + \dim J(X) + \dim S_{n_2,d_2} + \dim P(H^1(X, \mathrm{Hom}(M_2, M_1))) =$$
$$= (n_1^2 - 1)(g-1) + g + (n_2^2 - 1)(g-1) + n_1 \cdot n_2 \left(\frac{d_2}{n_2} - \frac{d_1}{n_1} + (g-1)\right) - 1.$$

If V is not superstable, then $\frac{d_1}{n_1} > \frac{d}{n} - \frac{n_2}{n}$ or $d_1 n_2 > d_2 n_1 - n_1 \cdot n_2$, that is, $1 > \frac{d_2}{n_2} - \frac{d_1}{n_1}$. Hence,

$$\dim S_0 < (n_1^2 + n_1 \cdot n_2 + n_2^2 - 1)(g-1) + n_1 \cdot n_2.$$

But $\dim S = (n^2 - 1)(g - 1)$. Therefore,

$$\mathrm{codim}_S S_0 > n_1 \cdot n_2 (g-2) \geqslant \frac{n^2}{4}(g-2),$$

as required.

Hence it follows that the superstable bundles form an open subset of the variety of classes of bundles.

This completes the description of the elm's in the language of sheaves.

II. Geometric interpretation of elm's. We now turn from vector bundles to their projectivizations. The embedding i in a.e.t. determines a birational morphism

$$P(\mathrm{elm}_x^k(g)(V)) \xrightarrow{\ i\ } P(V),$$

Which decomposes in the following way:

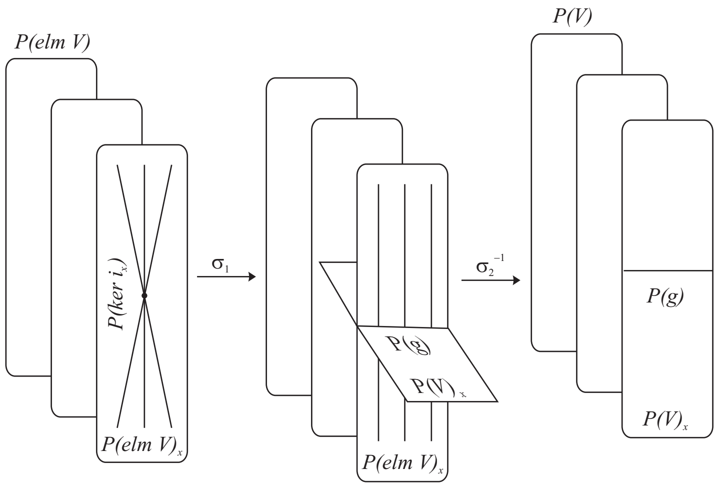

Рис. 1

σ_1 is a blow-up on $P(\mathrm{elm}V)$ with center $P(\ker i_x) \subset P(\mathrm{elm}V)_x$. After this blow-up two components will lie over x: the old fibre $P(\mathrm{elm})_x$ and the result of the blow-up. The old fibre is a "ruled" variety, which can be retracted by σ_2^{-1} onto the subspace $P(g) \subset P(V)_x$.

If V is a two-dimensional bundle, then $P(V)$ is a ruled surface and $\mathrm{elm}_x(p)$, $p \in P(V)$, is the well-known elementary transformation of ruled surfaces ([18], Chapter V).

Every ruled surface is obtained from the trivial $P^1 \times X$ by a finite collection of elementary transformations. It is easy to see that the same is true for bundles of arbitrary dimension ([16]).

Moreover, in the same way as we can construct the Jacobian as the symmetric gth power of a curve, so we can construct S by ordering the elementary operations ([16]).

III. Arithmetic interpretation of elm's. Suppose that a matrix divisor is given by an assignment E_x, $x \in X$. Then

$$\operatorname{elm}_{x_0}(E_x) = \begin{cases} E_x, & x \neq x_0, \\ \begin{pmatrix} 1 & 0 \\ 0 & \tau_{x_0} \end{pmatrix} E_{x_0}, \end{cases}$$

Where τ_{x_0} is a local parameter at $x_0 \in X$. The operation of multiplication by $\begin{pmatrix} 1 & 0 \\ 0 & \tau_{x_0} \end{pmatrix}$ is defined on assignments, but not on classes of assignments of matrix divisors; it does not commute with multiplication on the left by A_x, a regular and regularly invertible matrix. But on the set of left cosets $\begin{pmatrix} 1 & 0 \\ 0 & \tau_{x_0} \end{pmatrix} A_{x_0}$ the operation is well-defined.

A choice of coset $\begin{pmatrix} 1 & \alpha \\ 0 & \tau \end{pmatrix}$, $\alpha \in \mathbf{C}$, is a choice of $g \in V_x$, and determines $\operatorname{elm}_{x_0}(g)$ (see [16]).

§ 2 Variations of elementary operations.

The symbol for an elementary operation $\operatorname{elm}^k_{x_0}(g)V$ is adorned with three suffixes, three continuous parameters: the point $x \in X$, the subspace $g \in G_k(V_x)$ and the bundle $V \in S$. We can vary the operation with respect to all these parameters, obtaining a family of bundles.

The most important special cases of variations are the following.

I. Minimal variation. We fix the bundle V and its subbundle V_1 in such a way that V/V_1 is one-dimensional. Then $\{\operatorname{elm}_x V_{1x}(V)\}$, $x \in X$, is a family of bundles parametrized by the curve X. This is the minimal variation. It is used in the inversion problem. The determinants of the bundles of the family are varied, and we shall "touch up" this family a little in the next chapter.

The minimal variation can be described in terms of extensions. We have a commutative diagram

$$
(18)\qquad
\begin{tikzcd}
 & 0 & 0 & 0 & \\
0 \arrow[r] & \mathrm{elm}_x(L) = L(-x) \arrow[r] \arrow[u] & L \arrow[r] \arrow[u] & \mathcal{O}_x \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & \mathrm{elm}_x(V_{1x})(V) \arrow[r] \arrow[u] & V \arrow[r] \arrow[u] & \mathcal{O}_x \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & V_1 \arrow[r] \arrow[u] & V_1 \arrow[r] \arrow[u] & 0 \arrow[u] & \\
 & 0 \arrow[u] & 0. \arrow[u] & &
\end{tikzcd}
$$

and if an extension of the middle vertical column is given by a cocycle $h \in H^1(X, V_1 \otimes L^*)$, then the extension of the left column is given by the cocycle $r_x(h) \in H^1(X, (V_1 \otimes L^*)(x))$, where $r_x = H^1(X, \mathcal{F}) \longrightarrow H^1(X, \mathcal{F}(x))$ is the epimorphism of adjunction of a point induced by the exact triple

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}(x) \longrightarrow \mathcal{F}(x)_x \longrightarrow 0\,.$$

II. Variation with fixed point $x \in X$**.** Let T be the base of a family of bundles on X, that is, a bundle $X \times T$ is given on W. We can then construct the new family $\{\mathrm{elm}_x^k(g)(W_t)\}$ of all elementary transformations with fixed point $x \in X$. The base of the new family is $G_k(W_x)$ and the bundle on $X \times G_k(W_x)$ defining the family is denoted by $\mathrm{ELM}_x^k(W)$. When $k = n-1$, the operation is simply denoted by $\mathrm{ELM}_x(W)$.

We now take the most non-trivial family consisting of one trivial n-dimensional bundle I_n on X, and apply the operation ELM_x to it $(n+1)(g-1)$ times:

$$\underbrace{\mathrm{ELM}_x(\,\cdots\,(\mathrm{ELM}_x}_{(n+1)(g-1)\text{ times}}(I_n))\,\cdots\,)$$

We obtain a family of bundles on X with determinant $(n+1)\times(g-1)x$. The base of this family is a variety $P_{(n+1)(g-1)}$, which can be decomposed into a tower of $(n-1)$-dimensional projective bundles

$$
(19)\qquad P_{(n+1)(g-1)} \xrightarrow{\pi} P_{(n+1)(g-1)-1} \longrightarrow \cdots
$$
$$
\cdots \longrightarrow P_i \xrightarrow{\pi_{i-1}} P_{i-1} \longrightarrow \cdots \xrightarrow{\pi_1} P_1 = P^{n-1},
$$

where $P_i \xrightarrow{\pi_{i-1}} P_{i-1}$ is an $(n-1)$-dimensional projective bundle and $P_1 = P^{n-1}$ is projective space. The precise inductive description of this tower is as follows.

Suppose $P_i \xrightarrow{\pi_{i-1}} P_{i-1}$ is already constructed. Then $P_{i+1} = P(I \oplus \Omega_{\pi_{i-1}})$, that is, the bundle $P_{i+1} \xrightarrow{\pi_i} P_i$ is the projectivization of the direct sum of a trivial bundle and the relative cotangent bundle.

We can multiply all the bundles of the resulting family by $L^{g-1}(x)$ and transform, that is, pass to the family

$$[\mathrm{pr}_X^* L^{g-1}(x) \otimes \mathrm{ELM}_x^{(n+1)(g-1)}(I_n)]^*$$

on $X \times P_{(n+1)(g-1)}$.

The mapping to the class of bundles of this family gives us a *rational* mapping

$$\varphi : P_{(n+1)(g-1)} \longrightarrow S_{n,g-1} .$$

This mapping is rational, because it is not defined on the non-stable bundles.

Note that $\dim P_{(n+1)(g-1)} = \dim S_{n,g-1}$, and it is easy to see that it is a mapping onto the whole of $S_{n,g-1}$.

CONJECTURE. *φ is a birational morphism.*

The case $n = 2$, $g = 2$ is very convenient for verification of this conjecture.

Consider a bundle V with determinant $3x$, where $x \in X$ is a fixed point. Then, by the Riemann-Roch theorem, V has a section. It is easy to see that V_{gen} has only one section and has no zeros. This means that V_{gen} is uniquely represented as an extension

$$0 \longrightarrow I \longrightarrow V_{\mathrm{gen}} \longrightarrow L^3(x) \longrightarrow 0,$$

that is, if $P(H^1(X, L(-3))) = P^3$ is the base of the family of extension, then there is a birational isomorphism $\varphi : P^3 \leftrightarrow S_{2,1}$.

The operation of adjunction of a point x to the sheaf gives us two epimorphisms:

$$H^1(X, L(-3x)) \xrightarrow{\pi_2} H^1(X, L(-2x)) \xrightarrow{\pi_1} H^1(X, L(-x)),$$

which determine two projections:

$$P^3 = P(H^1(X, L(-3x))) \xrightarrow{\pi_2} P(H^1(X, L(-2x))) \xrightarrow{\pi_1} P(H^1(X, L(-x))) = P^1.$$

It is easy to see that this is just the chain (19). Thus, our conjecture is true for $n = 2$, $g = 2$. These arguments can easily be generalized to the case when $g - 1 \equiv n - 1 \mod n$.

Our construction has an obvious generalization. Let $k_1, \ldots, k_N$ be a sequence of positive integers $< n$ such that

$$\sum_{i=1}^{N} k_i(n - k_i) = (n^2 - 1)(g - 1).$$

We consider the family of bundles on X

$$\underbrace{\mathrm{ELM}_x^{k_N} \cdots \mathrm{ELM}_x^{k_1}}_{N \text{ times}}(I_n)$$

The determinants of the bundles of this family are $\sum_{i=1}^{N} k_i - Nn$. The base of this family is a manifold G_N, which decomposes into a tower of Grassmannizations

$$G_N \xrightarrow{\pi_N} G_{N-1} \longrightarrow \cdots \xrightarrow{\pi_1} G_1 = G_{k_1}^{n-1}.$$

Again we get a rational morphism $\varphi : G_N \longrightarrow S_{n,d}$, where $d = \sum_{i=1}^{N} k_i$.

It is easy to see that we can find number k_i such that

$$\sum_{i=1}^{N} k_i(n - k_i) = (n^2 - 1)(g - 1) \quad \text{and} \quad \sum_{i=1}^{N} k_i \equiv \text{ (any residue)(mod } n) \ .$$

Hence the rationality of $S_{n,d}$ follows for any n and d.

For $(n, d) = 1$ the rationality of $S_{n,d}$ was proved by Newstead [12].

The variation of the elm's with a fixed point will appear again in Chapter V, But meanwhile we turn to the minimal variation and the "inverse problem".

CHAPTER 4

The geometry of the inversion problem

§ 1 Construction of the minimal family.

Let L_1 be any one-dimensional bundle on X of degree 1. Then the family $\{\mathrm{elm}_x(L_1)\}$, $x \in X$, is parametrized by the curve X and

1) the mapping $\varphi : X \longrightarrow \mathrm{Pic}\ X$ induced by this family is an embedding;
2) $\varphi_* : H_1(X, \mathbf{Z}) \longrightarrow H_1(\mathrm{Pic}\ X, \mathbf{Z})$ is an isomorphism;
3) the mapping φ solves the inversion problem, that is, the family of bundles $(\mathrm{var}\varphi)^*(\Theta_0)$, where Θ_0 is the Poincaré bundle on Pic X, and varφ) are all possible variations of φ, is *universal*.

In the many-dimensional case the construction is more complicated, but the idea is the same. Let $E \subset E_0$; then in the family of bundles $\{\mathrm{elm}_x(E_x)(E_0)\}$ the determinants change, and by extending each bundle by a one-dimensional one

$$0 \longrightarrow \mathrm{elm}_x(E_x)(E_0) \longrightarrow \tilde{E} \longrightarrow L(x) \longrightarrow 0,$$

we obtain a family with constant determinant.

But $\{\mathrm{elm}_x(E_x)(E_0)\}$ is itself an extension (see. (18)):

$$\begin{tikzcd}
 & 0 & 0 & 0 & \\
0 \arrow[r] & (E_0/E)(-x) \arrow[r] \arrow[u] & E_2 \arrow[r] \arrow[u] & L(x) \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & \mathrm{elm}_x(E_x)(E_0)) \arrow[r] \arrow[u] & \tilde{E} \arrow[r] \arrow[u] & L(x) \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & E \arrow[r] \arrow[u] & E \arrow[r] \arrow[u] & 0 \arrow[u] & \\
 & 0 \arrow[u] & 0 \arrow[u] & &
\end{tikzcd}$$

so that $\tilde{E}$ is an extension of a constant bundle E and a non-trivial two-dimensional one.

Only this two-dimensional bundle requires a careful description; incidentally, it solves the inversion problem for two-dimensional bundles .

A bundle on $X \times X$ giving a family with a mapping into the classes $\varphi : X \longrightarrow S$ solving the inversion problem is not unique, from the very concept. We look for the bundle V on $X \times X$ among the symmetric bundles.

Definition 8. A bundle V on $X \times X$ is called *symmetric*, if $i^*V = V \otimes L$, where i is the involution that interchanges the direct factors, and L is a one-dimensional bundle.

Definition 9. A bundle V is *absolutely symmetric*, if $i^*V = V$.

Consider on $X \times X$ the extension

$$0 \longrightarrow L(-\Delta) \longrightarrow V_0 \longrightarrow L(\Delta) \longrightarrow 0, \tag{20}$$

where Δ is divisor of the diagonal on $X \times X$, and the space $H^1(X \times X, L(-2\Delta))$ giving all such extensions. Let

$$0 \longrightarrow -2\Delta \longrightarrow -\Delta \longrightarrow -\Delta\,|_{\Delta} \longrightarrow 0 \tag{21}$$

be the exact triple of the adjunction and

$$0 \longrightarrow \Omega X \longrightarrow R^1\mathrm{pr}_i(-2\Delta) \longrightarrow R^1\mathrm{pr}_i(-\Delta) \longrightarrow 0 \tag{22}$$

a piece of the exact sequence of the direct image functor relative to the projection pr_i on a factor of the direct product.

From

$$0 \longrightarrow -\Delta \longrightarrow \mathcal{O}_{X \times X} \longrightarrow \mathcal{O}_{\Delta} \longrightarrow 0$$

and

$$\begin{tikzcd}
0 \arrow[r] & R^0\mathrm{pr}_i\mathcal{O}_{X\times X} \arrow[r] \arrow[d, equal] & R^0\mathrm{pr}_i\mathcal{O}_{\Delta} \arrow[r] \arrow[d, equal] & R^1\mathrm{pr}_i(-\Delta) \arrow[r] & R^1\mathrm{pr}_i(\mathcal{O}_{X\times X}) \arrow[r] & 0 \quad , \\
 & \mathcal{O}_X & \mathcal{O}_X & & &
\end{tikzcd}$$

it follows that $R^1\mathrm{pr}_i(-\Delta) = R^1\mathrm{pr}_i(\mathcal{O}_{X\times X}) = I_g$ is the trivial g-dimensional bundle.

Also it is easy to see that

$$R^1\mathrm{pr}_i(-2\Delta) = \Omega X \oplus I_g$$

and that the direct decomposition

$$H^1(X \times X, L(-2\Delta)) = H^0(X, R^1\mathrm{pr}_i(-2\Delta)) = H^0(X, \Omega X) \oplus H^0(X, I_g)$$

corresponds to the decomposition into the symmetric and antisymmetric subspaces relative to the involution i that interchanges the direct factors in $X \times X$:

$$H^0(X, \Omega X) = H^0(X \times X, L(-2\Delta))^+, \quad H^0(X, I_g) = H^0(X \times X, L(-2\Delta))^-.$$

We now choose the trivial two-dimensional bundle $I_2 \in I_g = R^1\mathrm{pr}_i(-\Delta)$ and consider the family of extensions (20) $\{V_h\}$, $h \in P(I_2) = P^1$, on $X \times X$. Let $\varphi : I_2 \longrightarrow H^1(X \times X, L(-2\Delta))$ be the corresponding embedding.

The bundle V on $X \times X \times P^1$ giving this family can be represented as the extension:

$$(23)\quad 0 \longrightarrow \mathrm{pr}^*_{X\times X}(L(-\Delta)) \longrightarrow V \longrightarrow \mathrm{pr}^*_{X\times X}(L(-\Delta)) \otimes \mathrm{pr}^*_{P^1}\tau^* \longrightarrow 0$$

where τ is antitautological bundle on P^1, that is, $\tau = H$ is the bundle of points on P^1.

The cocycles giving such extensions, decomposed according to the Künneth formula:

$$H^1(X \times X \times P^1, \mathrm{pr}^*_{X\times X}(L(-2\Delta)) \otimes \mathrm{pr}^*_{P^1}\tau^*) =$$
$$= H^1(X\times X, L(-2\Delta))\otimes H^0(P^1,\tau) = \mathrm{Hom}(H^0(P^1,\tau)^*, H^1(X\times X, L(-2\Delta)) \,,$$

can be interpreted as linear mappings $P^1 \longrightarrow P(H^1(X \times X), L(-2\Delta))$.

As a cocycle giving (23) we take the *embedding* $\varphi : P(I_2) \longrightarrow P(H^1(X \times X, L(-2\Delta)))$.

The family so constructed consists of symmetric inequivalent bundles. We complete the family as far as the n-dimensional bundles and consider the extension on $X \times X \times P^1$:

$$(24)\quad 0 \longrightarrow \mathrm{pr}^*_1 \oplus M \otimes \mathrm{pr}^*_2 M I_{n-3} \otimes \mathrm{pr}^*_1 M' \otimes \mathrm{pr}^*_2 M' \longrightarrow E \longrightarrow V \longrightarrow 0$$

where M and M' are certain one-dimensional bundles chosen so that (24) gives the symmetric (or antisymmetric) cocycle of $\deg E$ congruent to a preassigned number d modulo n and that the general bundle $E|_{(X\times x,p)}$ is stable.

§ 2 The second Chern class.

The resulting bundle can be interpreted as:

1) a family of bundles $\{E|_{X\times x\times p}\}$, on X, parametrized by the ruled surface $X \times P^1$ (then we have a rational map $\varphi : X \times P^1 \longrightarrow S$);

2) a family of bundles $\{E|_{x\times X\times P^1}\}$, on the ruled surface $X\times P^1$, parametrized by the curve X.

We calculate the Künneth component $c^{3,1}$ of E in (24):

$$c^{3,1} \in H^3(X \times P^1, \mathbf{Z}) \otimes H^1(X, \mathbf{Z}).$$

From (24) $c_2(E) = c_2(V)$, and from (23) $c_2(V) = \Delta^2 \times P^1 + (\Delta \times P^1) \cdot (X \times X \times p)$, where p is a fixed point of P^1. Hence

$$c^{3,1}(V) = (\Delta \times p) = (\Delta \times P^1) \cdot (X \times X \times p). \tag{25}$$

COROLLARY. *The Poincaré bundles U_x on S are all distinct.*

For $E|_{(x \times X \times P^1)} = \varphi^*(U_x)$. But $c_2(E|_{(x \times X \times P^1)}) = (x \times p) + \text{const}$, so that the bundles for different x are distinct. Theorem 2 is now proved.

Proposition 11. (Ramanan [13]) *$\varphi^* : H^3(S, \mathbf{Z}) \longrightarrow H^1(X, \mathbf{Z})$ is an isomorphism.*

PROOF. Let U be a universal family on $X \times S$ and $c^{3,1} \in H^3(S, \mathbf{Z}) \otimes H^1(X, \mathbf{Z})$. By the duality $(H^1(X, \mathbf{Z}))^* = H^1(X, \mathbf{Z})$ we can then regard $c^{3,1}$, as a homomorphism $H^1(X)$ into $H^3(S)$. We consider the commutative diagram

$$\begin{array}{ccc} H_1(X,\mathbf{Z}) & \xrightarrow[X\times S]{c^{3,1}(U)} & H^3(S,\mathbf{Z}) \\ \Big\| {\scriptstyle 1^*} & \Big\uparrow {\scriptstyle 1\times\varphi} & \Big\downarrow {\scriptstyle \varphi^*} \\ & X\times\overbrace{X\times P^1} & \\ H_1(X,\mathbf{Z}) & \xrightarrow{c^{3,1}(E)} & H^3(P^1\times X,\mathbf{Z}) \end{array} \tag{26}$$

induced by the equation $E = (1 \times \varphi)^* U \otimes L$.

We decompose $c^{3,1}(E)$ into the product of $c^{1,1}(\Delta \times P^1)$ and τ:

$$\begin{tikzcd} H^1(X,\mathbf{Z}) \arrow[rr, "c^{3,1}(E)"] \arrow[dr, "c^{1,1}(\Delta\times P^1)"'] & & H^3(P^1\times X,\mathbf{Z}) \\ & H^3(P^1\times X,\mathbf{Z}) \arrow[ur, "\tau"'] & \end{tikzcd}$$

where $c^{1,1}(P^1 \times \Delta)(\gamma)[\gamma'] = [\gamma \smile \gamma' \cdot P^1 \times \Delta]_{P^1 \times X \times X} = (\gamma \smile \gamma')_X$, and τ is multiplication by the two-dimensional form of the fibre P^1. Clearly, $c^{1,1}(\Delta \times P^1)$ is an isomorphism and τ is an isomorphism, hence $c^{3,1}(E)$ is an isomorphism. We know that $H^3(S, \mathbf{Z})$ (theorem on $h^{*,1}$) and $H^3(P^1 \times X, \mathbf{Z})$ are free $\mathbf{Z}$-modules of rank $2g$. Therefore, it is sufficient to show that φ^* is an epimorphism. But $\varphi^{*-1} = c^{3,1}(U) \cdot (c^{3,1}(E))^{-1}$. This completes the proof.

COROLLARY. $J^3(S) \cong J(X)$.

COROLLARY. *The curve X can be reconstructed uniquely in terms of S.*

We return to the beginning of § 1, the mapping $\varphi : X \hookrightarrow \operatorname{Pic} X$, which solves the inversion problem. For φ, the properties 2) and 3) are equivalent.

CONJECTURE. *Let $X \times P^1 \overset{\varphi}{\hookrightarrow} Z$ be a mapping such that*

$$H_3((X \times P^1), \boldsymbol{Z}) \longrightarrow H_3(S, \boldsymbol{Z})$$

is an isomorphism. Then the mapping $\varphi|_{X \times p_0}$ solves the inversion problem, that is the family $(\varphi|_{X \times p_0})^*(U_{x_0})$ is universal (U_{x_0} is any Poincaré bundle).

CHAPTER 5
The Narasimhan–Ramanan theorem

§ 1 The double bundle.

In Chapter III, § 2 we defined the operation $\mathrm{ELM}_x^k(V)$ from an arbitrary bundle V on $X \times T$. We now apply it to the universal bundle U on $X \times S$.

Then $\mathrm{ELM}_x^k(U)$ is a bundle on $X \times G_k(U_x)$, that is, the Grassmannization of the Poincaré bundle U_x is the base of the new family. The determinant of the bundles of the resulting family is changed to $-(n-k)x$.

If $n \geqslant 3$, we can always choose k so that $d+k$ and n are coprime.

If $S = S_{n,d}$, let $S_k = S_{n,d+k}$. The resulting family defines a rational map $\varphi_k : G_k(U_x) \longrightarrow S_k$.

Proposition 12. *If E is a superstable bundle, then $\varphi_k^{-1}(E) = G_{n-k}(E_x)$.*

The proof follows immediately from the inversion formula (13) of Chapter III, § 1.

Let $\overline{G_k(U_x)}$ be the maximal open set such that $\overline{\pi(G_k(U_x))}$ is contained in an open set of superstable bundles and $\overline{\varphi(G_k(U_x))}$ in the set of superstable bundles. Let $\bar{S} = \pi\overline{(G_k(U_x))}$ and $\bar{S}_k = \varphi\pi(G_k(U_x))$. Then we have the effect of a double bundle

$$
\begin{tikzcd}
 & G_k(\bar{U}_x) = G_{n-k}(\bar{U}_k x) \arrow[dl] \arrow[dr] & \\
\bar{S} & & \bar{S}_k
\end{tikzcd}
\tag{27}
$$

where $\bar{U}_x = U_x|_{\bar{S}}$ and $\bar{U}_{kx} = U_{kx}|_{\bar{S}_k}$.

Proposition 13. $\operatorname{codim}_{G_k(U_x)} \left(G_k(U_x) - \overline{G_k(U_x)} \right) \geqslant \frac{n^2}{4}(g-3)$.

PROOF. By definition

$$
\begin{aligned}
\operatorname{codim}_{G_k(U_x)} \left(G_k(U_x) - \overline{G_k(U_x)} \right) &= \operatorname{codim}_S \left(S - \bar{S} \right) + \dim G_k \geqslant \\
&\geqslant \frac{n^2}{4}(g-2) + k(n-k) \geqslant \frac{n^2}{4}(g-3).
\end{aligned}
$$

(We have used Proposition 10).

The diagram (27) is called a double bundle. It was first discovered by Newstead ($g = 2$, $n = 2$, [10]).

THE INVERSION THEOREM. $\Theta_\pi = \Theta^*_{\pi_k}$.

The proof will be given in the next section.

We now show that the Narasimhan–Ramanan theorem follows from this.

PROOF OF THE NARASIMHAN–RAMANAN THEOREM. We first need a technical lemma.

HARTOGS' THEOREM. *Let S be a compact non-singular manifold, and $\bar{S}$ an open part such that* $\operatorname{codim}_S(S - \bar{S}) \geqslant m$. *Then for any sheaf $\mathcal{F}$ $H^i(\bar{S}, \mathcal{F}) \longrightarrow H^i(S, \mathcal{F})$ is an isomorphism for $i \leqslant m - 2$.*

The proof is not difficult and is in [3].

Hence for $i \leqslant \frac{n^2}{4}(g-3) - 2$ we obtain

$$H^i(G_k(U_x), \Theta_\pi) = H^i(\bar{G}_k(U_x), \Theta_\pi) = \\ = H^i(\overline{G_{n-k}(U_{kx})}, \Theta^*_{\pi_k}) = H^i(G_{n-k}(U_{kx}), \Theta^*_{\pi_k}),$$

and the Narasimhan–Ramanan theorem follows at once from Proposition 4 (Chapter I, § 2).

§ 2 The inversion theorem.

We consider $X \times G_k(U_x)$ and calculate the bundle $\mathrm{ELM}^k_x(U)$,which gives this family.

1) Let $\tau_{U^*_x}$ be the antitautological bundle on $G_k(U_x)$ and

$$\pi^*(U_x) \xrightarrow{\alpha} \tau_{U^*_x} \longrightarrow 0$$

be the epimorphism G.e.t. (Chapter I, § 2).

We identify $\tau_{U^*_x}$ with a skyscraper over $x \times G_k(U_x)$ in the direct product $X \times G_k(U_x)$. Then α uniquely determines the epimorphism

$$(1 \times \pi)^*(U) \xrightarrow{\tilde{\alpha}} \mathrm{pr}^*_G \tau_{U^*_x} \otimes \mathrm{pr}^*_X \mathcal{O}_x \longrightarrow 0, \\ \tilde{\alpha}|_{x \times G_k} = \alpha.$$

2) The kernel of $\tilde{\alpha}$ is the required bundle $\mathrm{ELM}^k_x(U)$, that is, there is an exact triple

$$(28) \quad 0 \longrightarrow \mathrm{ELM}^k_x(U) \xrightarrow{i} (1 \times \pi)^*(U) \xrightarrow{\tilde{\alpha}} \mathrm{pr}^*_G \tau_{U_x} \otimes \mathcal{O}_x \longrightarrow 0,$$

the restriction of which to (X, g) is a.e.t.

$$0 \longrightarrow \mathrm{elm}^k_x(g)(U_s) \longrightarrow U_s \longrightarrow \mathcal{O}^{n-k}_x \longrightarrow 0$$

We restrict (28) to $(x, G_k(U_x))$, that is, we multiply it by the torsion sheaf $\mathrm{pr}_X^* \mathcal{O}_x$:

(29)
$$0 \longrightarrow \mathrm{Tor}_1^{O_{X\times G}}(\tau_{U_x}, \mathrm{pr}_X^*\mathcal{O}_x) \longrightarrow (\mathrm{ELM}_x^k)_x \longrightarrow \pi^* U_x \xrightarrow{\alpha} \tau_{U_x} \longrightarrow 0$$

As a resolution of $\mathrm{pr}_X^*\mathcal{O}_x$ we use triple

$$0 \longrightarrow \mathrm{pr}_X^* L(-x) \longrightarrow \mathcal{O}_{X\times G} \longrightarrow \mathrm{pr}_X^*\mathcal{O}_x \longrightarrow 0.$$

Hence we have an exact quadruple

$$\begin{array}{ccccccccc} 0 \longrightarrow & \mathrm{Tor}_1^{O_{X\times G}}(\tau_{U_x}, \mathrm{pr}_X^*\mathcal{O}_x) & \longrightarrow & \tau_{U_x} & \longrightarrow & \tau_{U_x} & \longrightarrow & \tau_{U_x} & \longrightarrow 0 \\ & & & \| & & \| & & \| & \\ & & & \mathrm{Tor}_0 & & \mathrm{Tor}_0 & & \mathrm{Tor}_0 & \end{array}$$

which splits into two isomorphisms:

$$0 \longrightarrow \mathrm{Tor}_1^{O_{X\times G}}(\mathrm{pr}_G^*\tau_{U_x^*}, \mathrm{pr}_X^*\mathcal{O}_x) \longrightarrow \tau_{U_x^*} \longrightarrow 0 \longrightarrow \tau_{U_x^*} \longrightarrow \tau_{U_x^*} \longrightarrow 0.$$

(29) can now be rewritten as:

(30) $$0 \longrightarrow \tau_{U_x^*} \longrightarrow (\mathrm{ELM}_x^k)_x \longrightarrow \pi^* U_x \xrightarrow{\alpha} \tau_{U_x^*} \longrightarrow 0,$$

and $(\mathrm{ELM}_x^k)_x$ can be represented as an extension

(31) $$0 \longrightarrow \tau_{U_x^*} \longrightarrow (\mathrm{ELM}_x^k)_x \longrightarrow \tau_{U_x^*} \longrightarrow 0,$$

since $\ker\alpha$ is the tautological extension.

We now consider $G_{n-k}(U_{kx}) \xrightarrow{\pi_k} S_k$ and its first Grassmannisation exact triple:

(32) $$0 \longrightarrow \tau^*_{U_{kx}} \longrightarrow \pi_k^* U_{kx} \xrightarrow{\alpha} \tau_{U_{kx}^*} \longrightarrow 0.$$

Fundamental assertion. *The extensions* (31) *and* (32) *are proportional on* $\overline{G_k(U_x)} = \overline{G_{n-k}(U_{kx})}$ *that is, there is a one-dimensional bundle* $\bar{L}$ *such that*

$$(31) \otimes L = (32).$$

In fact, $\mathrm{ELM}_x^k|_{X\times\bar{G}} = \pi_k^*(U_k) \otimes \mathrm{pr}_S^* L^*$, therefore $(\mathrm{ELM}_x^k)_x = \pi_k^*(\bar{U}_k) \otimes L^*$. Here from the geometric argument itself (see the inversion formula (13) of Chapter III) the fibres of the subbundle τ_{U_x} in $(\mathrm{ELM}_x^k)_x$ are the kernels of elementary transformations.

It follows that

$$\begin{array}{ccccccc} 0 \longrightarrow & \tau_{U_x^*} \otimes L & \longrightarrow & (\mathrm{ELM}_x^k)_x \otimes L & \longrightarrow & \tau^*_{U_x} \otimes L & \longrightarrow 0 \\ & \| & & \| & & \| & \\ 0 \longrightarrow & \tau^*_{U_{kx}} & \longrightarrow & \pi_k^* U_{kx} & \longrightarrow & \tau_{U_{kx}^*} & \longrightarrow 0. \end{array}$$

Thus,

$$\begin{array}{ccc} \tau^*_{U^*_x} \otimes \tau^*_{U_x} & = & \tau_{U_{kx}} \otimes \tau_{U^*_{kx}} \\ \| & & \| \\ \Theta^*_\pi & = & \Theta_{\pi_k} \end{array} ,$$

according to formula (2) in Chapter I, § 2. This proves the inversion theorem.

As we wanted to exemplify the method, we did not pay attention to obtaining the best-possible bounds (such as Proposition 10). The Narasimhan–Ramanan theorem is, in fact, true for $n = 2$, $g \geqslant 2$ [8], and not only for $n \geqslant 3$, $g \geqslant 4$, as our bounds show].

In their own proof of the theorem in [8], the authors apply a "two-level" construction

$$\mathrm{ELM}_x(\mathrm{ELM}^*_x(U))$$

and go back to S.

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On the classification of rank 2 vector bundles over an algebraic curve of arbitrary genus

In this paper rank 2 algebraic bundles over an algebraic curve X of any genus g are classified. To do this we construct a category of quasi bundles and a functor (the rigidity functor) from this category to the category of rank 2 bundles, mapping finitely many ($\leqslant 2g$) quasi-bundles to the same bundle. For this category of quasi-bundles a universal problem of the Grothendick type is solved and the corresponding moduli space is computed.

Izv. AN SSSR. Ser. Math. (1964) 28:1 p. 21–52.

Introduction.

The present paper contains an attempt to classify vector bundles over algebraic curve of arbitrary genus. Known results, established in the classification problem, are contained in papers of Atiyah [2] and Grothendick [6]. In the first paper the classification over an elliptic curve is given, in the second one the classification over a rational curve. In the present paper one studies rank 2 vector bundles over a curve X of arbitrary genus. For this we start from the following formulation of the classification problem. Construction of a "classifying space" or a universal family of bundles would be the best (ideal) solution. This is understood as a family of rank 2 bundles over curve X, parameterised by an algebraic variety $\mathcal{B}$ (so as a rank 2 bundle $\overline{\mathcal{B}}$ over $X \times \mathcal{B}$), such that any family of rank 2 bundles $\mathcal{E} \longrightarrow E$, parameterised by an algebraic variety E, is uniquely represented in the form $\mathcal{E} = f^*(\mathcal{B}|_{f(E)})$, where $f : E \longrightarrow \mathcal{B}$ is a regular map. In other words the situation can be described if one says that the functor $\mathcal{F}$, attaching to any algebraic variety E the set of all families of rank 2 bundles over X, parameterised by E, is representable (see [4]).

But really this functor $\mathcal{F}$ is not representable and, hence, such a universal family $\mathcal{B}$ does not exist. However in the present paper one proves that the universal family itself does exist if one changes the notion of bundle itself. For this one introduces the notion of exceptional line sub-bundle of rank 2 bundle (§1). One proves that any rank 2 bundle over a curve X of genus g has at most $2g$ exceptional sub-bundles. A new object, consisting of a rank 2 bundle and some of its exceptional sub-bundle, is called a quasi-bundle. One establishes that for the quasi-bundles a universal family does exist whose base $\mathcal{B}$ is the union of finitely many algebraic varieties. Each of these varieties corresponds to integer values of four invariants of the bundle: $n(E), k(E), d(E)$ and $\alpha(E)$.

Among bases of universal families $\mathcal{K}(n, k, d, \alpha) \longrightarrow K(n, k, d, \alpha)$, which correspond to bundles with given values of the invariants, $K(g, 1, 0, 0)$ and $K(g-1, 0, 1, 0)$ have the maximal dimension.

The quasi-projective variety $K(g, 1, 0, 0)$, describing the main class of quasi-bundles with fixed determinant of even degree, is an open subset of the projective bundle, associated with vector bundle $I_{3g-4} \oplus T^3(X)$ over X, where I_{3g-4}

is $(3g-4)$-dimensional trivial bundle over X, $T(X)$ is the tangent bundle of X and $T^3(X)$ is the third tensor power of $T(X)$.

The quasi-projective variety $K(g-1,0,1,0)$, which describes the main class of quasi-bundles with fixed determinant of odd degree, is a $(3g-3)$-dimensional projective space without a subvariety. Further, the variety $K(g-1,0,1,0)$ is the moduli space of the main class of bundles (not only quasi-bundles) with invariants $(g-1,0,1,0)$.

One has to remark that classification of rank 2 bundles with trivial determinant is equivalent to bi-regular classification of ruled surfaces. The last statement follows from the paper of Nagata [7]. Evidently, it would be interesting to translate the present paper into the geometrical language of paper [7].

The author uses the case to express his deep gratitude to I.R. Shafarevich for his advice, remarks and help, given during the writing time of this paper.

Everywhere in what follows the main field k is an algebraically closed field of characteristics $\neq 2$.

The base of the bundles is a nonsingular complete algebraic curve X of arbitrary rank g.

The main notation is the same as in the paper [2]. The symbol $\mathcal{E}$ denotes the sheaf of section germs of the bundle E.

Divisors are denoted by Greek letters. $L(\eta)$ denotes the line bundle with divisor η; $\eta(L)$ denotes the divisor of line bundle L; L_s is the line bundle, defined by section s; η_s is the zero divisor of section s; $\mathcal{L}(\eta)$ is the space of functions, comparable with η; $|L|$ is the linear system $|\eta(L)|$.

CHAPTER 1
Invariants of bundles

The aim of this chapter is to show that for any rank 2 bundle there exists a finite number of line sub-bundles, which possess two properties: the height minimality and exceptionality. Two subsequent paragraphs are devoted to the study of these properties.

§ 1 Height.

Every bundle over an algebraic curve is reducible, so it has a line sub-bundle; possibly more than one such sub-bundle. Let us study which line sub-bundles a given bundle E could contain.

Definition 1. The height of a divisor ξ with respect to a point $\mathfrak{D} \in X$ is the minimal integer number n such that $\xi\mathfrak{D}^n \sim \eta$, where $\eta \geqslant 1$ so it is an effective divisor. The height of a divisor ξ is denoted as $h_{\mathfrak{D}}(\xi)$. As the height of a line bundle $L(\eta)$ one takes $h_{\mathfrak{D}}(\xi)$.

In what follows we deal only with the height with respect to a permanently fixed point $\mathfrak{D}$, which is not a Weierstrass point.

Remark 1. It is not hard to see that $h_{\mathfrak{D}}(\xi)$ is an invariant of the equivalence class, and the representation of a divisor ξ in the shape $\eta\mathfrak{D}^{-h_{\mathfrak{D}}(\xi)}$ uniquely realizes the choice from each equivalence class.

Indeed, if

$$\xi \sim \eta\mathfrak{D}^{-h(\xi)} \sim \eta'\mathfrak{D}^{-h(\xi)},$$

then $\eta \sim \eta'$ and hence $\dim \mathcal{L}(\eta) \geqslant 2$ and $\dim \mathcal{L}(\eta D^{-1}) \geqslant 1$, from which it follows $\xi \sim \eta''\mathfrak{D}^{-h(\xi)+1}$, and gets a contradiction with the definition of $h(\xi)$.

Definition 2. The index of a divisor ξ is $\deg \eta$, where $\eta\mathfrak{D}^{-h(\xi)} \sim \xi$. The index is denoted as $k(\xi)$.

It is not hard to see that if $\deg \xi = n$ then $h(\xi) \leqslant g - n$.

Lemma 1. Let J be the Jacobian variety of a curve X. Then the set of all $\sigma \in J$ such that $h(\sigma) \leqslant i$, $(0 \leqslant i \leqslant g)$, is an i-dimensional subvariety of J (we denote it as G_i).

Proof. Indeed, let $S^g(X)$ be the g-th symmetric product of the curve X, represented by divisors of the shape $c_1 \dots c_g\mathfrak{D}^{-g}$ and let $\phi : S^g(X) \longrightarrow J$ be the canonical map, identifying equivalent divisors. Consider the variety $S^i(X) \subset S^g(X)$ consisting of divisors of the shape $c_1 \dots c_i\mathfrak{D}^{-i}$. The Variety

$\phi(S^i(X))$ has dimension i. The set of exceptional divisors in $S^i(X)$ is a subvariety and it is proper if there exists at least one non special divisor. Such a divisor, for example, is given by the i-th power of any point which is not a Weierstrass point. The variety $\phi(S^i(X))$ consists of divisors of height $\leqslant i$ and only of such divisors. In analogy with the Poincare divisor the subvariety $\phi(S^i(X))$ of the Jacobian is called the i-dimensional Poincare cycle.

Lemma 2. Let J_k be the homogeneous space of classes of divisors of degree k. Then the subvariety $G_{k,i} = \{\sigma \in J_k, h(\sigma) \leqslant i\}$ is transferred to the $(i+k)$-dimensional Poincare cycle under the natural map $J_k \longrightarrow J$.

PROOF. The map $\psi : I \longrightarrow I_k, \psi(\sigma) = \sigma \cdot \mathfrak{D}^k$ transfers G_{i+k} to $G_{k,i}$. Then our statement follows from this fact and Lemma 1.

Remark 2. It follows from Remark 1 that $G_{k,i}/G_{k,i-1}$ is bi-regular equivalent to $S^{k-i}(X)$, where $\tilde{S}^n(X) = S^n(X) - S_n$ and $S^n(X)$ is the n-th symmetric power of the curve X, while

$$S_n = (\mathfrak{D} \cdot S^{n-1}(X)) \cup \{\sigma \in S^n(X), \dim \mathcal{L}(\mathfrak{D}) > 1\}.$$

Definition 3. The height of a bundle E is $\min\limits_{L \subset E} h(L)$ taken over all possible line sub-bundles L. The height of a bundle E is denoted $h(E)$.

In other words, $h(E)$ is an integer number, such that

(1)
$$\dim H^0\left(X, E \otimes L\left(\mathfrak{D}^{h(E)-1}\right)\right) = 0, \quad \dim H^0\left(X, E \otimes L\left(\mathfrak{D}^{h(E)}\right)\right) = i > 0.$$

In the set of all sub-bundles of the bundle E we distinguish the subset $Sp^{h_0}(E)$ of all sub-bundles of minimal height. The next lemma shows that $Sp^{h_0}(E)$ consists of either a single point or the points passing through a projective line (see §2).

Lemma 3. In formula (1) one has $1 \leqslant i \leqslant 2$.

PROOF. Assume, that $i \geqslant 3$. Let us choose a basis

$$s_1, \ldots, s_i \in \Gamma\left(X, E \otimes L\left(\mathfrak{D}^{h(E)}\right)\right).$$

Since the fiber of E has dimension 2, there exists a linear combination

$$s = \sum_{j=1}^{i} \alpha_j s_j,$$

such that $s(\mathfrak{D}) = 0$, so there is an effective sub-bundle with a divisor, containing $\mathfrak{D}$. But in this case

$$\dim H^0\left(X, E \otimes L\left(\mathfrak{D}^{h(E)-1}\right)\right) > 0,$$

which contradicts the first condition from the definition of $h(E)$.

Definition 4. Let us denote by $c(E)$ the divisor which is the greatest common divisor for all divisors of sections of the bundle E. In other words $c(E)$ denotes the divisor of common zeros of all sections of the bundle E.

Each section $s \in \Gamma(X, E)$ defines some sub-bundle L_s of the bundle E (see [2]) and $L_{\alpha s} = L_s$ ($\alpha \neq 0$).

Consider the space $\Gamma(E) = H^0(X, E)$ and the corresponding projective space $\mathbf{P}(\Gamma(E))$. It's clear that any $L_s \geqslant L(c(E))$ (the symbol $L \geqslant M$ means that $L \otimes M$ is equivalent to an effective line bundle or, equivalently, that there exists a nontrivial homomorphism $M \longrightarrow L$).

Let L and L' be different line sub-bundles of the bundle E.

Definition 5. Consider the set of points $\mathcal{P}_i$ of our base X, in which L and L' coincide. Take the product of these as divisors. The resulting divisor we will call the divisor - support of the intersection L and L' and denote it by $\xi(L, L')$. It is clear that $\xi(L, L') \geqslant 1$. The multiplicities for the points which one takes are defined by Lemma 5.

For complete correctness of the definition we need to show that the set of points from our base X, over which L and L' coincide, is finite (see Lemma 5).

Obviously, if $s, s' \in \Gamma(E)$ define sub-bundles L_s and $L_{s'}$ respectively then $\mathcal{P} \in \xi(L_s, L_{s'})$ if and only if

$$s(\mathcal{P}) = \alpha s'(\mathcal{P}), \quad \alpha \in k, \quad \alpha \neq 0.$$

Lemma 4. If $\xi(L, L') = 0$, i.e. L and L' do not intersect each other, then $E = L \oplus L'$.

The Lemma is obvious.

Lemma 5. Any distinct sub-bundles L and L' of the bundle E coincide over finitely many points. At the same time

$$\xi(L, L') \in |L^{-1} \otimes (L')^{-1} \otimes \det E|, \tag{2}$$

where $|M|$ denotes the complete linear system of effective divisors, which are equivalent to a given divisor, defining M.

Proof. We have the following diagram, where the top line is exact:

$$\begin{tikzcd}
0 \arrow[r] & L \arrow[r, "i"] & E \arrow[r, "j"] & L^* \otimes \det E \arrow[r] & 0 \\
 & & L' \arrow[u, "i'"] \arrow[ur, "ji'"'] & & \\
 & & 0 \arrow[u] & &
\end{tikzcd} \tag{3}$$

The homomorphism ji' vanishes on the intersection of L and L' and is identically zero if and only if $L = L'$. This homomorphism is a section of the sheaf

$$\mathrm{Hom}(L', L^* \otimes \det E) = (L')^* \otimes L^* \otimes \det E,$$

and the zero-set of this section is formed by the points, in which L and L' coincide.

Let s and $s' \in \Gamma(E)$ and $L_s \neq L_{s'}$.

Lemma 6. $\xi(L_{\alpha s+\alpha' s'}, L_{\beta s+\beta' s'}) = \xi(L_s, L_{s'})$ for all except a finite number of pairs $(\alpha, \alpha'), (\beta, \beta')$.

This Lemma follows directly from definition 5.

Denote the divisor of a section s by the symbol η_s. Let $s, s' \in \Gamma(E)$ and $L_s \neq L_{s'}$. Then it is easy to check the following statement.

Lemma 7. $\eta_{\alpha s+\alpha' s'}\xi(L_{\alpha s+\alpha' s'}, L_{\beta s+\beta' s'})\eta_{\beta s+\beta' s'} = \eta_s \xi(L_s, L_{s'})\eta_{s'}$ for any (α, α') and (β, β').

Thus if s, s', s'' and s''' lie on one line in $\mathbf{P}(\Gamma(E))$ then

$$\eta_s \xi(L_s, L_{s'})\eta_{s'} = \eta_{s''}\xi(L_{s''}, L_{s'''})\eta^{'''}.$$

Due to Lemma 5 the following important relationship occurs:

$$\eta_s \xi(L_s, L_{s'})\eta_{s'} \in \ |\det E|. \tag{4}$$

Corollary 1. If a non-decomposable bundle E with $\det E = L(\mathfrak{D}^d), d = 0, 1$, admits a sub-bundle L with $h(L) \leqslant \left[\frac{g-d}{2}\right]$, where $[\alpha]$ is the integer part of number α, then $h(L) = h(E)$ and $Sp^{h(E)}(E)$ is a single point.

Proof. All sub-bundles with $h(L) \leqslant n$ are defined by sections of the bundle $E \otimes L(\mathfrak{D}^n)$. According to (2), for the existence of at least two such sub-bundles one needs that $|\det E \otimes L^2(\mathfrak{D}^n)|$ should contain at least a point. But if $n \leqslant \left[\frac{g-d}{2}\right]$ then

$$|\det E \otimes L(\mathfrak{D}^{2n})| = |\mathfrak{D}^k|, k \leqslant g.$$

And, since $\mathfrak{D}$ is not a Weierstrass point, one deduces that $|\mathfrak{D}|$ is empty.

Corollary 2. If E is irreducible and contains at least two effective sub-bundles L and L' then either the degree of L or the degree of L' is less than $\left[\frac{\deg E-1}{2}\right]$. In other words, $\deg c(E) \leqslant \left[\frac{\deg E-1}{2}\right]$.

These statements easily follow from Lemma 5 and formula (4).

Let us apply now the results obtained to bundles with fixed determinant $L(\mathfrak{D}^d), d = 0, 1$.

Lemma 8. $h(E) \leqslant g - d$.

Proof. Indeed, according to the Riemann–Roch theorem (see [2])

$$\dim H^0(X, B) \geqslant \deg B - r(1-g),$$

where r is the dimension of the fiber (in our case $r = 2$). Let

$$B = E \otimes L(\mathfrak{D}^n);$$

then

$$\deg B = d + 2n,$$

so if $d + 2n + 2 - 2g > 0$, then $n \geqslant h(E)$. This gives the statement of the lemma.

§ 2 Exceptional sub-bundles.

Denote by the symbol $Sp(E)$ the set of line sub-bundles of a bundle E and call it the spectrum of the bundle. It is quite natural to try to introduce some algebraic structure on this set. The set $Sp(E)$ is stratified by the heights of line sub-bundles. Denote by $Sp^n(E)$ the set of sub-bundles of heights $\leqslant n$. Each element $L \in Sp^n(E)$ is defined by a section $s_L \in \Gamma(E \otimes L(\mathfrak{D}^n))$ such that

$$\alpha s_L = s'_L, \quad \alpha \neq 0,$$

so the proportional sections define the same sub-bundle. Consider the space $\Gamma(E \otimes L(\mathfrak{D}^n) \otimes L^*(c(E \otimes L(\mathfrak{D}^n)))$ and the corresponding projective space $\mathbf{P}_n$. We have a map

$$\mathbf{P}_n \xrightarrow{S_n} Sp^n(E),$$

and S_n is a map onto $Sp^n(E)$. If this map is one–to–one then one introduces the structure of a projective space on $Sp^n(E)$.

Definition 6. A component $Sp^m(E)$ is called regular if the map S_m is one–to–one.

It was proven in §1 that at least one regular component in $Sp(E)$ does exist, namely the one is given by $Sp^{h(E)}(E)$.

Regular components are projective spaces.

On the other hands, it is clear that the component $Sp^n(E), n > h(E) + g$, is irregular.

Let N_n be the set of sections of any bundle E which have zeros and M_n be the set of sections possessing $\Gamma(X, L_s) \geqslant 2$. Obviously, $N_n \supset M_n$.

The subsequent proposition proves that N_n is a proper algebraic subvariety of the space P_n.

Proposition 1. Let E be a bundle such that $\Gamma(E) \ni s_0, L_{s_0} \cong 1$. Then the set N of such s for which $L_s > L_{s_0}$, forms a proper homogeneous algebraic subvariety of the space $\Gamma(E)$.

Proof. It's sufficient to consider the kernel of the map of two fibered spaces:

$$0 \longrightarrow \bar{N} \longrightarrow V \xrightarrow{\alpha} E, \tag{4'}$$

where $V = \Gamma(E) \times X$ and the map α sends each section to its value at the point of base (see (2)). It's clear that

$$N = proj_{\Gamma(E)}(\bar{N}),$$

where $proj_{\Gamma(E)}$ denotes the projection of variety V to $\Gamma(E)$. The properness of the subvariety N follows from the existence of $s_0 \notin N$.

Denote as $\chi(E)$ the set of equivalence classes of set $Sp(E)$ and as ϕ the map $Sp(E) \xrightarrow{\phi} \chi(E)$, identifying equivalent line bundles.

Corollary 1. Either $\phi^{-1}(\zeta)$ consists of a single point or it is a projective space without a subvariety, i.e. it consists of infinitely many points.

It is sufficient to apply Proposition 1 to $E \otimes L^*(\zeta)$ which gives us the statement of the corollary.

Definition 7. Elements of the set $Sp(E)$ such that $\phi^{-1}(\phi(L)) = L$, i.e the ones which are uniquely defined by the equivalence classes, are called exceptional elements of the spectrum or exceptional sub-bundles.

The following obvious proposition holds:

Proposition 2. A pair (L, E), where L is an exceptional sub-bundle of E, uniquely determines a point $h \in \mathbf{P}(H^1(X, L^2 \otimes \det E^*)$ and is uniquely determined by such a point.

The above proposition is a well-known statement about extensions of the form

$$0 \longrightarrow L \longrightarrow E \longrightarrow L^* \otimes \det E \longrightarrow 0 \tag{5}$$

(see [2] and [3]).

Thus, if we prove the existence of the exceptional sub-bundles, then the classification problem for the bundles would be reduced to finding some canonical exceptional sub-bundles and solving the extension problem.

Let us prove first of all that for each bundle the exceptional sub-bundles do exist. Introduce some invariants of a bundle class:

- $h(E)$ (see Definition 3).
- $L\Big(c\big(E \otimes \mathfrak{D}^{h(E)}\big)\Big) = L(E)$ (clearly $\dim \Gamma(L(E)) = 1$).
- If $\dim \Gamma\big(E \otimes L(\mathfrak{D}^{h(E)})\big) = 2$ then put

$$\xi\Big(L\big(c\,(E \otimes L(\mathfrak{D}^{h(E)}))\big)\,,\, L'\big(c(E) \otimes L(\mathfrak{D}^{h(E)})\big)\Big) = \xi(E)\ ,$$

  where L and L' are distinct sub-bundles of E. If $\dim \Gamma\big(E \otimes L(\mathfrak{D}^{h(E)})\big) = 1$ then put $\xi(E) = X$.

It is not hard to prove the correctness of the definition for these invariants; we prove only the correctness of the definition for $\xi(E)$. Indeed, $\dim \Gamma\big(E \otimes L(\mathfrak{D}^{h(E)})\big)$ equals either 1 or 2 (Lemma 3). If $\dim \Gamma\big(E \otimes L(\mathfrak{D}^{h(E)})\big) = 2$ then the correctness of the definition for $\xi(E)$ follows from Lemma 6.

The following theorem is the main one for the present paper. Our idea of the classification is based on it.

Theorem 1.

I. Any non-decomposable bundle E has an exceptional sub-bundle.

II. If $\xi(E) = X$, then $L(E)$ is an exceptional sub-bundle.

III. If $\xi(E) \neq X$ then there exist only a finite number N of exceptional sub-bundles of the minimal height. Let $L(\eta_i \mathfrak{D}^{h(E)})$, $i = 1, \dots, N$, be such sub-bundles. Then:

(1) $N \leqslant 2h(E) - 2\deg L(E) + \deg E \leqslant 2g$;

(2) $\dim \mathcal{L}(\eta_i) = 1$, $i = 1, \ldots, N$, where $\mathcal{L}(\eta)$ is the space of the functions over our curve X, comparable with divisor η;

(3) $(\eta_i, \eta_j) = 1$, $i \neq j$, where (η_i, η_j) is the greatest common divisor of η_i and η_j;

(4) $\prod\limits_{i=1}^{N} \eta_i = \xi(E)$.

Proof. It is clear that all sub-bundles of the minimal height are defined by sections of the bundle $B = E \otimes L(\mathfrak{D}^{h(E)})$. According to Lemma 3 $\dim \Gamma(B)$ equals either 1 or 2. If the first case occurs, then $L(E)$ is the unique sub-bundle of the minimal height and hence is an exceptional sub-bundle.

If the second case takes place, let us take a basis s_1, s_2 in $\Gamma(B)$ such that

$$L_{s_1} \cong L_{s_2} \cong L(E).$$

Obviously, $\dim \Gamma(L(E)) = 1$ and therefore if

$$L_s = L_{s'}\ , \quad s, s' \in \Gamma(B)$$

then $s = \alpha s', \alpha \neq 0$. The sections s_1 and s_2 are linearly dependent over points, contained in the divisor

$$\xi(E) = \mathcal{P}_1 \ldots \mathcal{P}_m \in \left| \det E \otimes L^2(\mathfrak{D}^{h(E)}) \otimes L^{-2}(E) \right|$$

(according to Lemma 5). Let α_i be elements of the field k such that

$$s_1(\mathcal{P}_i) = \alpha_i s_2(\mathcal{P}_i).$$

Then the vector $\{\alpha_i\}$ is an invariant of the extension class, and the sub-bundles $L_i = L_{s_1 - \alpha_i s_2}$ are the desired ones. The properties (1) — (4) follow from this arguments.

The next two lemmas are corollaries of our general considerations for the concrete case when $\det E = L(\mathfrak{D}^d)$ where $d = 0$ or 1.

Lemma 9. If $L \in E$ and $\deg L > 0$ then L is an exceptional sub-bundle. If $d = 0$ so $\det E \cong 1$, $L \subset E$ and $\deg L = 0$, then either L is an exceptional sub-bundle or $L^2 = 1$ and $E = L \oplus L$.

Proof. If $E \supset L'$, $L' \cong L$, then according to Lemma 5 the space $|L^{-2} \otimes \det E|$ is not empty. From this point the statement of the lemma obviously follows.

Lemma 10. If E is non-decomposable and $L \subset E$ then $\deg L \leqslant g - 1$.

Proof. Indeed, there is an extension (5) where $\det E = L(\mathfrak{D}^d)$. Due to the duality

$$\dim H^1(X, L^2 \otimes L(\mathfrak{D}^{-d}) = \dim H^0(X, K \otimes L^{-2} \otimes L(\mathfrak{D}^d))$$

and, if $\deg L \geqslant g$, then

$$\dim H^1(X, L^2 \otimes L(\mathfrak{D}^{-d})) = 0,$$

since $\deg K - 2\deg L + d < 0$. Therefore

$$E = (L \oplus L \otimes L(\mathfrak{D}^d)).$$

§ 3 Quasi-bundles.

Theorem 1 makes it possible to reduce the classification of bundles to the classification of extensions. When classifying bundles we have to fix some invariants. First of all, one fixes the determinant of the bundles so one considers the category of bundles with fixed determinant $C(L)$. Obviously, in the case of rank 2 bundles it is sufficient to consider two categories $C(d)$ where $d = 0$ or 1, and $\det E = L(\mathfrak{D}^d)$ if $E \in C(d)$. The category of bundles with determinant L of even degree is derived from $C(0)$ by the multiplication of each bundle on $\sqrt{L}$ where $\sqrt{L}$ denotes one from two possible bundles M such that $M^2 = L$. Analogously, one gets the category of bundles with any odd determinant from $C(1)$. In all what follows the determinant is fixed.

Statement I of Theorem 1 in Chapter 1, where one describes the method of computations for exceptional bundles, and Proposition 2 in §2 give us the possibility to classify all bundles from the category $C(d, L)$ — the bundles with a fixed exceptional sub-bundle L. However, doing this we fix some continuous invariant and this is undesirable.

Statements II and III of Theorem 1 give us the possibility to classify all bundles up to a finite number in the category $C(d, n, k, \alpha)$ where (d, n, k, α) are integer valued bounded invariants (so the domain of values for d, n, k, α is just a finite set). To make the expression "up to a finite number" more precise we have to introduce the following notion.

Definition 8. An extension is a pair $0 \longrightarrow L \longrightarrow E$ where L is a fixed sub-bundle of E.

The set of extensions form a category EC, in which the morphisms are defined by commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & L \arrow[r] \arrow[d] & E \arrow[d] \\
0 \arrow[r] & L' \arrow[r] & E'
\end{tikzcd}$$

Each extension (l, E) has the following integer valued invariants: $\deg\det E$, $h(L)$, $\deg L + h(L)$; moreover, in the future we will introduce once more important integer valued invariant, $\alpha(L, E)$, whose definition is placed below (see Chapter II).

The symbol $EC(n, k, d, \alpha)$ denotes the category of extensions which consists of pairs $0 \longrightarrow L \longrightarrow E$ such that $E \subset C(d)$ so $\det E = L(\mathfrak{D}^d)$, $h(L) = n, k(L) = k$ and $\alpha(L, E) = \alpha$.

Definition 9. An extension $0 \longrightarrow L \longrightarrow E$ is called a quasi-bundle if L is an exceptional sub-bundle of the minimal height.

It is obvious that quasi-bundles form a subcategory in the category EC. This subcategory is denoted by QS. The subcategory of quasi-bundles for the category $EC(n,k,d,\alpha)$ is denoted as $QC(n,k,d,\alpha)$. It follows from statements II and III of Theorem 1 that, although two quasi-bundles with equivalent rank 2 bundles could be non isomorphic in the category QC, one has only a finite number of such quasi-bundles ($\leqslant 2g$).

Definition 10. A family of extensions is an algebraic family of rank 2 bundles $\mathcal{E} \xrightarrow{\rho} E$ and its subfamily of line bundles $\mathcal{L} \xrightarrow{\rho'} E$ such that

$$0 \longrightarrow (\rho')^{-1}(e) \longrightarrow \rho^{-1}(e)$$

is an extension for each $e \in E$.

A family of quasi-bundles is a family of extensions

$$(\mathcal{L}, \mathcal{E}) \xrightarrow{(\rho',\rho)} E$$

such that $0 \longrightarrow (\rho')^{-1}(e) \longrightarrow \rho^{-1}(e)$ is a quasi-bundle.

Let $\mathcal{V}ar$ be the category of algebraic varieties and $F : \mathcal{V}ar \longrightarrow \mathrm{Ens}$ is a functor to the category of sets which sends each variety V to the set of families of rank 2 extensions of the type (n,k,d,α) with the base V. We will prove that F is representable so there exists a family

$$\mathcal{E}(n,k,d,\alpha) \xrightarrow{\rho'} E(n,k,d,\alpha),$$

such that

$$F(V) = \mathrm{Hom}(V, E(n,k,d,\alpha))$$

(see [4]).

On the other hand, we will prove (Chapter III), that the subset $\mathcal{M}$ of the base $E(n,k,d,\alpha)$ of this universal family, consisting of non quasi-bundles, is a proper algebraic subvariety $E(n,k,d,\alpha)$. It follows that the family $\overline{\mathcal{E}}(n,k,d,\alpha) \longrightarrow (E(n,k,d,\alpha) - \mathcal{M})$, given by the restriction of $\mathcal{E}(n,k,d,\alpha)$ to $E(n,k,d,\alpha) - \mathcal{M}$, belongs to the category of families of quasi-bundles and is the universal object for the corresponding universal problem in this category.

CHAPTER 2
Construction of "universal" families and solution of the universal problem for the families of extensions.

§ 1 Matrix divisors.

An algebraic bundle over a variety V is defined by a covering $\{U_i\}$ of the variety V and matrices with functional entries $\phi_{i,j}$ where the functions are regular and regularly invertible on $U_{ij} = U_i \cap U_j$, satisfying the relationship $\phi_{ij}\phi_{jk}\phi_{ki} = 1$. In other words, a bundle is defined by a 1-dimensional cocycle of the sheaf $\mathrm{GL}(n, \mathcal{O})$ — the sheaf of matrices over the sheaf of the germs of regular functions. The sheaf $\mathrm{GL}(n, \mathcal{O})$ is a subsheaf of the sheaf $\mathrm{GL}(n, \mathbb{R})$ — the sheaf of matrices over the sheaf of the germs of rational functions. Hence every 1-dimensional cocycle with coefficients in $\mathrm{GL}(n, \mathcal{O})$ can be considered as a 1-dimensional cocycle with coefficients in $\mathrm{GL}(n, \mathbb{R})$. But every cocycle with coefficients in the sheaf $\mathrm{GL}(n, \mathbb{R})$ splits so it is the coboundary of a zero dimensional cochain $\{f_i\} \in C^0(\mathrm{GL}(n, \mathbb{R}), \{U_i\})$.

For us it is more convenient to define a bundle using the corresponding 0-dimensional cochain $\{f_i\} \in C^0(\mathrm{GL}(n, \mathbb{R}), \{U_i\})$ such that $\delta(\{f_i\}) \in Z^1(\mathrm{GL}(n, \mathcal{O}), \{U_i\})$.

Definition 11. A 0-dimensional cochain $\{f_i\}$ of the sheaf $\mathrm{GL}(n, \mathbb{R})$, the boundary of which is a 1-dimensional cocycle $\{f_i f_j^{-1}\}$ of the sheaf $\mathrm{GL}(n, \mathcal{O})$, is called a matrix divisor.

Let us define an equivalence relation for matrix divisor such that the equivalence classes of matrix divisors would coincide with the equivalence classes of the bundles.

Definition 12. Matrix divisors $\{f_i\}$ and $\{f_i'\}$ are equivalent if there exists a 0-dimensional cochain $\{a_i\}$ of the sheaf $\mathrm{GL}(n, \mathcal{O})$ and a matrix $c \in H^0(V, \mathrm{GL}(n, \mathbb{R}))$ such that $f_i = a_i f_i' c$ on each U_i.

If the base variety V is an algebraic curve then the notion of matrix divisor is equivalent to the following one.

Definition 13. Let us assign to each point x of the curve X the element $E_x \in \mathrm{GL}(n, \mathbb{R})_x$, i.e. the matrix of the germs of rational functions at this point x such, that only for a finite number of points $E_x \neq E^n$, where E^n is the identity matrix. This assingment is called a matrix divisor on the curve. Two matrix divisors E and E' are equivalent if at each point $x \in X$

$$E_x = A_x E'_x F,$$

where A_x is a regularly invertible matrix of the germs of regular functions at x and F is a functional matrix — the same for all x.

Exactly in this form algebraic bundles over a curve appeared first time in the paper of A. Weil [9].

It's not hard to check that a matrix divisor over a curve defines a matrix divisor in the sense of Definition 11.

We will use the notion of matrix divisor over a curve since it simplifies the computations.

Let E be a rank 2 bundle over a variety V and suppose that E admits a line sub-bundle L, defined by a divisor D. Then the following exact sequence takes place:

$$0 \longrightarrow L \longrightarrow E \longrightarrow L^* \otimes \det E \longrightarrow 0,$$

and E is uniquely determined by a 1 - dimensional cocycle with coefficients in

$$\mathrm{Hom}(L^* \otimes \det E, L) = L^2 \otimes \det E^*.$$

We will use the notion of matrix divisor and for this it is necessary to give a new interpretation of 1-dimensional cocycle with coefficients in $L(D)$.

To do this let us use the known exact sequence

$$0 \longrightarrow \Omega \longrightarrow R \longrightarrow R/\Omega \longrightarrow 0, \tag{6}$$

where Ω is the sheaf of germs of regular functions, R is the sheaf of germs of rational functions and R/Ω is the sheaf of germs of principle parts.

Take the tensor product of (6) with $L(D)$

$$0 \longrightarrow L(D) \longrightarrow R \longrightarrow R/L(D) \longrightarrow 0 \tag{6'}$$

and write down the exact sequence of the triple (6')

$$\begin{aligned} 0 \longrightarrow H^0(V, L(D)) \longrightarrow H^0(V, R) \longrightarrow H^0(V, R/L(D)) \longrightarrow \\ \longrightarrow H^1(V, L(D)) \longrightarrow H^1(V, R)\ . \end{aligned} \tag{7}$$

It is not hard to see that $H^1(V, R) = 0$.

Sections of the sheaf $R/L(D)$ are called the systems of principal parts with respect to a divisor D. Every such section is defined by a 0-dimensional cochain c of the sheaf R such that $\delta(c) \in Z^1(L(D))$. Denote the space of the chains which possess this property as $\overline{C}^0(R, D)$. Obviously,

$$H^0(V, R) = Z^0(R) \subset \overline{C}^0(R, D)$$

for every D. Moreover, two co-chains c and c' define the same section of the sheaf $R/L(D)$ if and only if $c - c' \in C^0(L(D))$. Thus from the sequence (7) it follows that

$$H^1(X, L(D)) = \overline{C}^0(R, D)/Z^0(R) + C^0(L(D))\,.$$

In what follows cocycles will be defined in the form of elements from $\overline{C}^0(R, D)$.

Lemma 11. Let S be a hyperplane section of V. Then every cocycle from $H^1(X, L(D))$ has a representative $c \in \overline{C}^0(R, D)$ of the following form: let $\{U_i\}$ be a covering of V and $C = \{f_i\}$. Then the functions f_i have non zero principle parts only on the divisor S.

PROOF. The systems of principle parts $\{f_i\}$ can be regarded as the system of principle parts on the variety $V - S$. But since $V - S$ is an affine variety one has $H^1(V - S, L(D)) = 0$ and, consequently there exists a function f such that $\{f - f_i\}$ has non zero principle part only on S and $\{f - f_i\} \sim \{f_i\}$.

The next paragraph is devoted to the concrete description of all possible matrix divisors over the curve X.

§ 2 Reduction to the normal form.

Consider all bundles which are nontrivial extensions

$$0 \longrightarrow L(\eta \cdot \mathfrak{D}^{-n}) \longrightarrow E \longrightarrow L(\eta^{-1}\mathfrak{D}^{n+d}) \longrightarrow 0.$$

According to [3], they form the projective space $P(H^1(X, L(\eta^2 \cdot \mathfrak{D}^{-2n-d})))$. Write down explicitly the corresponding matrix divisors. To do this first let us compute $H^1(X, E(\eta^2 \cdot \mathfrak{D}^{-2n-d}))$. By the definition

$$H^1(X, E(\mathfrak{D}^{-2n-d}\eta^2)) = R(X)/R(\mathfrak{D}^{-2n-d}\eta^2) + k(X),$$

where $R(X)$ is the space of distributions in the Shevally sense over the curve X, $R(\mathfrak{D}^{-2n-d}\eta^2)$ — the space of distributions, comparable with the divisor $\mathfrak{D}^{-2n-d}\eta^2$, and $k(X)$ is the field of functions over the curve X (see, f.e., [8]).

Lemma 12. For any point $\mathcal{P} \in X$ and any integer number n there is a number m such that there exists a function $f(n, \mathcal{P}) \in \mathcal{L}(\mathcal{P}^n\mathfrak{D}^m)$ with $\nu_{\mathcal{P}}(f(n, \mathcal{P})) = n$, where $\nu_{\mathcal{P}}(f)$ is the order of the function at $\mathcal{P}$.

PROOF. Let us choose m such that $n+m \geqslant 2g$. Then by the Riemann–Roch theorem

$$\dim \mathcal{L}(\mathcal{P}^{n-1}\mathfrak{D}^m) = n + m - g\ ,$$
$$\dim \mathcal{L}(\mathcal{P}^n\mathfrak{D}^m) = n + m - g + 1\ .$$

This proves the lemma.

Corollary 1. In every class $R(X)/R(\eta^2\mathfrak{D}^{-2n-d})+k(X)$ it is possible to choose a representative r such that $r_{\mathcal{P}}=0$ if $\mathcal{P}\neq\mathfrak{D}$ and $r_{\mathfrak{D}}=\sum\limits_{-m}^{2n+d-1}\alpha_i\tau_{\mathfrak{D}}^i$, where $\tau_{\mathfrak{D}}$ is some fixed once and for all local parameter.

Proof. Any distribution r is not comparable with 0 modulo $R(\eta^2\mathfrak{D}^{-2n-d})$ only in a finite set of points $\mathcal{P}_1,\dots,\mathcal{P}_l$. Hence according to Lemma 12 we can choose such a linear combination ϕ of functions of the shape $f(\mathcal{P}_i,k)$ that

$$r_{\mathcal{P}_i}+\phi=0 \qquad \mathrm{mod}\ R(\eta^2\mathfrak{D}^{-2n-d}).$$

Thus in any class from $R(X)/R(\eta^2\mathfrak{D}^{-2n-d})+k(X)$ one can choose such a representative r that $r_{\mathcal{P}}=0$ if $\mathcal{P}\neq\mathfrak{D}$ and $r_{\mathfrak{D}}=\sum\limits_{-m}^{2n+d-1}\alpha_i\tau_{\mathfrak{D}}^i$, and since $\mathfrak{D}$ is not a Weierstrass point then for any $n>g$ there exists a function $f(n,\mathfrak{D})\in\mathcal{L}(\mathfrak{D}^n)$ such that

$$\nu_{\mathfrak{D}}(f(n,\mathfrak{D}))=-n.$$

Hence it is always possible to choose such r that

$$r_{\mathfrak{D}}=\sum_{-g}^{2n+d-1}\alpha_i\tau_{\mathfrak{D}}^i.$$

Definition 14. Fix a local parameter $\tau_{\mathfrak{D}}=\tau$ at a point $\mathfrak{D}\in X$. Then for any function $f=\sum\limits_{-k}^{\infty}\alpha_i\tau^i$ one has

$$[f]_n=\sum_{-k}^{n-1}\alpha_i\tau^i.$$

The symbol $[\mathcal{L}(\xi)]_n$ denotes the linear space of all $[f]_n, f\in\mathcal{L}(\xi)$.

Lemma 13. $[\mathcal{L}(\xi)]_n=\mathcal{L}(\xi)/\mathcal{L}(\xi\mathfrak{D}^{-n-\nu_{\mathfrak{D}}(\xi)})$.

Proof. Associate to any function $f\in\mathcal{L}(\xi)$ the segment $[f]_n$ of the power series. This map is linear. For a function $f\in\mathcal{L}(\xi)$ $[f]_n=0$ if and only if $f\in\mathcal{L}(\xi\mathfrak{D}^{-n-\nu_{\mathfrak{D}}(\xi)})$.

Let us finish the computation of $H^1(X,L(\mathfrak{D}^{-2n-d}\eta^2))$.

Lemma 14. Let $R_n(\tau)$ be the linear subspace of segments of the power series of the form $\sum\limits_{-g}^{2n+d-1}\alpha_i\tau^i$. Denote $H^1(X,L(\mathfrak{D}^{-2n-d}\eta^2))$ as $H(n,\eta)$. Then the following exact sequence exists

(8)

$$0\longrightarrow\mathcal{L}(\eta^2\mathfrak{D}^{-2n-d})\xrightarrow{i}\mathcal{L}(\mathfrak{D}^g\eta^2)\xrightarrow{p}R_n(\tau)\xrightarrow{j}H(n,\eta)\longrightarrow 0,$$

where $p(f)=[f]$.

Proof. According to Corollary 1 of Lemma 12, in each class

$$R(X)/R(\mathfrak{D}^{-2n-d}\eta^2)+k(X)$$

we can choose as the representative some vector from $R_n(\tau)$. Two vectors h and h' would be cohomologous if and only if

$$h - h' = [f]^{\mathfrak{D}}_{2n+d}, \quad f \in \mathcal{L}(\mathfrak{D}^g \eta^2).$$

Thus we get the desired exact sequence (8). We choose a section j' of this sequence and identify $H(n,\eta)$ with $j'(H(n,\eta))$. Elements of the space $H(n,\eta)$ will be denoted by

$$h = \sum_{i=1}^{m} \alpha_i \beta_i(\tau),$$

where $\{\beta_i(\tau)\}$ is a basis of the space $j'(H(n,\eta))$.

Theorem 2. Each extension class of the triple (7) can be realized by a matrix divisor of the following shape:

at a point $\mathfrak{D}$

$$\begin{pmatrix} \tau^{-n} & 0 \\ 0 & \tau^{n+d} \end{pmatrix} \begin{pmatrix} 1 & h(\tau) \\ 0 & 1 \end{pmatrix},$$

at a point $C_i \in \eta (i = 1, \dots, k; \eta = C_1 \dots C_k)$

$$\begin{pmatrix} \tau_{C_i} & 0 \\ 0 & \tau_{C_i}^{-1} \end{pmatrix}, \tag{9}$$

at the remaining points

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

where $h(\tau) \in H(n,\eta)$ and τ_{C_i} is a local parameter at the point C_i.

Proof. It was proven in [3] that any element of (7) has the form

$$\begin{pmatrix} A_{\mathcal{P}} & 0 \\ 0 & B_{\mathcal{P}} \end{pmatrix} \begin{pmatrix} 1 & h_{\mathcal{P}} \\ 0 & 1 \end{pmatrix}$$

at each point $\mathcal{P} \in X$ where $A_{\mathcal{P}}$ and $B_{\mathcal{P}}$ are the distributions of the divisors $\eta \mathfrak{D}^{-n}$ and $\eta^{-1} \mathfrak{D}^{n+d}$ at the point $\mathcal{P}$ and $h_{\mathcal{P}}$ is the germ of the cocycle

$$h \in H^1(X, \operatorname{Hom}(L(\eta^{-1}\mathfrak{D}^{n+d}), L(\eta \mathfrak{D}^{-n})))$$

at the point $\mathcal{P}$. Using Lemma 14, we get the statement of our lemma.

We denote matrix divisors of the shape (9) as $E^h(n,\eta)$.

§ 3 Algebraic structure.

Extensions with the fixed sub-bundle $L(\eta\mathfrak{D}^{-n})$ were classified in § 2. The set of the extension classes $P(H(n,\eta))$ has the structure of a projective space.

Consider the set

$$E(n,k,d) = \bigcup_{\eta\in\tilde{S}^k(X)} (H(n,\eta)).$$

Recall that $\tilde{S}^k(X)$ is bi-regular equivalent to the variety of the classes of divisors of height n and index k:

$$\tilde{S}^k(X) = S^k(X) - S_k,$$

where $S^k(X)$ is the k-th symmetric power of X, and $S_k = \mathfrak{D}\cdot S^{k-1}(X) \cup \{\sigma \in S^k(X), \dim \mathcal{L}(\sigma) > 1\}$ (see Chapter 1, §1).

We would like to endow $E(n,k,d)$ with the structure of an algebraic variety compatible with the structure of $H(n,\eta)$ for every η and $\tilde{S}^k(X)$. Namely, we will endow $E(n,k,d)$ with the structure of an algebraic variety for which there exists a morphism $\pi(n,k,d) : E(n,k,d) \longrightarrow \tilde{S}^k(X)$ such that the triple

$$(E(n,k,d), \pi, \tilde{S}^k(X))$$

would be a locally trivial algebraic bundle and

$$\pi^{-1}(n,k,d)(\eta) = H(n,\eta).$$

Remark. It is not hard to see that such an algebraic structure is defined uniquely so for any other structure $E'(n,k,d)$ possessing the same properties the set $E(n,k,d)$ should be bi-regular equivalent to $E'(n,k,d)$ as an algebraic bundle over $\tilde{S}^k(X)$.

Consider the exact sequence (8):

$$0 \longrightarrow \mathcal{L}(\eta^2\mathfrak{D}^{-2n}) \longrightarrow \mathcal{L}(\mathfrak{D}^g\eta^2) \xrightarrow{[\,]} R_n(\tau) \longrightarrow H(n,\eta) \longrightarrow 0\ ,$$

and assume $k < n$; then $\mathcal{L}(\eta^2\mathfrak{D}^{-2n-d}) = 0$ and the sequence has the form

$$0 \longrightarrow \mathcal{L}(\mathfrak{D}^g\eta^2) \longrightarrow R_n(\tau) \longrightarrow H(n,\eta) \longrightarrow 0\ . \tag{10}$$

We would like to find such bundles $\mathcal{L}(k), R(k)$ and $\mathcal{H}(n,k,d)$ over $\tilde{S}^k(X)$ that there is a following exact sequence

$$0 \longrightarrow \mathcal{L}(k) \longrightarrow R(k) \longrightarrow \mathcal{H}(n,k,d) \longrightarrow 0, \tag{11}$$

which fiberwise coincides with the sequence (10).

For this we can take $R_n(\tau) \times \tilde{S}^k(X)$ as $R(n)$ and it remains just to endow the set

$$\mathcal{L}(k) = \bigcup_{\eta\in\tilde{S}^k(X)} \mathcal{L}(\mathfrak{D}^g\eta^2)$$

with the structure of an algebraic bundle over $\tilde{S}^k(X)$.

Theorem 3. The set $\mathcal{L}(k) = \bigcup_{\eta \in \tilde{S}^k(X)} \mathcal{L}(\mathfrak{D}^g \eta^2)$ admits the structure of an algebraic vector bundle $\mathcal{L}(k) \to \tilde{S}^k(X)$ with a fixed inclusion $0 \to \mathcal{L}(k) \xrightarrow{i} R(n)$, where $R(n) = R_n(\tau) \times \tilde{S}^k(X)$ and locally $i = [\,]$.

Proof. We construct $\mathcal{L}(k)$ straight away as a sub-bundle of $R(n)$. Consider

$$[\mathcal{L}(\mathfrak{D}^g \eta^2)]_{2n+d}$$

and choose there a basis $s_1, \ldots, s_{2k+1}$,

$$s_i = \sum_{-g}^{2n+d-1} \alpha_j^i \tau^i.$$

Obviously,

$$\dim \mathcal{L}(\mathfrak{D}^g \eta^2) = 2k + 1$$

for all η, since $\mathfrak{D}$ is not a Weierstrass point. Hence such a basis always exists. Moreover, there exists a non zero minor $\mu_{i_1,\ldots,i_{2k+1}}$ of order $2k+1$ in the matrix $\|\alpha_j^i\|$. Therefore the space $[\mathcal{L}(\mathfrak{D}^g \eta^2)]_{2n+d}$ could be identified with the space spanned by $\{\tau_j^i\}^{2n+d}, j = 1, \ldots, 2k+1$.

Consider a map $\phi : \tilde{S}^k(X) \longrightarrow \mathbf{P}_N$, where $\mathbf{P}_N$ is the N-dimensional projective space, $N = C_{g+2n+d}^{2k+1}$ defined as follows:

$$\phi(\eta) = \{\mu_{i_1,\ldots,i_{2k+1}}\},$$

where $i_1, \ldots, i_{2k+1}$ run through all collections of the numbers $-g, \ldots, 2n-1$; in other words, the coordinates of a point $\phi(\eta)$ in the space $\mathbf{P}_N$ are defined by all possible minors of order $2k+1$ of the matrix $\|\alpha_j^i(\eta)\|$. It is clear, that after the change of the basis $\{s_i\}$ to a new one all the coordinates are multiplied by the same number, namely by the determinant of the transformation matrix from the first basis to the second one. Moreover, it is not hard to see that the map ϕ is rational. Now if we consider the covering

$$U_{i_1,\ldots,i_{2k+1}} = \{\eta \in \tilde{S}^k(X), \mu_{i_1,\ldots,i_{2k+1}}(\eta) \neq 0\}$$

and the transition functions $\phi_{i_1,\ldots,i_{2k+1},j_1,\ldots,j_{2k+1}}$, transforming the basis

$$\{\tau^{i_l}\}, l = 1, 2, \ldots, 2k+1,$$

to the basis

$$\{\tau^{j_l}\}, l = 1, 2, \ldots, 2k+1,$$

then we get the desired bundle over $\tilde{S}^k(X)$. This bundle is denoted as $\mathcal{L}(k)$. Thus, we have derived the desired exact sequence of bundles

$$0 \longrightarrow \mathcal{L}(k) \xrightarrow{\;i\;} R(n) \xrightarrow{\;j\;} \mathcal{H}(n,k,d) \longrightarrow 0.$$

Proposition 3. The sequence (11) splits so there exists a section

$$S : \mathcal{H} \longrightarrow R(n)$$

and $R(n) = \mathcal{L}(k) \oplus \mathcal{H}(n,k,d)$. The existence of the section follows from the fact that

$$S^k(X)/\mathfrak{D}S^{k-1}(X)$$

is an affine variety and hence

$$H^1(\tilde{S}^k(X), \mathrm{Hom}(\mathcal{H}, \mathcal{L}(k)) = 0.$$

We fix some section S in the triple (11) and identify $\mathcal{H}(n,k,d)$ and $S(\mathcal{H}(n,k,d))$.

Thus, we have proved the following theorem.

Theorem 4. The set $E(n,k,d) = \bigcup_{\eta \in \tilde{S}^k(X)} (H(n,\eta))$ admits the structure of an algebraic variety with the following properties:

(1) there exists a map $\pi{:}E(n,k,d) \longrightarrow \tilde{S}^k(X)$ such that $(E(n,k,d), \pi, \tilde{S}^k(X))$ is a locally trivial algebraic bundle with $\pi^{-1}(\eta) = H(n,\eta)$;

(2) $(E(n,k,d), \pi, \tilde{S}^k(X))$ is a sub-bundle of the bundle R equals to $R = R_n(\tau) \times \tilde{S}^k(X)$ and there exists a projection $p : R \longrightarrow E(n,k,d)$ of the bundle R to the sub-bundle $E(n,k,d)$.

In the future we will use varieties

$$R_N = \tilde{S}^k(X) \times R_n^N(\tau),$$

where

$$R_n^N(\tau) = \sum_{-N}^{2n+d-1} \alpha_i \tau^i.$$

Let us define a canonical map $\phi_N : R_n^N(\tau) \longrightarrow R_n(\tau)$ in the following way:

$$0 \longrightarrow k \xrightarrow{i} \mathcal{L}(\mathfrak{D}^N) \xrightarrow{[\,]} R_n^N(\tau) \xrightarrow{\phi} R_n(\tau) \longrightarrow 0.$$

The map $\tilde{\phi}_N : R_N \longrightarrow R, \tilde{\phi}(\sigma,h) = (\sigma, \phi(h))$ defines a canonical projection $\tilde{\phi} : R_N \longrightarrow R$ where R is regarded as a sub-bundle of R_N.

The varieties R_N, R and $E(n,k,d)$ will serve us as the bases of families of bundles.

Let us study $E(g,1,0)$. As will be shown below, this variety has maximal dimension over all $E(n,k,d)$. We have an exact sequence

(11') $$0 \longrightarrow \mathcal{L}(\mathfrak{D}^g C^2) \longrightarrow R_g(\tau) \longrightarrow H(g,C) \longrightarrow 0.$$

Proposition 4. Let E be a rank n bundle over $X - \mathfrak{D}$. Then $E \cong I_{n-1} \oplus \det E$, where I_{n-1} is rank $n-1$ trivial bundle.

Indeed, any bundle over $X - \mathfrak{D}$ is thin, since $X - \mathfrak{D}$ is an affine variety. From this fact, using a well known argument of Atiyah [2] applied to the exact sequence (4'), one gets the exact sequence

$$0 \longrightarrow I_{n-1} \longrightarrow E \longrightarrow \det E \longrightarrow 0,$$

which splits, so $E = I_{n-1} \oplus \det E$.

Thus, any bundle over $X - \mathfrak{D}$ is uniquely determined by its determinant.

From the exact sequence (11) it follows that

$$\det H(g) = \det{}^* \mathcal{L}(g).$$

Let us compute $\det \mathcal{L}(g)$.

Proposition 5. a) Any function from the space $\mathcal{L}(\mathfrak{D}^g C^2)$ is defined up to an additive constant by its principal part at the point C;

b) there exists a function, which has a given principal part at C.

Indeed, let ϕ and $\phi' \in \mathcal{L}(\mathfrak{D}^g C^2)$ have the same principal part at the point C. Then $\phi - \phi' \in \mathcal{L}(\mathfrak{D}^g)$ and since $\mathfrak{D}$ is not a Weierstrass point we have $\mathcal{L}(\mathfrak{D}^g) = k$ and $\phi = \phi' + \alpha$.

On the other hand, according to the Riemann inequality

$$\dim \mathcal{L}(\mathfrak{D}^g C^2) \geqslant 3, \quad \dim \mathcal{L}(\mathfrak{D}^g C) \geqslant 2,$$

and since $\mathfrak{D}$ is not a Weierstrass point

$$\mathcal{L}(\mathfrak{D}^g C) \supset \mathcal{L}(\mathfrak{D}^g) = k.$$

It follows that there exists a function ϕ with any non zero principal part at C.

Corollary 1. $\mathcal{L}(g) \cong M_0^{-2}, (M_0^{-2})_x = m_x^{-2}/k$, where m_x is the maximal ideal of the local ring $\mathcal{O}_x$ at any point $x \in X - \mathfrak{D}$ and k is constants.

Corollary 2. a) There is an exact sequence

$$0 \longrightarrow M_0^{-1} \longrightarrow \mathcal{L}(g) \longrightarrow M_{-1}^{-2} \longrightarrow 0,$$

where $(M_0^{-1})_x = m_x^{-1}/k, (M_{-1}^{-2})_x = m_x^{-2}/m_x^{-1}$;

b) $\det \mathcal{L}(g) = M_{-2}^{-3}$ where $(M_{-2}^{-3})_x = m_x^{-3}/m_x^{-2}$;

c) $(M_{-2}^{-3})^* = M_4^3$, where $(M_4^3)_x = m_x^3/m_x^4$;

d) $\det H(n, 1) = T^3$, where T is the tangent bundle, $T_x = m_x/m_x^2$.

Thus the variety $\mathbf{P}(E(g, 1, 0))$ is completely described.

§ 4 Construction of universal family.

We are going to construct a family of extensions

$$\mathcal{E}(n,k,d) \xrightarrow{\rho} E(n,k,d)$$

such that

$$\rho^{-1}(\eta,h) = E^h(n,\eta). \tag{12}$$

According to Theorem 4, it is sufficient to construct a family with property (12) over $R = \tilde{S}^k(X) \times R_n(\tau)$ and to consider its restriction to $E(n,k,d)$.

Lemma 15. There exists a family $\mathcal{R}(n,k,d) \xrightarrow{\rho} R$ of rank 2 extensions such that

$$\rho^{-1}((\eta,h)) = E^h(n,\eta).$$

Proof. To prove this we have to construct an extension

$$0 \longrightarrow L(D) \longrightarrow \mathcal{R}(n,k,d) \longrightarrow L(D') \longrightarrow 0 \tag{13}$$

over $X \times \tilde{S}^k(X) \times R_n(\tau)$ such that

$$\mathcal{R}(n,k,d)_{(x,\eta,h)} = E^h(n,\eta)_x.$$

Consider the following divisors

$$\tilde{D}_1 = \sum_{i=2}^{k} \Delta(X_1 \times X_i) \times \prod_{j=2, j\neq i}^{k} X_j,$$
$$\tilde{D}_2 = \mathfrak{D} \times \prod_{i=2}^{k} X_i,$$

in the direct product $X_1 \times X_2 \times \cdots \times X_{k+1}$ of X with itself, where $\Delta(X_1 \times X_i)$ is the diagonal in $X_1 \times X_i$.

We introduce natural maps $\phi : X^k \to \tilde{S}^k(X)$ and $\tilde{\phi} : X \times X^k \to X \times S^k(X)$, $\tilde{\phi}(x,y) = (x, \phi(y))$.

The divisor $\phi(\tilde{D}_i)$ on $X \times S^k(X)$ is denoted by $D_i, i = 1,2$.

Consider the following coverings of our curve X:

$$U_1' = X - \sum_{i=1}^{p} \mathcal{P}_i,$$

where $\mathcal{P}_i$ is the divisor of zeros and poles of the function τ (which is the local parameter we have chosen above), distinct from $\mathfrak{D}$, and

$$U_2' = X - C,$$

where C is an arbitrary point, distinct from $\mathcal{P}_i$ and $\mathfrak{D}$.

In this covering the divisor D_2 is defined by functions f_1 and f_2: $f_1 = \tau$, $f_2 = 1$.

The covering $\{U_i'\}, i = 1, 2$ defines the covering $\{U_i' \times S^k(X)\}$ of the variety $X \times S^k(X)$. This covering is denoted by $\{U_i'\}$ too.

Let $\{U_i''\}, i = 1, 2, \ldots, m$, be a covering of the variety $X \times S^k(X)$ such that in U_i'' the divisor D_1 is given by the equation $\tau_{1i} = 0$ and on $U_i'' \cap U_j''$ the function $\tau_{1i}\tau_{1j}^{-1}$ is regular and regular invertible.

Let $\{U_{ij}'''\}$ be a covering of the variety $X \times S^k(X)$ such that

$$U_{ij}''' = U_i' \cap U_j'', \quad i = 1, 2, \quad j = 1, \ldots, m.$$

From this covering one can derive in a natural way a covering of the variety $X \times S^k(X) \times (R_n(\tau))$, namely

$$U_{ij} = U_{ij}''' \times R_n(\tau).$$

Now we construct an extension of the shape (13), where instead of D and D' we take the following divisors:

$$\begin{gathered} D = (D_1 - nD_2) \times (R_n(\tau)), \\ D' = (-D_1 + (n+d)D_2) \times (R_n(\tau)). \end{gathered}$$

This extension is defined in the form of a matrix divisor. For every element U_{ij} of the covering, constructed above, we associate the following functional matrix f_{ij}:

$$f_{1j} = \begin{pmatrix} \overline{\tau}_{1j}\overline{\tau}^{-n} & 0 \\ 0 & \overline{\tau}_{1j}^{-1}\tau^{n+d} \end{pmatrix} \begin{pmatrix} 1 & \sum\limits_{-g}^{2n+d-1} \overline{\alpha}_i \overline{\tau}^i \\ 0 & 1 \end{pmatrix}, \quad f_{2j} = \begin{pmatrix} \overline{\tau}_{1j} & 0 \\ 0 & \overline{\tau}_{1j}^{-1} \end{pmatrix},$$

where $\overline{\tau}_{1j}, \tau$ and α_k are the following functions on $X \times S^k(X) \times R_n(\tau)$:

$\overline{\tau}_{1j}(x, \eta, h) = \tau_{1j}(x, \eta)$ is the function defined above, $\overline{\tau}(x, \eta, h) = \tau(x)$ and $\overline{\alpha}_k(x, \eta, h) = \alpha_k(h)$.

It is not hard to check that this matrix cochain is a matrix divisor, i.e. that on $U_{ij} \cap U_{i'j'}$ the matrix $f_{ij} \not\cong f_{i'j'}^{-1}$ is regular and regularly invertible.

It is even easier to see that the restriction of this matrix divisor to any curve $(X \times \eta \times h)$ gives the matrix divisor $E^h(n, \eta)$. The family of extensions we have constructed is denoted by

$$\mathcal{R}(n, k, d) \xrightarrow{\rho} S^k(X) R_n(\tau).$$

The symbol $\mathcal{E}(n, k, d)$ denotes the restriction of this family to $E(n, k, d)$.

It is easy to establish that the family $\mathcal{R}(n, k, d)$ is equivalent to a family which is induced by the canonical projection $p : R \longrightarrow E(n, k, d)$ and the family

$$\mathcal{E}(n, k, d) \longrightarrow E(n, k, d)\ .$$

We denote by $\mathcal{R}_N(n,k,d)$ the family on R_N which is induced (via the canonical projection $R_N \xrightarrow{\tilde{\phi}_N} R$) by the family $\mathcal{R}(n,k,d)$. This family will play an important role for the solution of the universality problem.

Now, there remains just a single problem for us — "the problem of pushing down". Namely, let varieties X and Y be given together with a bundle E over X and a regular map $\phi : X \longrightarrow Y$. Is it possible to find such a bundle E' over Y that E would be equivalent to the lifting of E' with respect to ϕ? In our case the situation is sufficiently simple: as X we have $X \times R_n(\tau)$, as Y we have $X \times \mathbf{P}(R_n(\tau))$, as ϕ we have a natural map $\phi(h) = \phi(\alpha, h), \quad \alpha = 0$, and as E we have our bundle $\mathcal{R}(n,k,d)$, which defines the family $\mathcal{R}(n,k,d)$. It is not hard to show, however, that it is impossible to construct such a bundle in our case. Because of this, we have to introduce another discrete invariant.

Definition 15. For any extension $0 \longrightarrow L \longrightarrow E$ we will denote by $\alpha(E,L)$ the integer number $g + \nu_{\mathfrak{D}}(h)$, where $E^h(L)$ is a matrix divisor of the shape (9), which defines E.

Obviously, if (E,L) is a quasi-bundle, then $\alpha(E,L)$ is its invariant. In what follows, we will consider (n,k,d,α)-families of extensions or quasi-bundles with

$$\alpha(E,L) = \alpha\ .$$

We will consider, further, the families $\mathcal{E}(n,k,d,\alpha) \longrightarrow E(n,k,d,\alpha)$, which are the restrictions of the family $\mathcal{R}(n,k,d)$ on $E(n,k,d) \cap R_n^{\alpha}(\tau)\tilde{S}^k(X)$, where R_n^{α} is a subspace of

$$R_n(\tau) = \left\{ \sum_{i=-g}^{2n+d-1} \alpha_i \tau^i \right\},$$

defined by the relationships

$$\alpha_j = 0, \quad j < \alpha, \quad \alpha_\alpha = 1.$$

There obviously are no equivalent extensions among the elements of the family $\mathcal{E}(n,k,d,\alpha)$.

Remark. It is clear that

$$\dim E(n,k,d,2k) > \dim E(n,k,d,\alpha)$$

for any $\alpha \neq 2k$ and that

$$\dim E(n,k,d,2k) = \dim E(n,k,d) - 1.$$

Moreover, it is clear that

$$E(n,k,d,2k) = \mathbf{P}(E(n,k,d)) - A(n,k,d,2k),$$

where $A(n,k,d,2k)$ is a proper subvariety of $\mathbf{P}(E(n,k,d))$.

§ 5 The solution of the universality problem for $EC(n,k,d)$.

In this paragraph we will prove the following main theorem.

Theorem 5. The universality problem for the category $EC(n,k,d,\alpha)$ is solvable and the family $\mathcal{E}(n,k,d,\alpha) \longrightarrow E(n,k,d,\alpha)$ is a universal object in the category of families of (n,k,d,α)-extensions.

PROOF. Consider the family $(\mathcal{L},\mathcal{M}) \longrightarrow M$ of (n,k,d)-extensions. Since the universality problem for families of line bundles is solvable and the Jacobian variety J of the curve X is the base of the universal family, the family $\mathcal{L}$ is induced by a morphism $\sigma : M \longrightarrow I_{n-k}$. But since we restrict the investigation to the case of (n,k,d)-extensions, $\sigma : M \longrightarrow G_{n-k,n}/G_{n-k,n-1}$.

The variety $G_{n-k,n}/G_{n-k,n-1}$, according to Remark 2 of § 1, Chapter 1, is bi-regular equivalent to $\tilde{S}^k(X)$. Thus, the family $\mathcal{L}$ is induced by the morphism $\sigma : M \longrightarrow \tilde{S}^k(X)$.

The family of (n,k,d)-extensions $(\mathcal{L},\mathcal{M}) \xrightarrow{\rho} M$ is defined by the extension

$$0 \longrightarrow L(D_1) \longrightarrow \mathcal{M} \longrightarrow L(D_2) \longrightarrow 0 \tag{14}$$

over $X \times M$.

Let us first study the case when M is an affine variety. In this case the divisor $(\mathfrak{D} \times M)$ of the variety $X \times M$ is a hyperplane section of it, that is $(X \times M)/(\mathfrak{D} \times M)$ is an affine variety.

Because of this, the divisor D_1 is equivalent to a divisor of the shape

$$D_1' + m_1(\mathfrak{D} \times H),$$

where D_1' is an effective divisor. But, since over every curve $(X \times m)$ the divisor D_1 cuts a divisor of the height n, $m_1 = n$, i.e.

$$D_1 \sim D_1' + n(\mathfrak{D} \times M) = \tilde{D}_1,$$

where D_1' is an effective divisor.

Analogous arguments show that

$$D_2 \sim D_1' + (n+d)(\mathfrak{D} \times M) = \tilde{D}_2.$$

Consequently we have an extension

$$0 \longrightarrow L(D_1' - n(\mathfrak{D} \times M)) \longrightarrow \mathcal{M} \longrightarrow L(-D_1' + (n+d)(\mathfrak{D} \times M)) \longrightarrow 0. \tag{15}$$

The classes of such extensions are in one-to-one correspondence with elements of the group

$$H^1(X \times M, L(2D_1' - (2n+d)(\mathfrak{D} \times M))).$$

Let us write down this extension in the form of a matrix divisor.

Let $\{U_i'\}$ be a covering of $X \times M$ such that in U_i' the divisor $(\mathfrak{D} \times M)$ is determined by the equation $\tau_i = 0$ where $\tau_1 = \overline{\tau}(x,m) = \tau(x)$ and $\tau_2 = 1$.

Let $\{U_i''\}, i = 1, \ldots, m$, be a covering of $X \times M$ such that in U_i the divisor D_1' is determined by the equation $\phi_i = 0$.

Consider a covering $\{U_{ij}\}$ of the variety $X \times M$ such that

$$U_{ij} = U_i' \cap U_j'', \quad i = 1, 2, j = 1, 2, \ldots, m.$$

We shall write down the divisor of the extension (14) with respect to this covering:

$$F_{ij} = \begin{pmatrix} \phi_j \tau_i^{-n} & 0 \\ 0 & \phi_j^{-1} \tau_i^{d+n} \end{pmatrix} \begin{pmatrix} 1 & f_{ij} \\ 0 & 1 \end{pmatrix}, \tag{16}$$

where $\{f_{ij}\} \in \overline{C}^0(R, \tilde{D}_1 \tilde{D}_2^{-1}, \{U_{ij}\})$ (the definition was given in §1 of the present Chapter).

According to Lemma 11, we can assume that the principal parts of the functions f_{ij} with respect to $\tilde{D}_1 \tilde{D}_2^{-1}$ are different from zero only on $S = \mathfrak{D} \times M$. Because of this fact, the matrix divisor has the form

$$F_{1j} = \begin{pmatrix} \phi_j \overline{\tau}^{-n} & 0 \\ 0 & \phi_j^{-1} \tau^{n+d} \end{pmatrix} \begin{pmatrix} 1 & f_{1j} \\ 0 & 1 \end{pmatrix}, \quad F_{2j} = \begin{pmatrix} \phi_j & 0 \\ 0 & \phi_j^{-1} \end{pmatrix},$$

where $f_{1j} = \sum\limits_{-N_j = l}^{2n+d-1} \alpha_{lj} \overline{\tau}^l$, α_{lj} are regular on U_{1j}.

Let $\{V_j\}$ a covering of $(\mathfrak{D} \times M) : V_j = U_{1j} \cap (\mathfrak{D} \times M)$. Consider a map $\gamma_j : V_j \longrightarrow (R^{N_j}(\tau) - \mathfrak{D})$, defined as follows:

$$\gamma_j(v) = \sum_{-N_j}^{2n+d-1} \alpha_{lj}(v) \tau^l.$$

Since $\{f_{1j}\} \in \overline{C}^0(\mathcal{M}, \tilde{D}_1 \tilde{D}_2^{-1}, \{U_{ij}\})$, all f_{1j} have the same principal part with respect to divisor $\tilde{D}_1 \tilde{D}_2^{-1}$. Because of this, $N_i = N_j = N$ and $\alpha_{lj}(v) = \alpha_{li}(v)$ on $V_i \cap V_j$.

Thus, there is a regular map $\gamma : M \longrightarrow R_n^N(\tau)$.

Consider a map $(\sigma, \gamma) : M \longrightarrow \tilde{S}^k(X) \times R_n^N(\tau) = R_N$, defined as follows:

$$(\sigma, \gamma)(m) = (\sigma(m), \gamma(m)).$$

Obviously, the family $(\mathcal{L}, \mathcal{M}) \longrightarrow M$ is induced by the map $(\sigma, \gamma) : M \longrightarrow R_N$ and the family $\mathcal{R}_N \longrightarrow R_N$ (the definition is given in the previous paragraph). But the family $\mathcal{R}_N$ itself is equivalent to the family which is induced by the projection $\tilde{\phi}(P) : R_N \longrightarrow E(n, k, d, \alpha)$ and the family $\mathcal{E}(n, k, d, \alpha) \xrightarrow{\rho} E(n, k, d, \alpha)$. Consequently, the family $(\mathcal{L}, \mathcal{M}) \longrightarrow M$ is equivalent to the family which is induced by the map $M \longrightarrow E(n, k, d, \alpha)$ and the family $\mathcal{E}(n, k, d, \alpha)$.

Now let us turn to the general case.

Let M be an arbitrary variety, $\{U_i\}$ is its covering by affine varieties and $U_i = M - S_i$.

In this case the family $\mathcal{L}$ is also induced by a morphism $\sigma : M \longrightarrow \tilde{S}^k(X)$.

There the extension (14) takes place, where

$$D \sim D_1 + n(\mathfrak{D} \times M) + ms_i;$$

here D_1 is an effective divisor, and we can take i to be any number since $s_i \sim s_j$. For each k we can represent the extension of $\mathcal{M}$ in the form of (16) where f_{ij}^k have non zero principal parts only on $(\mathfrak{D} \times M) + (X \times s_k)$. The restriction of the family $\mathcal{M} \longrightarrow M$ to V_k is induced by the map $(\sigma, \gamma_k \tilde{\phi}_{N_k} P) : V_k \longrightarrow E(n, k, d)$. It remains to check that

$$(\sigma, \gamma_k \tilde{\phi}_{N_k} P) = (\sigma, \gamma_l \tilde{\phi}_{N_l} P)$$

over $(X \times V_k) \cap (V_l \times X)$, so that

$$\gamma_k \tilde{\phi}_{N_k} P = \gamma_l \tilde{\phi}_{N_l} P$$

over $V_k \cap V_l$.

But it is obvious, since the co-chains $\{f_{ij}^k\}$ and $\{f_{ij}^l\}$ defining the same extension of $\mathcal{M}$, are equivalent, so there exists a function f such that

$$\{f_{ij}^k - f_{ij}^l\} = \{f\}.$$

But this means exactly that the distributions defined by the restrictions of $\{f_{ij}^k\}$ and $\{f_{ij}^l\}$ to the curve $(X \times v)$, where $v \in V_k \cap V_l$, differ by a function distribution so they are cohomologous to each other (see §2 of the present Chapter). Thus, the family $(\mathcal{L}, \mathcal{M}) \longrightarrow M$ is equivalent to the family which is induced by the morphism $(\sigma, \psi) : M \longrightarrow (E(n, k, d, \alpha))$. This proves that the family $\mathcal{E}(n, k, d, \alpha) \xrightarrow{\rho} E(n, k, d, \alpha)$ is the universal object in the category of families of (n, k, d, α)-extensions.

CHAPTER 3
Weak independence of the invariants

§ 1 Properties of quasi-bundles.

Not every extension is a quasi-bundle.

Proposition 6. An extension $0 \longrightarrow L \xrightarrow{i} E$ is not a quasi-bundle if and only if either there exists a homomorphism $i' : L \longrightarrow E, i' \neq i$ or there exists a homomorphism $L(\mathfrak{D}^{-n+1}) \longrightarrow E$ where $n = h(L)$.

The proof follows easily from the definition of quasi-bundles and the fact that the height does not increase under homomorphisms.

Corollary. An extension $0 \longrightarrow 0 \longrightarrow E$ is not a quasi-bundle if and only if either $\dim \Gamma(E \otimes L(\mathfrak{D}^{n-1})) > 0$ or $\dim \Gamma(E \otimes L^*) > 1$.

Theorem 6. Let $\mathcal{M}(n, k, d)$ be the set of the extensions from $E(n, k, d)$, which are not quasi-bundles. Then $\mathcal{M}(n, k, d)$ is a homogeneous algebraic variety.

PROOF. Consider the family $\mathcal{E}(n, k, d) \longrightarrow E(n, k, d)$ and the family

$$\mathcal{E}(n, k, d) \otimes L(\mathfrak{D}^{n-1}) \xrightarrow{\pi} E(n, k, d),$$

defined as follows:

$$\pi^{-1}(\eta, h) = E^h(n, \eta) \otimes L(\mathfrak{D}^{n-1}).$$

The set $\{a \in E(n, k, d) : H^0(X, \pi^{-1}(a)) \geqslant 1\}$ is an algebraic subvariety of $E(n, k, d)$ due to the semi-continuity principle for algebraic families (see [5]). The statement of the theorem follows from the analogous arguments for bundles of the height n with $c(E) = \eta$.

To compute the codimension of the variety $\mathcal{M}(n, k, d)$ and to construct it explicitly we need to study more carefully matrix divisors of the type $E^h(n, \eta)$.

§ 2 Sections of a matrix divisor.

The notion of a matrix divisor over a curve is equivalent to the notion of a bundle. It remains to define a notion which is equivalent to the notion of a section of a bundle. The subsequent definition fills this gap. Here we use the geometric terminology in order to emphasize the relationship of the consequent arguments with the material of §1 and §2, Chapter 1.

Definition 16. A section of a matrix divisor E is a pair of functions $g_1, g_2 \in k(X)$ such that the components of the vector

$$E_x \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} = (u_x^1, u_x^2)$$

are regular at a point x of our curve X.

It is obvious how one constructs, starting with a section of a matrix divisor, a section of the corresponding bundle and vice versa.

Zeros of the section $\begin{pmatrix} g_1 \\ g_2 \end{pmatrix}$ are given by the mutual zeros of u_x^1 and u_x^2. Zeros of the section $\begin{pmatrix} g_1 \\ g_2 \end{pmatrix}$ form a divisor η_s.

Let us study sections of a matrix divisor $B = \tau^m E^h(n,\eta)$. We shall denote by L_k the line bundle of the extension $E^h(n,\eta)$. Write down the following conditions on g_1 and g_2:

at the point $\mathfrak{D}$

$$\begin{pmatrix} \tau^{m-n} & 0 \\ 0 & \tau^{m+n+d} \end{pmatrix} \begin{pmatrix} 1 & h(\tau) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} = (\tau^{m-n}(g_1 + hg_2), \tau^{m+n} g_2),$$

at a point $C \in \eta$

$$\begin{pmatrix} \tau_C & 0 \\ 0 & \tau_C^{-1} \end{pmatrix} \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} = (\tau_C g_1, \tau_C^{-1} g_2), \tag{17}$$

and at the other points

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} = (g_1, g_2).$$

Let us first of all write down the conditions on g_2. We have: $g_2 \in \mathcal{L}(\mathfrak{D}^{m+n+d}\eta^{-1})$, so g_2 defines a point in the linear system $|\mathfrak{D}^{m+n+d}\eta^{-1}|$ and is determined itself by this point uniquely. Hence a linear map is defined

$$\Gamma(B) \xrightarrow{\phi} \Gamma(\mathfrak{D}^{m+n+d}\eta^{-1}) = \Gamma(L(\mathfrak{D}^m) \otimes L^*(\eta\mathfrak{D}^{-n}) \otimes L(\mathfrak{D}^d)).$$

The geometrical meaning of this map is given by the following

Lemma 16. Let $s = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}$ be a section of B and $(g_2) = \xi\eta\eta_s\mathfrak{D}^{-m-n-d}$. Then $\xi = \xi(L_k, L_s)$.

PROOF. Sections of the sub-bundle L_k are given by vectors $\begin{pmatrix} g \\ 0 \end{pmatrix}$. Therefore, in order to have

$$\alpha \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} = \begin{pmatrix} g \\ 0 \end{pmatrix}$$

at a point $x \in X$, it is necessary and sufficient that

$$\nu_x(g_2) > \nu_x(\mathfrak{D}^{m+n+d}\eta^{-1}),$$

and this gives the statement of the lemma.

Before we write down the conditions on g_1 let us introduce a couple of new notations and notions.

Definition 17. A defect of a function $f \in \mathcal{L}(\zeta)$ at a point $x \in X$ is the number

$$d_x^\zeta(f) = \nu_x(\zeta) + \nu_x(f).$$

This number is denoted as $d_x^\zeta(f)$.

The symbol $g.c.d.(\xi, \xi')$ is used for the divisor which consists of the mutual points of effective divisors ξ and ξ'.

The symbol $\overline{\xi\xi'}$ denotes the minimal mutual multiple of divisors ξ and ξ'.

Now let us write down the conditions on g_1:

1) $g_1 \in \mathcal{L}(\mathfrak{D}^{g+n+m+d}\eta)$,

2) $g_1 + hg_2 = 0 \mod \tau^{n-m-1}$ or

2') $\frac{g_1}{g_2} \equiv -h \mod \tau^{2n+d-1-d_{\mathfrak{D}}^{\zeta}(g_2)}, \zeta \sim \mathfrak{D}^{n+m+d}\eta^{-1}$.

In the future, speaking about a defect of a function g_2 we will have in mind the defect in the linear spaces $\mathcal{L}(\mathfrak{D}^{n+m+d}\eta^{-1})$. Therefore, if $d_{\mathfrak{D}}(g_2) = 0$ then

$$h = -\left[\frac{g_1}{g_2}\right]_{2n+d}.$$

Thus, we have proved the following theorem, which establishes the necessary and sufficient conditions on $h = \sum_{-g}^{2n+d-1} \alpha_i \tau^i$ which a matrix divisor $B = E^h(n, \eta) \otimes L(\mathfrak{D}^m)$ must satisfy to have a section.

Theorem 7. A divisor of the form B has a section s such that $n_s \cdot \xi(L_k, L_s) = \xi$ if and only if there exists a function for which

(1) $f \in \mathcal{L}(\mathfrak{D}^g \eta^2 \xi \cdot \mathfrak{D}^{-\nu_{\mathfrak{D}}(\xi)})$,

(2) $h \equiv f \mod \tau^{2n+d-1-\nu_{\mathfrak{D}}(\xi)}$.

The next lemma makes it possible to compute the divisor - support of the intersection of two line bundles - in terms of the matrix divisors.

Lemma 17. Let $s = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}$ and $s' = \begin{pmatrix} g_1' \\ g_2' \end{pmatrix}$ be sections of B. Then

$$\phi = g_1 g_2' - g_1' g_2 \in \mathcal{L}(\mathfrak{D}^{2m+d}).$$

Further, let $\zeta = (\phi)_0$ be the divisor of zeros of ϕ. Then

$$\eta_s \xi(L_s, L_{s'})\eta_{s'} = \mathfrak{D}^{\max(\mathfrak{D}, n-k-m)}\zeta,$$

where $k = \deg \zeta$.

PROOF. We have:

$$\phi = g_2 g_2' \left(\frac{g_1}{g_2} - \frac{g_1'}{g_2'} \right).$$

According to Lemma 16,

$$(g_2)_0 = \xi, \quad g_2 \in \mathcal{L}(\mathfrak{D}^{m+n+d} \eta^{-1})$$

and

$$\frac{g_1}{g_2} \in \mathcal{L}(\mathfrak{D}^g \eta^2 \xi), \quad \frac{g_1'}{g_2'} \in \mathcal{L}(\mathfrak{D}^g \eta^2 \xi'),$$

where $\xi' = (g_2')_0$. From Theorem 7 it follows that

$$\left(\frac{g_1}{g_2} - \frac{g_1'}{g_2'} \right) \in \mathcal{L} \left(\frac{\eta^2 \overline{\xi \xi'}}{\mathfrak{D}^{2n+d-\max(d_{\mathfrak{D}}(g_2), d_{\mathfrak{D}}(g_2'))}} \right),$$

from which one deduces that

$$\left(\frac{g_1}{g_2} - \frac{g_1'}{g_2'} \right) g_2 g_2' \in$$
$$\in \mathcal{L} \left(\mathfrak{D}^{2(m+n+d) - d_{\mathfrak{D}}(g_2) - d_{\mathfrak{D}}(g_2') - 2n - d + \max(d_{\mathfrak{D}}(g_2), d_{\mathfrak{D}}(g_2'))} \right) \in \mathcal{L}(\mathfrak{D}^{2m+d}).$$

This proves the first statement of the lemma.

The equality $\alpha s(\mathcal{P}) + \beta s'(\mathcal{P}) = 0$ holds if and only if the following system of homogeneous equations

$$\begin{gathered} \alpha u^1_{\mathcal{P}}(\mathcal{P}) + \beta (u')^1_{\mathcal{P}}(\mathcal{P}) = 0, \\ \alpha u^2_{\mathcal{P}}(\mathcal{P}) + \beta (u')^2_{\mathcal{P}}(\mathcal{P}) = 0 \end{gathered}$$

has a nontrivial solution, so if and only if

$$D(\mathcal{P}) = \begin{vmatrix} u^1_{\mathcal{P}}(\mathcal{P}) & (u')^1_{\mathcal{P}}(\mathcal{P}) \\ u^2_{\mathcal{P}}(\mathcal{P}) & (u')^2_{\mathcal{P}}(\mathcal{P}) \end{vmatrix} = 0.$$

But if $\mathcal{P} = \mathfrak{D}$ then $D(\mathcal{P}) = \tau^{m-n} \phi(\mathfrak{D})$; on the other hand if $\mathcal{P} \neq \mathfrak{D}$ then $D(\mathcal{P}) = \phi(\mathcal{P})$. From this fact it follows the second statement of the lemma. Thus the lemma is completely proven.

In §1, Chapter 1, we have seen that

$$\xi(L_s, L_{s'}) = \xi(L_{\alpha s + \alpha' s'}, L_{\beta s + \beta' s'})$$

for almost all (α, α') (so on whole the projective line (α, α') except a finite number of points) and

$$\eta_s \xi(L_s, L_{s'}) \eta_{s'} = \eta_{\alpha s + \alpha' s'} \xi(L_{\alpha s + \alpha' s'}, L_{\beta s + \beta' s'}) \eta_{\beta s + \beta' s'}$$

for every $(\alpha, \alpha'), (\beta, \beta')$.

Let us solve the inverse problem, namely let

$$\eta_s\xi(L_s, L_{s'})\eta_{s'} = \eta_s\xi(L_s, L_{s''})\eta_{s''}. \tag{18}$$

What could be said then about s''? The answer is given by

Theorem 8. Let s, s' and s'' be sections of B and assume and assume that they satisfy relation (18). Then $s'' = \alpha s' + s'''$, where $s''' \in \Gamma(L_s)$.

Proof. Let $s = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}$, $s' = \begin{pmatrix} g_1' \\ g_2' \end{pmatrix}$,$s'' = \begin{pmatrix} g_1'' \\ g_2'' \end{pmatrix}$ and assume that the relationship (18) is satisfied. This means that divisors $(g_1g_2' - g_1'g_2)$ and $(g_1g_2'' - g_1''g_2)$ coincide with each other (see Lemma 17) and, consequently

$$\alpha(g_1g_2' - g_2g_1') = (g_1g_2'' - g_2g_1'');$$

implies that

$$\frac{g_1}{g_2} = \frac{\alpha g_1' - g_1''}{\alpha g_2' - g_2''}.$$

Since $\begin{pmatrix} g_1 \\ g_2 \end{pmatrix}$, $\begin{pmatrix} g_1' \\ g_2' \end{pmatrix}$ and $\begin{pmatrix} g_1'' \\ g_2'' \end{pmatrix}$ are sections of the same bundle, then

$$\begin{array}{ll} g_1 & \alpha g_1' - g_1'' = g_1''' \in \mathcal{L}(\mathfrak{D}^{g+n+m+d}\eta) \\ g_2 & \alpha g_2' - g_2'' = g_2''' \in \mathcal{L}(\mathfrak{D}^{n+m+d}\eta^{-1})\ . \end{array}$$

Therefore our problem is reduced to the following: find a function ϕ such that

$$\begin{array}{c} g_1, g_1\phi \in \mathcal{L}(\mathfrak{D}^{g+n+m+d}\eta), \\ g_2, g_2\phi \in \mathcal{L}(\mathfrak{D}^{n+m+d}\eta^{-1}). \end{array} \tag{20}$$

Let

$$(g_2) = \xi\eta\mathfrak{D}^{-n-m-d+d_{\mathfrak{D}}(g_2)}.$$

Then it is obvious that $\phi \in \mathcal{L}(\xi)$. On the other hand, if

$$(g_1) = \xi'\mathfrak{D}^{-(g+n+m+d)+d_{\mathfrak{D}}(g_1)}\eta^{-1},$$

then $\xi' = \xi \cdot \zeta$ since $g_1\phi \in \mathcal{L}(\mathfrak{D}^{g+n+m+d}\eta)$. It means that if the function ϕ does exist then the divisor of the poles $(\phi)_\infty$ belongs to the divisor of zeros of the section $s = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}$, and vice versa, if $\dim \mathcal{L}(\eta_s) > 1$, then taking any function from $\mathcal{L}(\eta_s)$ we get the relationship (20). Obviously, the section

$$s''' = \begin{pmatrix} g_1\phi \\ g_2\phi \end{pmatrix}$$

belongs to $\Gamma(L_s)$, Consequently,

$$s''' = \begin{pmatrix} g_1\phi \\ g_2\phi \end{pmatrix} = \begin{pmatrix} \alpha g_1' - g_1'' \\ \alpha g_2' - g_2'' \end{pmatrix} = \alpha s' - s'',$$

so $s''' = \alpha s' - s''$, which is what we wanted to prove.

§ 3 Computation of the codimension of variety $\mathcal{M}(n,k,d)$.

In this paragraph one proves the following theorem, which is extremely important for the classification of the bundles.

Theorem 9. $\operatorname{codim}_{\mathbf{P}}(E(n,k,d))\mathcal{M}(n,k,d) > 0$.

Moreover, one gives an effective method to construct

$$\mathcal{M}_\eta(n,k,d) = \mathcal{M}(n,k,d) \cap (H(n,\eta))$$

and, in particular, one proves that the irreducible components of $\mathcal{M}(n,k,d)$ are rational varieties.

The subset $\mathcal{M}(n,k,d)$ which consists of extensions but not quasi-bundles, splits into two parts:

$$\mathcal{M}(n,k,d) = \mathcal{M}'(n,k,d) \cup \mathcal{M}''(n,k,d).$$

The first part $\mathcal{M}'(n,k,d)$ consists of such $E^h(n,\eta)$ which contain a homomorphic image $L(\mathfrak{D}^{n-1})$, that is $h(E^h(n,\eta)) < n$. The second part $\mathcal{M}''(n,k,d)$ consists of such $E^h(n,\eta)$ which contain a homomorphic image $L(\eta\mathfrak{D}^{-n})$ other than L_k. Both these sets are varieties. The variety $\mathcal{M}''(n,k,d)$ has big codimension in $E(n,k,d)$ and can be computed in exactly the same way as $\mathcal{M}'(n,k,d)$. Because of this we shall compute only $\mathcal{M}'(n,k,d)$. Denote by $\mathcal{M}'_\eta(n,k,d)$ the intersection $\mathcal{M}'(n,k,d) \cap H(n,\eta)$.

Let us study when bundles $E^h(n,\eta) \otimes L(\mathfrak{D}^{-n+1})$ have sections, that is when $m = n-1$. Then from the results of §1, Chapter 1, it will follow that for every divisor $E^h(n,\eta)$ the divisor-support of the intersection L_k and $\psi(L(\mathfrak{D}^{-n+1}))$ (where ψ is a homomorphism) belongs to the linear system $|\mathfrak{D}^{2n-1+d}\eta^{-1}|$. Let $\xi \in |\mathfrak{D}^{2n-1+d}\eta^{-1}|$. Then, according to Theorem 7 all the divisors $E^h(n,\eta)$ containing $\psi(L(\mathfrak{D}^{-n+1}))$ and such that

$$\eta_s\xi(\psi(L_s(\mathfrak{D}^{-n+1})), L_k) = \xi,$$

form a linear subspace in $H(n,\eta)$ which is given by $h \in H(n,\eta)$ such that

$$h \equiv f \qquad \operatorname{mod} \tau^{2n+d-1-\nu_{\mathfrak{D}}(\xi)},$$

where f is an arbitrary function from $\mathcal{L}(\mathfrak{D}^g\eta^2\xi\mathfrak{D}^{\nu_{\mathfrak{D}}(\xi)})$.

Consider the direct product

$$P(\mathcal{L}(\mathfrak{D}^{g+n+m+d}\eta)) \times \mathbf{P}(\mathcal{L}(\mathfrak{D}^{n+m+d}\eta^{-1}))$$

and define the following map $\Phi : P \longrightarrow R_n(\tau)$,

$$\Phi(g,g') = \left[\frac{g}{g'}\right]_{2n+d-d_{\mathfrak{D}}(g')} + \sum_{2n+d-d_{\mathfrak{D}}(g')}^{2n+d-1} \alpha_i\tau^i, \tag{21}$$

where α_i are any numbers.

Lemma 18. The function Φ is regular on

$$\mathcal{L}(\mathfrak{D}^{g+n+m+d-1}\eta) \times \{\mathbf{P}(\mathcal{L}(\mathfrak{D}^{n+m+d-1}\eta^{-1})/\mathbf{P}(\mathcal{L}(\mathfrak{D}^{n+m+d-2}\eta^{-1})\}.$$

Proof. Let

$$R_n^1(\tau) = \tau^g R_n(\tau),$$

where

$$R_n(\tau) = \tau^{-g}k[[\tau]] \qquad \mathrm{mod}\ \tau^{2n+d}.$$

Then $R_n^1(\tau)$ can be regarded as a vector space over k and as a ring. Consider the map $\tilde{\Phi} : R_n(\tau) \times R_n^1(\tau) \longrightarrow R_n(\tau)$,

$$\tilde{\Phi}(a, b) = a \cdot b^{-1} \qquad \mathrm{mod}\ \tau_{\mathfrak{D}}^{2n+d}.$$

This map is rational. Let

$$b^{-1} = \left(\sum_{i=0}^{g+2n+d-1} x_i\tau^i\right)^{-1} = \sum_{i=0}^{\infty} \beta_i\tau^i,$$

where

$$\beta_0 = \frac{1}{x_0}, \quad b_1 = -\frac{x}{x_0^2}, \ldots, \beta_k = \frac{P_k(x_0, x_1, \ldots, x_k)}{x_0^{k+1}}$$

and

$$a = \sum_{-g}^{2n+d-1} y_i\tau^i.$$

Then

$$ab^{-1} = \sum_{-g}^{2n+d-1} \gamma_i\tau^i$$

and

$$\gamma_k = \sum_{i+j=k} \frac{y_i P_j(x_0, \ldots, x_i)}{x_0^{j+1}}.$$

It follows that the map $\tilde{\Phi}$ is rational and regular on

$$R_n(\tau) \times R_n^1(\tau)/\tau R_n^1(\tau).$$

Restricting it to

$$\tau^{n+m+d-1}\left[\mathcal{L}(\mathfrak{D}^{g+n+m+d-1}\eta)\right]_{2n+d} \times \tau^{m+n+d-1}\left[\mathcal{L}(\mathfrak{D}^{n+m+d-1}\eta^{-1})\right]_{2n+d},$$

we get the statement of the lemma.

Lemma 19. The function Φ is linear and injective in the second argument.

Proof. The linearity of Φ follows from the linearity of $[\,]_{2n+d}$ (see §4). Let

$$g, g' \in \mathcal{L}(\mathfrak{D}^{g+n+m+d-1}\eta), \quad f \in \mathcal{L}(\mathfrak{D}^{m+n+d-1}\eta^{-1})$$

and $\Phi(g,f)=\Phi(g',f)$. Then

$$\frac{g}{f}-\frac{g'}{f}\in\mathcal{L}\left(\frac{\eta^2\xi}{\mathfrak{D}^{2n+d-d_{\mathfrak{D}}(f)}}\right),$$

where $\xi'=(f_0)$. But

$$\frac{\eta^2\xi}{\mathfrak{D}^{2n+d}-d_{\mathfrak{D}}(f)}\sim\frac{\eta}{\mathfrak{D}},$$

so

$$\dim\mathcal{L}\left(\frac{\eta^2\xi}{\mathfrak{D}^{2n+d-d_{\mathfrak{D}}(f)}}\right)=\dim\mathcal{L}\left(\frac{\eta}{\mathfrak{D}}\right)=0$$

due to the choice of η (see § 4, Chapter 1).

Lemma 20. Let $\mathcal{M}'_\eta(n,k,d)=\mathcal{M}'(n,k,d)\cap H(n,\eta)\subseteq H(n,\eta)$. Then

$$\operatorname{codim}_{H(n,\eta)}\mathcal{M}'_\eta(n,k,d)\geqslant 2g-2n-d+1-\dim\mathcal{L}(K\mathfrak{D}^{-2n-d+1}\eta),$$

where K is the canonical divisor.

Proof. Indeed, according to §4, as $H(n,\eta)$ we can choose any vector space, complementary to the space $[\mathcal{L}(\mathfrak{D}^g\eta^2)]_{2n+d}$. Hence

$$\operatorname{codim}_{H(n,\eta)}\mathcal{M}'_\eta(n,k,d)=\operatorname{codim}_{R_n(\tau)}\Phi(P).$$

Since Φ is a rational map,

$$\dim\Phi(P)\leqslant\dim P=\dim\left|\mathfrak{D}^{2n+d-1}\eta^{-1}\right|+\dim\left|\mathfrak{D}^{g+2n+d-1}\eta\right|+1.$$

The dimension of the last linear system equals to $2n+d-1+k$ since $\mathfrak{D}$ is not a Weierstrass point. It follows that

$$\operatorname{codim}_{R_n(\tau)}\Phi(P)\geqslant g-k-\dim\left|\mathfrak{D}^{2n-1+d}\eta^{-1}\right|.$$

Applying the Riemann–Roch theorem we get

$$\operatorname{codim}_{H(n,\eta)}\mathcal{M}'_\eta(n,k,d)\geqslant 2g-2n-d+1-\dim\mathcal{L}\left(K\mathfrak{D}^{-2n-d+1}\eta\right).$$

Proof of Theorem 9. Let us prove that codim$\mathcal{M}'(n,k,d)$ in $E(n,k,d)$ is greater than or equal to $2g-2n+1-d$. We need to compute the dimension

$$\dim\bigcup_{\substack{\eta\in S^k(X)\\ \xi\in|\mathfrak{D}^{2n+d-1}\eta^{-1}|}}\mathbf{P}(\mathcal{L}(\mathfrak{D}^g\eta^2\xi\mathfrak{D}^{-\nu_{\mathfrak{D}}(\xi)}))=r.$$

Note that

$$r\leqslant\dim\bigcup_{\zeta\in|\mathfrak{D}^{2n-1+d}|}\mathcal{L}(\mathfrak{D}^g\eta\zeta),$$

where $\eta\xi'=\zeta$. Indeed, for any $\zeta\in\left|\mathfrak{D}^{2n+d-1}\eta^{-1}\right|$ one has:

$$\xi\cdot\eta=\zeta\in\left|\mathfrak{D}^{2n-1+d}\right|,\quad\mathcal{L}(\mathfrak{D}^g\eta^2\xi)=\mathcal{L}(\mathfrak{D}^g\eta\zeta),$$

where $\eta = c_1^{\beta_1} \dots c_n^{\beta_n}$, if $\zeta = c_1^{\alpha_1} \dots c_n^{\alpha_n}$; here $\beta_i \leqslant \alpha_i$.

Then, we have

$$\dim \bigcup_{\zeta \in |\mathfrak{D}^{2n-1+d}|} (\mathcal{L}(\mathfrak{D}^g \eta\zeta)) = 4n + d + k - 1 - g,$$

since $\mathfrak{D}$ is not a Weierstrass point, and for any $\zeta = c_1^{\alpha_1} \dots c_n^{\alpha_n}$ there exists just a finite number of η's; this number is less than or equal to $\prod_{i=j}^{n} (\alpha_i + 1)$, hence

$$\operatorname{codim} \mathcal{M}(n, k, d) \geqslant 2g - 2n + 1 - d.$$

The next lemma studies the component of the maximal dimension in the set of bundle classes.

Lemma 21. If $n = g$, $k = 1$ and $d = 0$ then $\operatorname{codim}_{H(g,C)} \mathcal{M}'_C(g, 1, 0) \geqslant 1$.

Proof. In this case $\dim \mathcal{L}(KC\mathfrak{D}^{-2g+1}) > 0$ if and only if $CK \sim \mathfrak{D}^{2g-1}$, where K is the canonical divisor of the base curve X. But this is impossible, since

$$\mathcal{L}(CK) \supset \mathcal{L}(K), \quad \dim \mathcal{L}(K) = g, \quad \dim \mathcal{L}(CK) = g,$$

thus $\mathcal{L}(CK) = \mathcal{L}(K)$. Consequently, the order of the divisor of poles for any function from $\mathcal{L}(CK)$ is less than or equal to $2g - 2$ and therefore the divisor $\mathfrak{D}^{2g-1}$ of degree $2g-1$ can not be the divisor of zeros of any of these functions.

Further, we write $\mathcal{M}(n, k, d, \alpha) = \mathcal{M}(n, k, d) \cap E(n, k, d, \alpha)$.

§ 4 Conclusions.

Thus, the set of classes of quasi-bundles degree d, height n, index k and $\alpha(E) = \alpha$ coincides with

$$E(n, k, d, \alpha)/\mathcal{M}(n, k, d, \alpha) = K(n, k, d, \alpha),$$

and $\operatorname{codim}_{K(n,k,d,\alpha)} \mathcal{M}(n, k, d, \alpha) \geqslant 1$. The variety $K(n, k, d, \alpha)$ is the base of family of (n, k, d, α)-quasi-bundles: $\mathcal{E}(n, k, d, \alpha) \longrightarrow K(n, k, d, \alpha)$, which is the universal object in the category of families of (n, k, d, α)-quasi-bundles.

The following lemma distinguishes the component of the maximal dimension in $K(n, d, k, \alpha)$.

Lemma 22. If $n > n'$ then

$$\dim E(n, 0, d, 2k) > \dim E(n, 1, d, 2k) > \dim E(n', k, d, 2k)$$

for any k.

Proof. Indeed,

$$\dim E(n, 1, d, 2k) - \dim E(n', k, d, 2k) = 2(n - n') - (1 - k') > 0.$$

Corollary 8. $\mathcal{M}(n,k,d) = \varnothing$ if $n \leqslant [\frac{g-d}{2}]$.

Therefore the following main theorem is proven.

Theorem 10. The component of maximal dimension of the set of classes of quasi-bundles of degree $d = 0$ is the following bundle over $(X - \mathfrak{D})$:

$$\mathbf{P}(I_{3g-4} \oplus T^3(X)),$$

where $I_{3g-4} = (X - \mathfrak{D}) \times E_{3g-4}$ is the trivial rank $3g-4$ bundle with no proper subvariety.

Remark. For quasi-bundles of degree $d = 1$ the component of maximal dimension of the variety of equivalence classes coincides with the component of maximal dimension of the set of equivalence classes of rank two bundles, namely, the following theorem holds.

Theorem 11. The component of maximal dimension of the set of equivalence classes of quasi-bundles of degree $d = 1$ is a $3g-3$-dimensional projective space with no subvariety. Moreover, there are no two quasi-bundles from this component with equivalent rank 2 bundles. Thus, this component is the component of maximal dimension in the set of equivalence classes of rank 2 bundles of degree $d = 1$.

The family $\mathcal{E}(g-1,0,1,0) \longrightarrow K(g-1,0,1,0)$ is the unique candidate to be the universal object in the category of families of degree $d = 1$ bundles.

A question arises wether or not it is possible to find a variety of classes of bundles — not of quasi-bundles — of degree 0, as was done for degree 1 bundles.

This variety can be obtained only in the case $g = 2$. In this case this component is a projective space (of dimension $3g-3 = 3$) without the Kummer surface of the curve X as well.

Varieties $K(n,k,d,\alpha)$ are open varieties and because of this the question arises whether we can glue them together to a joint closed variety, that is the question about the topology in the union $\bigcup\limits_{n,k,d,\alpha} E(n,k,d,\alpha)$.

Here the fact that the varieties $E(n,k,d,\alpha)$ are open is extremely important. The next proposition shows that for the representability of the canonical functor, giving the classification of the bundles, restrictions and fixing of some invariants are necessary.

Proposition 7. Let $\mathcal{V}ar$ be the category of algebraic varieties and

$$\mathcal{V}ar \xrightarrow{F} \mathcal{E}ns$$

be the functor from $\mathcal{V}ar$ to the category of sets $\mathcal{E}ns$ which associates to each variety V the set of families of rank 2 algebraic bundles with base V. Then F is not representable (in the sense of Grothendick).

Indeed, let us suppose that there exists a variety V and a family of bundles $\mathcal{V} \longrightarrow V$ which is a "universal object", a solution of the "universality problem" in the category of families of algebraic bundles in the sense of Grothendick (see

[4]). In this case a family of bundles $\mathcal{E} \longrightarrow E$ should be induced by a morphism $E \xrightarrow{\phi} V$.

Consider a family $\mathcal{H}(n,\eta) \xrightarrow{\pi} H(n,\eta)$, which is the restriction of the family

$$\mathcal{E}(n,k,d,2k) \longrightarrow K(n,k,d,2k)$$

onto some $H(n,\eta_0) = V_0$. V_0 is a linear space and it is easy to see that there are entire algebraic curves inside V_0 consisting of equivalent bundles. But there exists a subvariety $D \subset V_0$ such that there are no equivalent bundles in $\mathcal{V} \in V_0 - D$ (since $(\eta_i(E), \eta_j(E)) = 1$; see §2, Chapter 1).

The family $\mathcal{H}(n,\eta)$ should be induced by a morphism $\phi : V_0 \longrightarrow V$, which is one-to-one on $V_0 - D$ and $\dim \phi(D) < \dim D$. But there is no such a morphism for a linear space. Therefore, to solve "the universality problem" in the category of families of algebraic bundles it is necessary to fix some of the invariants.

How are the sets $K(n,k,d,\alpha)$ related to each other, or, in other words, what is the topology of $\bigcup\limits_{n,k,d,\alpha} E(n,k,d,\alpha)$? The only thing one can say about it is that in the topology of $E(n,k,d,\alpha)$ elements of $E(n,k+1,d,\alpha)$ and $E(n-1,k,d,\alpha)$ are not separated from decomposable bundles.

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Vector bundles of finite rank over infinite varieties

This article contains a proof of the conjecture of Schwarzenberger that vector bundles on infinite-dimensional projective space $\mathbb{P}_\infty$ split as the sum of line bundles, and a generalisation to quasi-homogeneous projective varieties.

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Introduction.

The nonsingular hyperplane section of a projective variety inherits many of its topological properties, whilst losing many of its geometric ones. The inverse operation — that of embedding a variety as a hyperplane section of a bigger one — is called a linear extension, and an infinite tower of extensions gives rise to an infinite variety — an object which is extremely convenient for checking some of the main conjectures of algebraic geometry.

The aim of the present article is, whilst leaving aside the infinitesimal or formal theory of extensions, to use elementary geometric properties of infinite projective varieties to solve a group of problems which have attracted the attention of algebraic geometers in recent years (see [2] and [3]).

The language of infinite varieties in the proof of Schwarzenberger's conjecture (Theorem 1, § 3.1) and its generalization (Theorem 2, § 3.1) allows us to avoid the cumbersome description of a large number of constants of the finite-dimensional theory, the only point of which is that they be "sufficiently big".

The article is divided into three chapters, each of which falls into two sections.

Chapter 1 gives a definition of an infinite projective variety (§ 1.1), and describes its simplest properties. Chapter 2 contains auxiliary results which we need for the study of vector bundles. Chapter 3 contains a proof of Schwarzenberger's conjecture (§ 3.1), and of its generalization (§ 3.2).

CHAPTER 1
Infinite variety

§ 1 Linear extensions and infinite variety.

Definition 1.1. Let X_1 be a nonsingular complete algebraic variety, and let X_0 be a nonsingular positive divisor on X_1 such that $X_1 - X_0$ is an affine variety. [1] The pair $(X_1 \supset X_0)$ is called a *Lefschetz pair*.

Definition 1.2. A Lefschetz pair $(X_2 \supset X_1)$ is called a *linear affine extension* of the pair $(X_1 \supset X_0)$ if the self-intersection class X_1^2 of X_1 in X_2 is linearly equivalent to X_0 (as divisors on X_1).

Definition 1.3. A polarized variety (X_1, X_0) is called *absolute* if the divisor class X_0 contains a representative X_0 such that the Lefschetz pair $(X_1 \supset X_0)$ admits an infinite extension.

Definition 1.4. An infinite tower of extensions

$$X_0 \subset X_1 \subset \cdots \subset X_n \subset \cdots$$

is called a *flag* over $X_0 \subset X_1$.

Definition 1.5. The flag $Y_0 \subset Y_1 \subset \cdots$ *contains* $X_0 \subset X_1 \subset \cdots$ if for each i we can find $j(i)$ for which

$$X_i \subset Y_{j(i)}. \tag{1.1}$$

If at the same time this embedding is strictly compatible with the filtration, that is, if

$$X_i \subset Y_{j(i)} \Rightarrow X_{i-1} \subset Y_{j(i)-1}, \tag{1.2}$$

then the flag $X_0 \subset X_1 \subset \cdots$ is *strictly contained* in $Y_0 \subset Y_1 \cdots$; notation: $X \overset{s}{\subset} Y$.

Definition 1.6. Two flags are *equivalent* if they are both strictly contained in a third.[2]

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[1]Translator's note: It seems reasonable to make the further restriction that the divisor X_0 be ample

[2]Translator's note: This equivalence is absurdly strong, since it implies that all flags of complete intersections are equivalent to $\mathbb{P}_i$; in the sequel the author seems to consider two flags X_i and Y_i as equivalent only if they coincide up to renumbering from some point on; that is, if, for some n and m, $X_i = Y_{i+n}$ for $i \geqslant m$.

Definition 1.7. An equivalence class of flags is called an *infinite variety* denoted X_∞.

The simplest example of an infinitive variety is the infinite-dimensional projective space $\mathbb{P}_\infty$, defined by a flag over $\mathbb{P}_0 \in \mathbb{P}_1$.

Definition 1.8. A variety X_∞ is *contained* in another Y_∞ if there exist flags $X_0 \subset X_1 \subset \cdots$ and $Y_0 \subset Y_1 \cdots$ defining X_∞ and Y_∞, and the first flag is contained in the second. If this inclusion is strict, then X_∞ is called a *subvariety* of Y_∞ *of finite codimension*, or a *strictly embedded subvariety*; notation: $X_\infty \overset{s}{\subset} Y_\infty$.

If the variety X_∞ is defined by the flag $X_0 \subset X_1 \subset \cdots$, then the embedding

$$\begin{array}{ccccccccccc} X_0 & \subset & X_1 & \subset & \cdots & \subset & X_n & \subset & X_{n+1} & \subset & \cdots \\ & & \cup & & & & \cup & & \cup & & \\ & & X_0 & \subset & \cdots & \subset & X_{n-1} & \subset & X_n & \subset & \cdots \end{array} \tag{1.3}$$

and the shift of the filtration by 1 defines a strict embedding $X_\infty \overset{s}{\subset} X_\infty$ of X_∞ into itself as a divisor.

Definition 1.9. Let X_∞ be defined by the flag $X_0 \subset X_1 \subset \cdots$. A system $\{E_i\}$ of vector bundles E_i on the X_i satisfying $E_{i|X_{i-1}} = E_{i-1}$ defines a *vector bundle* E on X_∞.

Definition 1.10. $\operatorname{Pic} X_\infty$ will denote the group of isomorphism classes of line bundles on X_∞.

As we have seen (1.3), every representation of X_∞ by a flag $X_0 \subset X_1 \subset \cdots$ defines an effective divisor on X_∞; that is, a positive element of $\operatorname{Pic} X_\infty$. Thus, the choice of a flag defining the variety gives us a choice of polarization on X_∞; that is, of an effective positive line bundle h on X_∞.

Thus a polarized variety (X_1, L) is absolute if we have an embedding $X_1 \subset X_\infty$ of X_1 into an infinite variety, such that $L = h|_{X_1}$, with h the effective polarization of X_∞.

Definition 1.11. The vector bundle E on an absolute variety X_1 is said to be *absolute* if E is the restriction of a vector bundle on X_∞.

The Lefschetz theorem allows us to speak of the cohomology of infinite varieties since for a flag $X_0 \subset X_1 \subset \cdots$, for every integer i there is some integer $j(i)$ such that $\operatorname{res} : H^i(X_j) \longrightarrow H^i(X_{j-1})$ is an isomorphism for $j > j(i)$. There is a similar assertion for sheaf (coherent) cohomology.

The simplest class of infinite varieties is the class of subvarieties of $\mathbb{P}_\infty$ of finite codimension.

Definition 1.12. An infinite variety X_∞ is called *projective* if $X_\infty \overset{s}{\subset} \mathbb{P}_\infty$.

Definition 1.13. An absolute variety (X_1, L) is called *projective* if $(X_1, L) \subset (X_\infty, h) \overset{s}{\subset} \mathbb{P}_\infty$.

As an example of absolute projective variety we can take any complete intersection in $\mathbb{P}_n$.

The following is a weak corollary of the results of Barth and Larsen [1].

Proposition 1.1. *For a projective infinite variety* $X_\infty \overset{s}{\subset} \mathbb{P}_\infty$*:*

1) $\operatorname{Pic} X_\infty = \mathbf{Z}$ *and* $\operatorname{Pic} X_\infty$ *is generated by* $O(1)$;

2) $H^i(X_\infty) = H^i(\mathbb{P}_\infty)$, *the isomorphism being given by restriction.*

These results also follow from the Lefschetz theorem if we take into account the following.

Proposition 1.2. *An absolute projective variety $X_n \subset \mathbb{P}_n$ is a complete intersection.*

This assertion was first announced by Hartshorne [3]. We will however not make any use of it, since the arguments we present are simpler, and Proposition 1.2 can itself be deduced from them.

There exist a fair number of simple examples of absolute varieties which are not projective. For example, a double space $X_\infty \xrightarrow{\varphi} \mathbb{P}_\infty$ can be defined by a system of double coverings $X_i \xrightarrow{\varphi_i} \mathbb{P}_i$, ramified in some flag $W_i \subset \mathbb{P}_i$ of hypersurfaces of even degree; X_∞ can be polarized by $\varphi^*(O(1))$.

Proposition 1.3. *For the double space $\varphi : X_\infty \longrightarrow \mathbb{P}_\infty$ we have* $\operatorname{Pic} X_\infty =$ $\boldsymbol{Z}$, *a generator being given by $\varphi^*(O(1))$.*

Indeed, for the ramification divisor W_∞ we have $W_\infty \overset{s}{\subset} X_\infty$ and $W_\infty \overset{s}{\subset} \mathbb{P}_\infty$, and both $X_\infty - W_\infty$ and $\mathbb{P}_\infty - W_\infty$ are affine varieties; so by the Lefschetz theorem, $\operatorname{Pic} X_\infty = \operatorname{Pic} W_\infty = \operatorname{Pic} \mathbb{P}_\infty$.

It follows from this that the infinite double space X_∞ is not a projective variety.

Proposition 1.4. *If $X_\infty \overset{s}{\subset} \mathbb{P}_\infty$, then for every $d \in \boldsymbol{Z}$ we have $H^i(X_\infty, I(d)) = 0$ for $i > 0$.*

Proof. It follows from Proposition 1.1 that for every i we have $H^i(X_\infty, I) = 0$. It follows from the short exact sequence that $H^i(X_\infty, I(d)) = 0$ for every i and $d \leqslant 0$. Let $X_0 \subset X_1 \subset \cdots$ be a flag defining X_∞. Then for every d there is an N_d such that $H^i(X_\infty, I(d)) \longrightarrow H^i(X_N, I(d))$ is an isomorphism for $N \geqslant N_d$.

Since $\operatorname{Pic} X_\infty = \mathbf{Z}$, the canonical class is given by $K_{X_i} = I(k_i)$, and by the adjunction formula one has $k_{i+1} = k_i - 1$. Hence we can find some N_0 such that $k_N < 0$ for all $N \geqslant N_0$. Hence, using Serre duality, $h^i(X_N, I(d)) = h^{N-i}(X_N, I(k_N - d)) = 0$ for $N \gg 0$, since $k_N - d < 0$. Q.e.d.

In the sequel we will not insist on the nonsingularity of our infinite varieties, making special mention if nonsingularity is necessary.

For infinite varieties the two following extremely simple assertions work just as well as the maximum principle for complete varieties.

Connectedness Principle. *If $X_\infty \overset{s}{\subset} \mathbb{P}_\infty \overset{s}{\supset} Y_\infty$ are two projective infinite varieties, then they intersect in some subvariety of finite codimension.*

Thus any reducible variety $X_\infty \overset{s}{\subset} \mathbb{P}_\infty$ (having a finite number of components) is connected.

Rigidity Principle. *A morphism of an infinite projective variety $X_\infty \overset{s}{\subset}$ $\mathbb{P}_\infty$ into any finite-dimensional variety Z is constant.*

Indeed, if $\varphi : X_\infty \longrightarrow Z$, then for any two points z_1 and z_2 of Z the fibers $\varphi^{-1}(z_1)$ and $\varphi^{-1}(z_2)$ are varieties of finite codimension in $\mathbb{P}_\infty$, and hence intersect.

We can include varieties such as the double space of Proposition 1.3 into our theory provided that we go over to weighted projective space (see [4]). Let $w = (d_0, d_1, \cdots, d_n)$ be a collection of integers, and let $\mathbb{P}_w$ be the weighted projective space[3] of index w. Then the pair $\mathbb{P}_w \subset \mathbb{P}_{(w,1)}$ is a Lefschetz pair in a certain sense. Let us partially order the set of indices by $w_1 \subset w_2 \Leftrightarrow w_2 = (w_1, 1)$ and consider the infinite sequence $w_1 \subset w_2 \subset \cdots \subset w_n \subset \cdots = w_\infty$. This defines a flag $\mathbb{P}_{w_1} \subset \mathbb{P}_{w_2} \subset \cdots \subset \mathbb{P}_{w_n} \subset \cdots = \mathbb{P}_{w_\infty}$, which can naturally be called *infinite weighted projective space.*

Definition 1.12'. An infinite variety X_∞ is called a *weighted projective variety* if $X_\infty \overset{s}{\subset} \mathbb{P}_{w_\infty}$.

The weighted projective space $\mathbb{P}_w$ has a standard covering by the standard projective space:

$$\varphi_w : \mathbb{P}_{[w]} \longrightarrow \mathbb{P}_w, \tag{1.4}$$

where $[w] = n$ is the dimension of $\mathbb{P}_w$ (see § 1 of [4]), which is a Galois covering with Abelian group $\prod_{i=0}^{n} \mu_{d_i}$.

We get a diagram of coverings:

$$\begin{array}{ccccccccccc}
\mathbb{P}_{w_1} & \subset & \mathbb{P}_{w_2} & \subset & \cdots & \subset & \mathbb{P}_{w_n} & \subset & \cdots & = & \mathbb{P}_{w_\infty} \\
\uparrow{\scriptstyle \varphi_{w_1}} & & \uparrow{\scriptstyle \varphi_{w_2}} & & & & \uparrow{\scriptstyle \varphi_{w_n}} & & & & \uparrow{\scriptstyle \varphi_{w_\infty}} \\
\mathbb{P}_{[w_1]} & \subset & \mathbb{P}_{[w_2]} & \subset & \cdots & \subset & \mathbb{P}_{[w_n]} & \subset & \cdots & = & \mathbb{P}_{w_\infty}
\end{array}$$

It follows that any weighted projective variety defines a projective variety $\widetilde{X}_\infty$:

$$\begin{array}{ccc}
X_\infty & \overset{s}{\subset} & \mathbb{P}_{w_\infty} \\
\uparrow{\scriptstyle \varphi_{w_\infty}} & & \uparrow{\scriptstyle \varphi_{w_\infty}} \\
\widetilde{X}_\infty & \overset{s}{\subset} & \mathbb{P}_{w_\infty}.
\end{array} \tag{1.5}$$

This finite covering allows us to carry over to the weighted projective case any assertion about nonsingular infinite projective varieties.

We note two simple properties of the covering (1.4):

$$\varphi^*_{w_\infty} : \operatorname{Pic} X_{w_\infty} \xrightarrow{\sim} \operatorname{Pic} \widetilde{X}_\infty \quad \text{is an isomorphism;} \tag{1.6}$$

$$R^0 \varphi_w O_{\widetilde{X}} = \bigoplus_{i=1}^{N} O_X(m_i), \text{ with } m_i \leqslant 0 \tag{1.7}$$

[3] Definition. $\mathbb{P}_w = \operatorname{Proj} k[x_0, \cdots, x_n]$, where the weighting of the ring $k[x_0, \cdots, x_n]$ is defined by $w(x_i) = d_i$.

From these facts one deduces at once the analogs of Proposition 1.1 and 1.4 and the Rigidity and Connectedness Principles for infinite weighted projective varieties.

We take special note only of the analog of Proposition 1.2, which identifies the class of absolute weighted projective varieties with the class of generalised complete intersections considered by Mori (see [4]).

Proposition 1.2'. *A nonsingular absolute weighted projective variety $X \subset \mathbb{P}_w$ is a generalised complete intersection.*

This follows easily from the fact that $\widetilde{X}$ is a complete intersection in $\mathbb{P}_{[w]}$.

Finally, the following seems to be very plausible:

CONJECTURE (*M. Reid*). *Any absolute X is a weighted complete intersection.*

§ 2 The linear connectivity of an infinite projective variety.

We have seen that infinite projective varieties $X_\infty \overset{s}{\subset} \mathbb{P}_\infty$ are similar in their topological properties to projective space. In this section we note a few geometrical properties of such varieties X_∞ which are similar to the properties of quadrics.

Lemma 1.1. *Through every point $P \in X_\infty \overset{s}{\subset} \mathbb{P}_\infty$ there passes an infinite-dimensional linear space $\mathbb{P}_\infty$, that is, X_∞ is swept out by infinite-dimensional linear spaces.*

REMARK. If we use Hartshornes result (Proposition 1.2), then the lemma follows from results of Predonzan [5].

PROOF. A projective variety $X_\infty \overset{s}{\subset} \mathbb{P}_\infty$ has two invariants $n = \operatorname{codim}_{\mathbb{P}_\infty} X_\infty$ and $d = \deg X_\infty$; both of these are integer invariants of a projective variety which are preserved under extension. Let

$$\begin{array}{ccccccccc} X_0 & \subset X_1 & \subset & \cdots & \subset X_i & \subset X_{i+1} & \subset \cdots \\ \cap & \cap & & & \cap & \cap & \\ \mathbb{P}_n & \subset \mathbb{P}_{n+1} & \subset & \cdots & \subset \mathbb{P}_{n+i} & \subset \mathbb{P}_{n+i+1} & \subset \cdots \end{array}$$

be the inclusion of flags which defines the projective variety $X_\infty \overset{s}{\subset} \mathbb{P}_\infty$. Let x be any point on X_∞ and $x \in X_i \subset \mathbb{P}_{n+i}$. Then the projectivised tangent spaces to X_j at x give a flag $(\mathbb{P}TX_i)_x \subset (\mathbb{P}TX_{i+1})_x \subset \cdots$ which defines the infinite projective space $(\mathbb{P}TX_\infty)_x$. Each point of $(\mathbb{P}TX_i)_x$ can be interpreted as a projective line $l \subset \mathbb{P}_{n+i}$ passing through x and tangent there to X_i; that is, the local intersection number $(X_i, l)_x$ of l with X_i at x is at least 2:

$$(\mathbb{P}TX_i)_x = \{l \subset \mathbb{P}_{n+i}; (X_i, l)_x \geqslant 2\}.$$

Consider the variety $(V_i^k)_x = \{l \subset \mathbb{P}_{n+i} \mid (X_i, l)_x \geqslant k+1\}$, that is, the variety of lines of $\mathbb{P}_{n+i}$ having local intersection number at least $k+1$ with X_i

at the point x. Then $(V_i^k)_x \subset (V_i^{k-1})_x$; $(V_i^1)_x = (\mathbb{P}TX_i)_x$; and $(V_i^d)_x$ is the variety of lines of $\mathbb{P}_{n+i}$ passing through x and contained entirely in X_i.

Lemma 1.2. $\operatorname{codim}_{(V_i^{k-1})_x}(V_i^k)_x \leqslant n$ and $V_i^k = V_{i+1}^k \cap (\mathbb{P}TX_i)_x$.

Proof. We can find n forms $f_1, \cdots, f_n$ in $\mathbb{P}_{n+i}$ such that X_i is given in a neighborhood of x as the locus of common zeros of the f_j; consider affine coordinates centered on x and write out each of the forms f_m as a sum of homogeneous forms of ascending degree:

$$f_m = f_m^1 + f_m^2 + \cdots \text{ with } \deg f_m^r = r\,.$$

The components f_m^i can be considered as forms on the projective space $(\mathbb{P}T\mathbb{P}_{n+i})_x$. It is then clear that

$$\bigcap_{m=1}^{n} \{f_m^1 = 0\} = (\mathbb{P}TX_i)_x\,, \quad \text{and} \quad \bigcap_{m=1}^{n} \Big(\bigcap_{l=1}^{k+1} \{f_m^l = 0\} \Big) = V_i^k\,.$$

The first part of the lemma follows from this. The second assertion is obvious, and the lemma is proved.

Corollary 1.1. *The flag $(V_i^k)_x \subset (V_{i+1}^k)_x \subset \cdots$ defines an infinite variety $(V_\infty^k)_x$.*

Corollary 1.2. *We have the following series of inclusions, each of codimension $\leqslant n$*

$$(V_\infty^d)_x \overset{s}{\subset} (V_\infty^{d-1})_x \overset{s}{\subset} \cdots \overset{s}{\subset} (V_\infty^1)_x = (\mathbb{P}TX_\infty)_x = \mathbb{P}_\infty.$$

In particular, each of the infinite varieties $(V_\infty^k)_x$ is a projective variety.

Corollary 1.3. *Any infinite projective variety $X_\infty \overset{s}{\subset} \mathbb{P}_\infty$ is swept out by lines.*

The variety V_x^d of lines lying on X_∞ and passing through x is an infinite projective variety $V_x^d \overset{s}{\subset} \mathbb{P}_\infty = (\mathbb{P}TX_\infty)_x$.

Lemma 1.3. *For any $\mathbb{P}_n \subset X_\infty \overset{s}{\subset} \mathbb{P}_\infty$, there is a $\mathbb{P}_{n+1}$ such that $\mathbb{P}_n \subset \mathbb{P}_{n+1} \subset X_\infty \overset{s}{\subset} \mathbb{P}_\infty$.*

Proof. We prove this by induction on n. For $n = 0$ the assertion has already been proved. Suppose that we know it to hold for $n-1$. Consider a $\mathbb{P}_n \subset X_\infty$, and a point $x \in \mathbb{P}_n$. Then $\mathbb{P}_n$ defines a $\mathbb{P}_{n-1}$ on $V_x^d \overset{s}{\subset} \mathbb{P}_\infty = (\mathbb{P}TX_\infty)_x$. By the induction hypothesis there exists a $\mathbb{P}_n$ such that

$$\mathbb{P}_{n-1} \subset \mathbb{P}_n \subset V_x^d \overset{s}{\subset} (\mathbb{P}TX_\infty)_x.$$

But if we take the cone over this $\mathbb{P}_n$ with vertex x we get a $\mathbb{P}_{n+1}$ with $\mathbb{P}_n \subset \mathbb{P}_{n+1} \subset X_\infty$, since $\mathbb{P}_n \subset V_x^d$.

This corollary implies Lemma 1.1, the proof of which is now complete.

Note however, that the inclusion $\mathbb{P}_\infty \subset X_\infty$ is not strict, since $\mathbb{P}_\infty$ will have infinite codimension in X_∞.

Thus any infinite projective variety is swept out by lines. However, the only variety in which every pair of points can be joined by a line is of course projective space. On an infinite projective variety, any pair of points can be joined by a broken line.

Definition 1.14. A connected, reducible curve $C = \bigcup_{i=1}^{N} C_i$ is called a *broken line* if the components C_i are lines in projective space and $C_i \cdot C_j = \varnothing$ if $|j-i| > 1$. The number of components is called the *length* of the broken line.

Lemma 1.4. *On an infinite projective variety $X_\infty \overset{s}{\subset} \mathbb{P}_\infty$ any pair of points x_1, x_2 can be joined by a broken line of length 2.*

PROOF. Indeed, consider the point $x \in X_\infty \overset{s}{\subset} \mathbb{P}_\infty$. The tangent linear space $(TX_\infty)_x$ can be considered as the cone over $(\mathbb{P}TX_\infty)_x$ with vertex x:

$$(TX_\infty)_x = \langle x, (\mathbb{P}TX_\infty)_x \rangle;$$

this subspace contains the cone $C(V_x^d) = \langle x, (V_x^d) \rangle$. This cone is the variety swept out by the lines of X_∞ through x. Hence

$$\operatorname{codim}_{\mathbb{P}_\infty} C(V_x^d) \leqslant \operatorname{codim}_{\mathbb{P}_\infty} (TX_\infty)_x + \operatorname{codim}_{(\mathbb{P}TX_\infty)_x} V_x^d \leqslant n + d + 1.$$

Hence $C(V_x^d)$ is a subvariety of finite codimension in $\mathbb{P}_\infty$, as is $C(V_{x_2}^d)$; but any two such subvarieties intersect according to the Connectedness Principle (see § 1.1).

COROLLARY 1.4. *If two points $x_1, x_2 \in X_\infty \overset{s}{\subset} \mathbb{P}_\infty$ cannot be joined by a line of X_∞, then the broken lines of length 2 joining them on X_∞ form a projective infinite variety.*

Indeed, if x_1 and x_2 cannot be joined by a line, each of the broken line of length 2 joining them is uniquely determined by the vertex at which the two lines meet. But the variety formed by these vertices is just the intersection $C(V_{x_1}^d) \cap C(V_{x_2}^d) \overset{s}{\subset} \mathbb{P}_\infty$.

Definition 1.15. A connected reducible surface $S = \bigcup_{i=1}^{N} S_i$ is called a *broken plane* if its components S_i are projective planes, and each $S_i \bigcap S_{i+1} = l$ is a line. The number of components is called the *length* of the broken line.

Lemma 1.5. *Let l_1 and l_2 be any intersecting pair of lines on $X_\infty \overset{s}{\subset} \mathbb{P}_\infty$. Then these lines can be joined by a broken plane of length 2.*

PROOF. Let x be the point of intersection of l_1 and l_2. In $V_x^d \overset{s}{\subset} (\mathbb{P}TX_\infty)_x$ the pair of points corresponding to the line l_1 and l_2 can be joined by a broken line of length 2. This broken line of V_x^d defines a broken plane of length 2 in X_∞, joining l_1 and l_2, q.e.d.

Lemma 1.6. *Let l_1 and l_2 be two lines on $X_\infty \overset{s}{\subset} \mathbb{P}_\infty$ which do not intersection. Then X_∞ contains two planes π_1 and π_2 such that $\pi_i \supset l_i$ and $\pi_1 \bigcap \pi_2 \neq \varnothing$.*

Proof. Let $x \in l \subset X_\infty \overset{s}{\subset} \mathbb{P}_\infty$. The line l defines a point $l \in V_x^d \overset{s}{\subset} (\mathbb{P}TX_\infty)_x$. Consider the variety $V_l^{d!} \overset{s}{\subset} (\mathbb{P}TV_x^d)_l$ of lines of V_x^d passing through l. Let

$$C(V_l^{d!}) = \langle l, V_l^{d!} \rangle$$

be the cone, swept out by the lines of V_x^d passing through the point l. Let

$$C(C(V_l^{d!}))_x = \langle x, C(V_l^{d!}) \rangle$$

be the cone swept out by the lines passing through x and lying in $C(V_l^{d!})$. Then the variety $C_2(l) = C(C(V_l^{d!}))_x \overset{s}{\subset} X_\infty$ (that is, the variety swept out by the planes passing through l) is independent of the point $x \in l$. It is a subvariety of X_∞ or of $\mathbb{P}_\infty$ of finite codimension.

Now if l_1, l_2 is a pair of lines, the subvarieties $C_2(l_1)$ and $C_2(l_2)$ intersect according the Connectedness Principle; this proves the lemma.

Corollary 1.5. *Any pair l_1, l_2 of lines on $X_\infty \overset{s}{\subset} \mathbb{P}_\infty$ can be joined by a broken surface of length 4.*

Indeed, according to Lemma 1.6, we can find a pair of planes π_1 and π_2 with $\pi_i \supset l_i$ and $\pi_1 \bigcap \pi_2 \neq \varnothing$. Let $x \in \pi_1 \bigcap \pi_2$ and choose any lines l_1' and l_2' through x with $l_i' \subset \pi_i$. If we apply Lemma 1.5 to this pair, we get a broken plane $S_1 \bigcup S_2$ joining l_1' and l_2'; the broken plane $\pi_1 \bigcup S_1 \bigcup S_2 \bigcup \pi_2$ then joins l_1 and l_2.

CHAPTER 2
The simplest families of vector bundles over $\mathbb{P}_1$

A vector bundle E over an infinite projective variety defines a family of vector bundles over $\mathbb{P}_1$, consisting of the restriction of E to the lines of the infinite variety. This family contains elementary subfamilies, and it is the purpose of this auxiliary chapter to describe the simplest properties of these elementary families.

§ 1 Vector bundles on F_1.

In this section we will study the simplest families of vector bundles on a pencil of lines.

Let $\sigma : F_1 \longrightarrow \mathbb{P}_2$ be the blowing-up of the point $P \in \mathbb{P}_2$. Then $p : F_1 \longrightarrow \mathbb{P}_1$ is a fibration with fiber $l \cong \mathbb{P}_1$, and $\sigma^{-1}(P) = S$ is a section of this fibration with $S^2 = -1$.

We will be considering n-dimensional vector bundles E on F_1 whose restriction to S is $E|_S = I_n(\delta)$, with δ an integer (I_n denotes the sum of n copies of the structure sheaf; that is, the trivial rank n bundle). Restricting E to the fibre l_P (for $P \in \mathbb{P}_1$) we get

$$E_P = E|_{l_P} = \bigoplus_{i=1}^{m} I_{k_i(P)}(d_i(E_P)), d_1(E_P) > d_2(E_P) > \cdots > d_m(E_P).$$

For each points $P \in \mathbb{P}_1$ apart from a finite number, the invariants $d_i(E_P)$, $k_i(E_P)$ and $m(P)$ of the vector bundle E_P are constants, d_i, k_i and m.

The constants which we will need are d_1, d_m, k_1 and δ. The difference $d_1 - d_m = D$ is always nonnegative.

Lemma 2.1. $h^0(F_1, E) \leqslant \frac{n}{2} d_1(d_1 + 1) + (d_1 + 1)(\delta + 1)$.

Proof. For any k we have an exact sequence

$$0 \longrightarrow E(-(k+1)S) \longrightarrow E(-kS) \longrightarrow E(-kS)|_S \longrightarrow 0. \tag{2.1}$$

Hence, since $E(-kS)|_S = I_n(k + \delta)$, we get the inequality

$$h^0(F_1, E(-kS)) - h^0(F_1, E(-(k+1)S)) \leqslant n(\delta + k + 1).$$

Summing this inequality from k down to 0, we get

$$h^0(F_1, E) - h^0(F_1, E(-(k+1)S)) \leqslant n\left(\delta(k+1) + \frac{k(k+1)}{2} + k + 1\right).$$

But, for $k = d_1$, we have $H^0(F_1, E(-(k+1)S)) = 0$; this gives us the required inequality.

Remark. The same short exact sequence (2.1) also shows that

$$\chi(E(-kS) = \chi(E) - n\left(\frac{k(k+1)}{2} + k\delta\right). \tag{2.2}$$

Lemma 2.2. *Let E be a vector bundle on F_1 such that $d_1 < 0$. Then for every $P \in \mathbb{P}_1$ we have $d_1(E_P) \leqslant h^1(F_1, E)$.*

Proof. The adjunction exact sequence for the fiber over P gives the cohomology sequence

$$0 \longrightarrow H^0(F_1, E) \longrightarrow H^0(F_1, E(l_P)) \longrightarrow H^0(F_1, E_P) \longrightarrow H^1(F_1, E);$$

but $H^0(F_1, E) = H^0(F_1, E(l_P)) = 0$, so that

$$h^0(\mathbb{P}_1, E_P) \leqslant h^1(F_1, E).$$

But one sees easily that $d_1(E_P) \leqslant h^0(\mathbb{P}_1, E_P) - 1$, which gives the required inequality.

Lemma 2.3. *Let E be a vector bundle over F_1 for which $d_1 = -1$; then for any $P \in \mathbb{P}_1$*

$$d_1(E_P) - d_1 \leqslant \frac{n}{2}D(D-1) - n\delta(D-1) - \chi(E). \tag{2.3}$$

Proof. $h^1(F_1, E) = h^0(F_1, E) + h^0(F_1, E^* \otimes K_{F_1}) - \chi(E)$; but $H^0(F_1, E) = 0$ and $K_{F_1} = -2S - 3l$, so that $d_1(E^* \otimes K_{F_1}) = D - 1$, and $\delta(E^* \otimes K_{F_1}) = -(\delta + 1)$. Applying the estimate of Lemma 2.1 to $E^* \otimes K_{F_1}$ gives the required inequality.

Now for any vector bundle E on F_1, the bundle $E(-(d_1+1)S)$ satisfies the hypothesis of Lemma 2.3; that is,

$$d_1(E(-(d_1+1)S)) = -1, \quad \text{and} \quad \delta(E(-(d_1+1)S)) = \delta + d_1 + 1$$

and, by (2.2),

$$\chi(E(-(d_1+1)S)) = \chi(E) - \frac{n}{2}(d_1+1)(d_1+2) - \delta(d_1+1).$$

Now note that for any line bundle L on F_1 we have

$$\begin{gathered} d_1((E \otimes L)_P) - d_1(E \otimes L) = d_1(E_P) - d_1(E), \\ D(E^*) = D(E) = D(E \otimes L). \end{gathered} \tag{2.4}$$

Hence

$$d_1(E_P) - d_1 \leqslant$$
$$\leqslant \frac{n}{2}\, D(D-1) - n(\delta + d_1 + 1)(D-1) - \chi(E) + \frac{n}{2}\,(d_1+1)(d_1+2) + n\delta(d_1+1).$$

From now on we will be considering vector bundles having $\delta = 0$; that is, bundles on F_1 of the form $E = \sigma^*(E')$ for some E' over $\mathbb{P}_2$. For these bundles we have the inequality

$$\text{(2.5)}\quad d_1(E_P) - d_1 \leqslant \frac{n}{2}\, D(D-1) - n(d_1+1)(D-1) + \frac{n}{2}\,(d_1+1)(d_1+2) - \chi(E)\ .$$

for every $P \in \mathbb{P}_1$. The constant D is nonnegative, and from now on we will usually consider the case $d_1 \geqslant 1$. The function on the right-hand side of this inequality is not monotonic, and is therefore not convenient for iteration; throwing away a term which is clearly negative, we arrive at the coarser inequality

$$\text{(2.6)}\qquad d_1(E_P) - d_1 \leqslant \frac{n}{2} D(D-1) + \frac{n}{2}(d_1+1)(d_1+4) - \chi(E)\ .$$

The function $D(E_P) = d_1(E_P) + d_1(E_P^*)$ is lower semicontinuous, since both of its components are. We estimate the amount it can jump by

Lemma 2.4. *Let E be a vector bundle over F_1 with $\delta(E) = 0$ and $d_1 > 0$. Then for every point $P \in \mathbb{P}_1$*

$$\text{(2.7)}\quad D(E_P) - D \leqslant nD(D-1) + n(d_1+1)(d_1+4) + $$
$$+ \frac{n}{2}\, D(D - 2d_1 + 3) - n(D - d_1 - 2)(D-1) - 2\,\chi(E)\ .$$

Proof. $D(E_P) = d_1(E_P) + d_1(E_P^*)$. Let us apply (2.6) to $d_1(E_P)$, and (2.5) to

$$d_1(E_P) - d_1(E) = d_1((E^* \otimes K_{F_1})_P) - d_1(E^* \otimes K_{F_1})\ ,$$

by (2.4). One sees easily that $d_1(E^*(K_{F_1} - S)) = D(E) - d_1(E) - 3$, and that $\delta = 0$. Then, (2.5) gives

$$(*)\quad d_1(E_P^*) - d_1(E^*) \leqslant \frac{n}{2}\, D(D-1) - n(D - d_1 - 2)(D-1) +$$
$$+ \frac{n(D - d_1 - 2)(D - d_1 - 1)}{2} - \chi(E)\ ,$$

since $\chi(E) = \chi(E^* \otimes K_{F_1}) = \chi(E^*(K_{F_1} - S))$. Adding $(*)$ to (2.6) and adding $n(d_1+1)$, a term which is clearly positive, gives us the required inequality.

The inequality that we have obtained allows us to estimate the jump in the degree of the maximal subbundle of the restriction of E to a fiber.

Lemma 2.5. *There exists a polynomial $F(X_1, X_2, X_3)$ in 3 variables, having the property that for any vector bundle E on $\mathbb{P}_3$, whose restriction down to a general line has the constants $d_1 = d_1(E|_{l_{gen}}) > 0$ and $D = D(E|_{l_{gen}})$ and* *whose restriction to some plane $\pi \subset \mathbb{P}_3$ has $\chi(E|_\pi) = \chi_2$, and for $l_1 \subset \mathbb{P}_3$ an arbitrary line, we have*

$$d_1(E|_l) \leqslant F(d_1, D, \chi_2).$$

PROOF. We put

$$\begin{aligned} f_1(X_1, X_2, X_3) &= \frac{n}{2}\, X_2(X_2-1) + \frac{n}{2}\,(X_1+1)(X_1+4) + X_1 - X_3\,, \\ f_2(X_1, X_2, X_3) &= n\, X_2(X_2-1) + n\,(X_1+1)(X_1+4) - \\ &\quad - n\,(X_2 - X_1 - 2)(X_2 - 1) + \\ &\quad + \frac{n}{2}\, X_2(X_2 - 2X_1 + 3) - 2\, X_3 + X_2 \end{aligned}$$

and set $F(X_1, X_2, X_3) = f_1(f_1, f_2, X_3)$; let us show that this polynomial has the desired property.

Let l_0 be some line such that $d_1(E|_{l_0}) = d_1$ and $D(E|_{l_0}) = D$; let $l_1 \subset \mathbb{P}_3$ be an arbitrary line. Let l be a line meeting l_i at $\mathbb{P}_i$ (for $i = 0, 1$), and let $\pi_i = (l, l_i)$ be the plane spanned by l and l_i. Let $\sigma_i : F_1 \longrightarrow \pi_i$ be the blowing up of $\mathbb{P}_i \in \pi_i$ and let $p_i : F_1 \longrightarrow \mathbb{P}_1$ be the ruling of F_1. On F_1 consider the vector bundle $\sigma_0^*(E|_{\pi_0})$. One sees easily that $\chi(E|_\pi) = \chi(\sigma_0^*(E|_\pi))$. Then, using the estimates (2.6) and (2.7), we have

$$d_1(E\,|_l) \leqslant f_1(d_1, D, \chi_2), \quad \text{and} \quad D(E\,|_l) \leqslant f_2(d_1, D, \chi_2).$$

Now consider the bundle $\sigma_1^*(E|_{\pi_1})$. Then we have nonnegative constants d_1' and D' such that

$$d_1(E|_{l_1}) \leqslant f_1(d_1', D', \chi_2)\,.$$

These are constants of the vector bundle $(E|_{l_{\text{gen}}})$, with l_{gen} the general line of π_1 passing through $\mathbb{P}_1$. Now note that if $a_1 \geqslant a_1'$ and $a_2 \geqslant a_2'$ (and the a_i' are positive), then $f_1(a_1, a_2, \chi_2 \geqslant f_1(a_1', a_2', \chi_2)$. Hence

$$\begin{aligned} d_1(E|_{l_1}) &\leqslant f_1\,(d_1(E|_l)\,,\, D(E|_l)\,,\, \chi_2) \leqslant \\ &\leqslant f_1\,(f_1(d_1, D, \chi_2)\,,\, f_2(d_1, D, \chi_2)\,,\, \chi_2) = F\,(d_1, D, \chi_2)\ . \end{aligned}$$

q.e.d.

COROLLARY 2.1. *A vector bundle E on $\mathbb{P}_\infty$ has an integer invariant*

$$d(E) = \max_{l \subset \mathbb{P}_\infty} d_1(E|_l).$$

§ 2 Vector bundles over ruled varieties.

Let $P \xrightarrow{\pi} B$ be a locally trivial fibration over B with fiber $\mathbb{P}_1$, and let $s : B \longrightarrow P$ be a fixed section of π, so that $s(B) = S$ is a divisor in P.

Let E be a vector bundle of rank n over P. We can view E as a family $\{E_b = E|_{\pi^{-1}(b)}\}_{b \in B}$ of vector bundles over $\mathbb{P}_1$. Let

$$E_{b_{\text{gen}}} = \bigoplus_{i=1}^{m} I_{k_i}(d_i), \text{ with } d_1 > d_2 > \cdots > d_m.$$

Lemma 2.6. *Let E be a vector bundle over P such that for every $b \in B$*

$$E_b = I_{k_1(d_1)} \oplus \Big(\bigoplus_{i=2}^{m_b} I_{k_i(b)}\big(d_i(b)\big)\Big).$$

Then the following assertions are true:

1) *$R^0\pi E(-d_1 S)$ is a locally free sheaf on B (and we use the same symbol to denote the corresponding vector bundle).*

2) $\operatorname{rk} R^0\pi E(-d_1 S) = k_1$.

3) *The standard map $\gamma : \pi^* R^0\pi E(-d_1 S) \longrightarrow E(-d_1 S)$ is everywhere nondegenerate.*

Proof. 1) and 2) follow immediately from the base change theorem. For 3) note that γ cannot vanish on an entire fiber of π, since

$$R^0\pi E(-d_1 S) \xrightarrow{R^0\gamma} R^0\pi E(-d_1 S)$$

is an isomorphism; by the base change theorem once more the fiber of $R^1\pi E(-d_1 S)$ at b is

$$H^0(\pi^{-1}b, E_b(-d_1)) ,$$

and the restriction of γ to the fiber over b is the standard map

$$\gamma_b : H^0(\mathbb{P}_1, E_b(-d_1)) \otimes O_P \longrightarrow E_b(-d_1) .$$

Since by hypothesis $E_b(-d_1)$ has no factors of positive degree, its sections are nowhere vanishing, so that the restriction γ_b is everywhere nondegenerate. This proves 3).

Lemma 2.7. *Let E be a vector bundle of rank n on P such that, for every $b \in B$, $H^1(\mathbb{P}_1, E_b) = 0$. Let $b_0 \in B$ be a point such that E_{b_0} contains as a factor a line bundle of degree $D \geqslant 0$. Let $B_D \ni b_0$ be the subvariety defined in a neighborhood of b_0 as the locus of points b such that E_b contains a line bundle of degree $\geqslant D$. Then*

$$\operatorname{codim}_B B_D \leqslant D(n-1).$$

Proof. By hypothesis $R^0\pi E$ is a locally free sheaf on B whose fibers can be identified with $H^0(\mathbb{P}_1, E_b)$. We can assume that the base B is affine. Let s_0

be a section of E_{b_0} having D transversal zeros, s_0 defining a subbundle of degree D. Then we can find a section s of E over P such that $s_0 = s|_{\pi^{-1}(b_0)}$. Let Δ_s be the subvariety of zeros of s. Then $\mathrm{codim}_P \Delta_s \leqslant n$, and Δ_s meets $\pi^{-1}(b_0)$ transversally in D points $\mathbb{P}_1, \cdots, \mathbb{P}_D$. We consider an analytic neighborhood U_0 of b_0 in B. Let $\Delta_{\mathbb{P}_i}$ be the branch of the subvariety Δ_s at $\mathbb{P}_i$ in $\pi^{-1}(U_0)$. Then for sufficiently small (U_0) the fiber over any point $b \in \bigcap_{i=1}^{D} \pi(\Delta_{\mathbb{P}_i})$ intersects Δ in D points, and hence

$$B_D \supseteq \bigcap_{i=1}^{N} \pi(\Delta_{\mathbb{P}_i}).$$

Hence, the lemma follows.

Let E be a vector bundle of rank n over $P \xrightarrow{\pi} B$. We put

$$d'(E) = \max_{b\in B} d_1(E_b^*),\ d(E) = \max_{b\in B} d_1(E_b),\ \text{and } D(E) = d(E) + d'(E)\ .$$

Lemma 2.8. *Let $B_d = \{b \in B, \quad d_1(E_b) = d(E)\}$. Then*

$$\mathrm{codim}_B B_D \leqslant (n-1)D(E),$$

provided that B_D is nonempty.

Proof. $E(d'(E)S)$ satisfies the conditions of the previous lemma; setting $D = D(E)$, we get the required assertion.

Lemma 2.9. *Let E be such that $d_1(E_b) = d(E)$ is constant. Let $B_m = \{b \in B,\ k_1(E_b) = k = \max_{b\in B} k_1(E_b)\}$. Then*

$$\mathrm{codim}_B B_m \leqslant k_1(n-1)D(E).$$

Proof. Again we can suppose that $d'(E) = 0$. Let $b \in B_m$. Then there exist k linearly independent sections $s_1, \cdots, s_k$ of E_{b_0} each of which has $D(E)$ zeros. Each of them extends to a section $\bar{s_i}$ of E over $\pi^{-1}(U)$for some neighborhood U of b_0. For U sufficiently small the restrictions of the $\bar{s_i}$ to the fiber over $b \in U$ are still linearly independent. Let $\mathbb{P}_i^j$ be the zeros of the section $\bar{s_j}$, and let $\Delta_{\mathbb{P}_i^j}$ be the branch of the variety of zeros of the $\bar{s_j}$ around $\mathbb{P}_i^j$. Then

$$B_m \supseteq \bigcap_{i=1}^{N} \bigcap_{j=1}^{k_1} \pi(\Delta_{\mathbb{P}_i^j}),$$

which gives the required assertion.

CHAPTER 3
Vector bundles of finite rank over infinite varieties

In this chapter we will prove several theorems on the structure of vector bundles of finite rank over infinite projective and quasi-projective varieties.

THEOREM 1. *Any rank n bundle over $\mathbb{P}_\infty$ is a direct sum of line bundles.*

The equivalent "finite" assertion is

THEOREM 1'. *An absolute vector bundle of rank n over $\mathbb{P}_k$ is a direct sum of line bundles.*

THEOREM 2. *Let $X_\infty \overset{s}{\subset} \mathbb{P}_\infty$ be a nonsingular infinite projective variety, and let E be a vector bundle on X_∞ of rank n. Then E is a direct sum of line bundles.*

The equivalent "finite" assertion is

THEOREM 2'. *An absolute rank n bundle E on an absolute projective variety is a direct sum of line bundles.*

THEOREM 3. *Let $X_\infty \overset{s}{\subset} \mathbb{P}_{w_\infty}$ be nonsingular, and let E be a vector bundle on X_∞ of rank n. Then E is a direct sum of line bundles.*

The proof is based on the old result of Grothendieck's that every vector bundle on $\mathbb{P}_1$ is absolute, and is a direct sum of line bundles, and on a series of standard arguments of van de Ven and Barth ([2], [7]).

§ 1 Vector bundles on $\mathbb{P}_\infty$.

We have seen that any infinite variety $X_\infty \overset{s}{\subset} \mathbb{P}_\infty$ is swept out by its lines.

Definition 3.1. A vector bundle E on $X_\infty \overset{s}{\subset} \mathbb{P}_\infty$ is said to be *linearly trivial* if when restricted down to any line $l \subset X_\infty$ it becomes trivial: $E|_l = I_n$.

Proposition 3.1 (van de Ven [7]). *A linearly trivial bundle on $\mathbb{P}_n$ is trivial.*

PROOF. Let us blow up some point $\mathbb{P}_0 \in \mathbb{P}_n$: $\widetilde{\mathbb{P}_n} \xrightarrow{\sigma} \mathbb{P}_n$. Then $\widetilde{\mathbb{P}_n}$ can be fibered in lines $\widetilde{\mathbb{P}_n} \xrightarrow{p} \mathbb{P}_{n-1} = (\mathbb{P}T\mathbb{P}_n)_{\mathbb{P}_0}$ and $\sigma^{-1}(\mathbb{P}_0) = \mathbb{P}_{n-1}$ is a section of p. Consider the natural map associated to p:

$$p^*(R^0p(\sigma^* E)) \xrightarrow{\gamma} \sigma^*(E) \; .$$

Restrict this isomorphism onto the section $\sigma^{-1}(\mathbb{P}_0)$; this gives us an isomorphism

$$\gamma|_{\sigma^{-1}(\mathbb{P}_0)} \; : \; R^0p(\sigma^*(E)) \xrightarrow{\sim} I_n \; ,$$

and hence an isomorphism $I_n \xrightarrow{\gamma} \sigma^* E$. Taking the image of this isomorphism under the direct image functor , we get a morphism $I_n \xrightarrow{R^0\gamma} E$ which can only degenerate at $\mathbb{P}_0$. But then it must be an isomorphism everywhere.

COROLLARY. *A linearly trivial vector bundle on $\mathbb{P}_\infty$ is trivial.*

Definition 3.2. A vector bundle E on $X_\infty \overset{s}{\subset} \mathbb{P}_\infty$ is sad to be *level* if for any two lines l_1 and l_2 of X_∞ we have $E|_{l_1} \cong E|_{l_2}$.

Proposition 3.2. *A level vector bundle on $\mathbb{P}_\infty$ is a direct sum of line bundles.*

Suppose that for $l \subset \mathbb{P}_\infty$ we have

$$E|_l = \bigoplus_{i=1}^{m} I_{k_i}(d_i), \text{ with } d_1 > d_2 > \cdots > d_m.$$

We have to establish such a decomposition on the whole of $\mathbb{P}_\infty$. We first construct a subbundle $E_1 \subset E$ such that

$$E_1|_l = I_{k_1}(d_1) \ \text{ and } \ (E/E_1)|_l = \bigoplus_{i=1}^{m} I_{k_i}(d_i);$$

that is, we will separate off inside E a subbundle which will be the maximal subbundle of each restriction to a line.

To construct a rank k_1 subbundle of E is the same thing as to construct a section of the Grassmanisation[4] $G_{k_1}(E)$, that is, to assign to each point $P \in \mathbb{P}_\infty$ a k_1-dimensional linear subspace of the fiber E_P. Let l be a line passing through P. Then from the pair (l, P) we can construct the subspace $I_{k_i}(d_i)_P \subset (E|_l)_P$ — the fiber of the maximal subbundle of $E|_l$. This subspace is independent of the particular line l; indeed, the regular map $\mathbb{P}_\infty = (\mathbb{P}T\mathbb{P}_\infty)_P \xrightarrow{\varphi} G(k_1, n) = G_k(E)_P$ is constant (by the Rigidity Principle of § 1.1).

This gives us the construction of a subbundle $E_1 \subset E$, with $E_1(-d_1)$ linearly trivial. Thus (by Proposition 3.1) $E_1 = I_{k_1}(d_1)$ over $\mathbb{P}_\infty$.

By induction on n we can assume that $E|_{E_1}$ is a direct sum of line bundles, and the required assertion then follows from the fact that any short exact sequence of bundles over $\mathbb{P}_\infty$ of the form

$$0 \longrightarrow I(d) \longrightarrow E \longrightarrow I(d') \longrightarrow 0\,,$$

is split, which follows since $H^1(\mathbb{P}_\infty, I(d - d')) = 0$. This same proposition was proved by van de Ven for rank 2 vector bundles [7].

For the proof of Theorem 1 in the general case the construction is the same, but instead of all the lines of $\mathbb{P}_\infty$, the construction uses only a certain (sufficiently large) part of them.

PROOF OF THEOREM 1. Let E a rank n vector bundle on $\mathbb{P}_\infty$. According to Corollary 2.1, there exist constants

$$d = \max_{l \in \mathbb{P}_\infty} d_1(E|_l), \ D = \max_{l \subset \mathbb{P}_\infty} d_1(E|_l) + \max_{l \subset \mathbb{P}_\infty} d_1(E^*|_l), \ k = \max_{\substack{l \subset \mathbb{P}_\infty \\ d_1(F|_l) = d}} k_1(F|_l)\ .$$

[4] for the definition, see [6]

Definition 3.3. $l \subset \mathbb{P}_\infty$ is said to be *exceptional* for E if

$$d_1(E|_l) = d, \quad k_1(F|_l) = k.$$

Let $(\mathbb{P}T\mathbb{P}_\infty)_P$ be the set of lines of $\mathbb{P}_\infty$ through P, and let D_P be the subvariety of lines through P and exceptional for E.

Lemma 3.1. *$D_P \overset{s}{\subset} \mathbb{P}_\infty = (\mathbb{P}T\mathbb{P}_\infty)_P$ is an infinite projective variety, and*

$$\operatorname{codim}_{\mathbb{P}_\infty} D_P \leqslant n(n-1)D.$$

Proof. P lies in some $\mathbb{P}_N$ for sufficiently large N. Let $G(1,N)$ be the Grassmannian of lines of $\mathbb{P}_N$. We can assume that $\mathbb{P}_N$ already contains some line l for which $d_1(E|_l) = d$ and $k_1(F|_l) = k$. Let $G_{d,k}(1,N)$ be the subvariety of lines satisfying this property. Then Lemmas 2.8 and 2.9 give the inequality

$$\operatorname{codim}_{G_(1,N)} G_{d,k}(1,N) \leqslant n(n-1)D.$$

It follows that for N large enough the intersection of $G_{d,k}$ with the $\mathbb{P}_{N-1}$ parameterising the lines through P is nonempty. The lemma follows.

We now carry through a general construction. Let l be a line through P exceptional for E. Then to the pair (l,P) we we can associate the subspace $I_k(d)_P \subset (E|_l)_P$, the fiber at P of the maximal subbundle of

$$E|_l = I_k(d) \oplus \left(\overset{m}{\underset{i=2}{\oplus}} I_{k_i}(d_i) \right)$$

with $d > d_2 > \cdots > d_m$. This subspace is independent of l. Indeed, the regular map

$$D_P \xrightarrow{\psi} G(k_1, n) = G_k(E)_P$$

is constant according to the Rigidity Principle and Lemma 3.1. Thus we have constructed a section of the Grassmannisation $G_k(E)$; that is, a subbundle $E_1 \subset E$. Carrying out an induction on the rank of E, we get a proof of Theorem 1.

§ 2 Vector bundles on infinite projective varieties.

Proof of Theorem 2. We carry out an induction on $\operatorname{rk} E = n$.

Lemma 3.2. *Any vector bundle on $X_\infty \overset{s}{\subset} \mathbb{P}_\infty$ is level* (see Definition 3.2.).

Proof. Let π be an arbitrary plane lying on $X_\infty \overset{s}{\subset} \mathbb{P}_\infty$. Then $E|_\pi = \bigoplus_{i=1}^{m} I_{k_i}(d_i)$, since $\pi \subset \mathbb{P}_\infty \subset X_\infty$ (by Lemma 1.3), and, according to Theorem 1, $E|_{\mathbb{P}_\infty} = \bigoplus_{i=1}^{m} I_{k_i}(d_i)$ (with $d_i > d_{i+1}$). Obviously, if $S = \bigcup_{i=1}^{N} S_i$ is a broken plane, then the restriction of E onto each component is the same. Hence the restriction of E onto all lines which lie on a broken plane is the same. But according to Corollary 1.5 each pair of lines can be joined by a broken plane in X_∞. This proves the lemma.

Lemma 3.3. *Let E be a rank n vector bundle on $X_\infty \overset{s}{\subset} \mathbb{P}_\infty$, and suppose that $E|_l = \bigoplus_{i=1}^{m} I_{k_i}(d_i)$, with $d_1 > \cdots > d_m$. Then there exists a subbundle $E_1 \subset E$ such that $E_1|_l = I_{k_i}(d_i)$ and $E/E_1 = \bigoplus_{i=2}^{m} I_{k_i}(d_i)$ for every line $l \subset X_\infty \overset{s}{\subset} \mathbb{P}_\infty$.*

Proof. We again apply our standard construction. Let $x \in X_\infty$, and let l be a line of X_∞ through x. To the pair (x, l) we associate the fiber $(I_{k_1}(d_1))_x \subset (E|_l)_x$. For a fixed x this subspace is independent of l, since the regular map of the infinite projective variety V_x^d into the Grassmannian $G(k_1, n)$ is constant by the Rigidity Principle; q.e.d.

Thus any vector bundle E on $X_\infty \overset{s}{\subset} \mathbb{P}_\infty$ can be represented as an extension of linearly trivial bundles twisted by line bundles.

Proposition 1.4 reduces the general case to that of linearly trivial bundles.

Lemma 3.4. *Let E be a rank n bundle on a nonsingular $X_\infty \overset{s}{\subset} \mathbb{P}_\infty$, and suppose that $E|_l = I_n$. Then E is the trivial vector bundle.*

Proof. The proof is analogous to the proof of Proposition 3.1, with the only difference that instead of considering the line joining x_0 to x we will consider broken lines of length 2.

Consider a point $x_0 \in X_\infty \overset{s}{\subset} \mathbb{P}_\infty$ at which the variety $V_{x_0}^d$ is nonsingular. One sees easily that such points do exist. Blow up this point on X_∞, giving $\sigma : \widetilde{X}_\infty \longrightarrow X_\infty$, with σ everywhere an isomorphism except for $x_0 \in X_\infty$, and $\sigma^{-1}(x_0) = \mathbb{P}_\infty = (\mathbb{P}TX_\infty)_{x_0} \subset \widetilde{X}_\infty$. Then the inverse image $\widetilde{C}(V_{x_0}^d) \subset \widetilde{X}_\infty$ is a nonsingular subvariety. It is a ruled variety $p : \widetilde{C}(V_{x_0}^d) \longrightarrow V_{x_0}^d$, with fibers lines $p^{-1}(l) = \mathbb{P}_1$. Consider the inverse image $\sigma^*(E) = \widetilde{E}$ on $\widetilde{X}_\infty$ under the map $\sigma : \widetilde{X}_\infty \longrightarrow X_\infty$, and its restriction on $\widetilde{C}(V_{x_0}^d)$.

Lemma 3.5. $\widetilde{E}|_{\widetilde{C}(V_{x_0}^d)}$ *is trivial.*

Proof. For every fiber $p^{-1}(l)$ we have $\sigma^*(E)|_{p^{-1}(l)} = E|_l = I_n$. Hence the standard map

$$\gamma : p^* R^0 p \widetilde{E}|_{\widetilde{C}} \longrightarrow \widetilde{E}|_{\widetilde{C}}$$

is an isomorphism (we have written $\widetilde{C} = \widetilde{C}(V_{x_0}^d)$). The fibration p has the section $s : V^d_{x_0} \longrightarrow \widetilde{C}(V^d_{x_0})$ given by the intersection of $\widetilde{C}(V^d_{x_0})$ and $\sigma^{-1}(x_0)$ in $\widetilde{X}$. Restrict γ to this section:

$$\gamma|_{s(V^d_{x_0})} : R^0 p\widetilde{E}|_{\widetilde{C}} \longrightarrow \widetilde{E}|_{s(V^d_{x_0})};$$

but the restriction $\sigma^*(E)|_{s(V^d_{x_0})}$ is trivial. Hence $R^0 p\widetilde{E}|_{\widetilde{C}}$ is trivial, and so $\widetilde{E}|_{\widetilde{C}}$ also; q.e.d.

Consider now the following diagram of varieties and maps:

$$\begin{tikzcd} & P\mathcal{V} \arrow[dl, "\sigma"'] \arrow[d, "p"] \\ \widetilde{X}_\infty \arrow[d, equal] & \mathcal{V} \arrow[d, "p_1"] \\ \widetilde{X}_\infty & \supset \widetilde{C}(V^d_{x_0}) \end{tikzcd} \qquad \begin{array}{ll} \text{the fiber} & p^{-1}(v) = \mathbb{P}_1 \\ \text{the fiber} & p_1^{-1}(c) = V^d_c \end{array}$$

where

(1) $\mathcal{V}$ is the variety of pairs (l, c), with l a line meeting $C(V^d_{x_0})$ and c the point of intersection. The map p_1 projects the pair (l, c) to $c \in \widetilde{C}(V^d_{x_0})$. The fiber $p_1^{-1}(c) = V^d_c$ is the variety V^d_c of lines of X_∞ through c.

(2) $P\mathcal{V}$ is the variety of pairs $\{x, l \ni c\}$, $l \ni c$ being a point of $\mathcal{V}$, and x a point on the line l. p is the projection onto $l \ni c$. This has a section $s : \mathcal{V} \longrightarrow P\mathcal{V}$ consisting of pairs $\{c, l \ni c\}$, for which $x = c$. The map σ is the projection to the point x.

Over a point x, the fiber of σ is the variety of broken lines of length 2 joining x and x_0. If $x \notin C(V^d_{x_0})$, then, as we have already seen (Corollary 1.4), the fiber $\sigma^{-1}(x) = C(V^d_{x_0}) \bigcap C(V^d_x)$ is an infinite projective variety, and is therefore connected by the Connectedness Principle. Hence to prove the triviality of $\widetilde{E}$ it is enough to show the triviality of $\sigma^*(\widetilde{E})$, since $R^0\sigma(\sigma^*(\widetilde{E})) = \widetilde{E}$, and the fact that $\widetilde{E}$ is trivial is enough to give the triviality of E on X_∞ (see the conclusion of the proof of Proposition 3.1).

Since $\sigma^*(\widetilde{E})|_{p_1^{-1}(l \ni c)} = E|_l = I_n$, the standard map $p^* R^0 p\sigma^*(\widetilde{E}) \xrightarrow{\gamma} \sigma^*(\widetilde{E})$ is everywhere an isomorphism. Restrict this isomorphism onto the section $s(\mathcal{V})$:

$$\gamma|_{s(V)} : R^0 p\sigma^*(\widetilde{E}) \longrightarrow \sigma^*(\widetilde{E})|_{s(\mathcal{V})},$$

but the restriction of $\sigma^*(\widetilde{E})$ to $s(\mathcal{V})$ is nothing other than the inverse image under p_1 of the restriction of $\widetilde{E}$ onto $\widetilde{C}(V^d_{x_0}) = \widetilde{C}$, i.e. $\sigma^*(\widetilde{E})|_{s(\mathcal{V})} = p_1^*(\widetilde{E}|_{\widetilde{C}})$. According to lemma 3.5, this vector bundle is trivial. Hence $\sigma^*(\widetilde{E})|_{s(\mathcal{V})} = I_n$, and the isomorphism $\gamma|_{s(\mathcal{V})}$ trivializes $R^0 p\sigma^*(\widetilde{E})$; but then the isomorphism γ trivializes $\sigma^*(\widetilde{E})$.

This proves the lemma, and hence also Theorem 2.

Proof of Theorem 3. Let $X_\infty \overset{s}{\subset} \mathbb{P}_\infty$, and let E be a vector bundle on X_∞ of rank n. Let us return to the covering (1.5). Then $\varphi^*_{\omega_\infty} E$ is a direct sum of line bundles. E is contained as a direct summand in the vector bundle

$$R^0\varphi_{\omega_\infty}(\varphi^*_{\omega_\infty} E) = \overset{n}{\underset{i=1}{\oplus}} O(d_i) R^0\varphi_\omega O_{X_{[\omega]}} = \Big(\overset{n}{\underset{i=1}{\oplus}} O(d_i)\Big) \otimes \Big(\overset{n}{\underset{i=1}{\oplus}} O(m_i)\Big)$$

according (1.6) and (1.7). Using the uniqueness of the decomposition of a vector bundle into a direct sum, we get that E is a direct sum of line bundles, q.e.d.

Modulo Reid's conjecture we get a complete description of vector bundles of finite rank on an arbitrary infinite variety.

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[5] Added by translator.

Symplectic structures on the varieties of moduli of vector bundles on algebraic surfaces with $p_g > 0$

The author studies the geometry and periods of components of moduli of sheaves on a regular algebraic surface.

Math. USSR Izvestiya, 1989, 33:1.

Introduction.

The subject and the title of this paper illustrate Arnold's paraphrase of Plutarch:

> ...just as every skylark must display its crest, so every area of mathematics will ultimately become symplecticized
>
> ([1], final paragraph of Chapter 14)

I dedicate this paper to Vladimir Igorevich Arnold on the occasion of his fiftieth birthday.

In algebraic geometry, the notion of a symplectic structure on a smooth algebraic variety has to be carefully defined, unless we want to restrict ourselves to a rather special, although very interesting, class of varieties similar to K3 surfaces.

DEFINITION 0.1. By an *algebraic symplectic structure* on a smooth algebraic variety B we understand any nonzero skew-symmetric homomorphism

$$TB \xrightarrow{\omega} T^*B = \Omega B \ : \quad \omega^* = -\omega \ , \tag{0.1}$$

of the tangent sheaf into the cotangent sheaf. An algebraic symplectic structure is called *nondegenerate*, if ω is an isomorphism at the generic point, and *everywhere nondegenerate*, if ω is an isomorphism.

Any holomorphic 2-form on a regular algebraic surface defines a nondegenerate symplectic structure, and if it is everywhere nondegenerate the surface is K3.

DEFINITION 0.2. By a *Poisson structure* on B we understand any nonzero skew-symmetric homomorphism

$$\alpha : T^*B \longrightarrow TB = \Omega B, \quad \alpha^* = -\alpha, \tag{0.1'}$$

with the same non-degeneracy specifications as in Definition 0.1.

The existence of a Poisson structure on a regular algebraic surface implies the rationality and "almost" minimality of the surface.

For a smooth B, one can describe the geometry of a sheaf F in a simple way analogous to the construction of the Jacobian of a curve: every sheaf has

1) discrete invariants: the class of the sheaf in a lattice,

2) continuous invariants: the component $M(F)$ of the variety of moduli, containing F, and

3) operations $F \rightsquigarrow F'$ inducing the equality $M(F) = M(F')$.

1. The discrete invariants of the sheaf. These are analogues of the degree of a divisor on a curve: R_m, which is a class in the big lattice $\{F\}_m \in V_Z(B)$, and the Mukai vector in the Mukai lattice $v(F) \in M(B)$ are described in§ 1 and § 2 of Chapter 1.

2. The component $M(F)$ of the variety of moduli. This is an analogue of the Jacobian of a curve. The existence problem for $M(F)$ is nontrivial (see § 1 of Chapter 1). "The theory of periods" describes connections between the Hodge structures of the base B and $M(F)$. In the compact modular case (see Definition 1.1.4, 3), the period matrix is replaced by the lattices of transcendental cycles $T(B)$ and $T(M(F))$ and the Mukai structure (see Definition 1.2.1, 2). The Mukai correspondence (1.2.28), (1.2.29) gives rise to a homomorphism of the Mukai structures (1.2.30). In particular, we have a homomorphism $\tau : H^{2,0}(B) \longrightarrow H^{2,0}(M(F))$ which assigns to each symplectic structure on B a symplectic structure on $M(F)$.

In§ 3 of Chapter 1, for a regular surfaces S we give geometric constructions of a symplectic structure on $M(F)$ based on a symplectic structure on S (Theorem 1.3.1), and a Poisson structure on $M(F)$ based on a Poisson structure on S (Theorem 1.3.1'). We also construct the local invariant of the sheaf F on S (Definition 1.3.1), which has no analogue for curves.

3. Modular operations. These are analogues of tensoring by an invertible sheaf and passing to the conjugate. In the curve case they establish isomorphisms between all the components of the Picard scheme. In the case of a regular surface the set of those operations is much richer, and we devote all of Chapter 2 to their description.

In§ 1 of that chapter we describe special sheaves on S with moduli varieties isomorphic to symmetric powers of S and relative Picard schemes of linear systems of curves on S. In § 2 we define and investigate the universal extension operation. With the aid of this operation we construct an infinite series of varieties of moduli of bundles on S, which are birationally equivalent to symmetric powers of S if $\rho(S) \geqslant 2$ (Theorem 2.2.2 and (2.2.25)). In particular, we obtain an infinite series of varieties of moduli of bundles birationally equivalent to S itself (see the remark after (2.2.25)). In§ 3 of Chapter 2 we investigate the universal division operation.

In Chapter 3 we discuss the principal difference between classification theories of vector bundles on surfaces with symplectic and Poisson structures.

Notational conventions. In our terminology:

Schemes are algebraic schemes over algebraically closed field k ($k = \mathbb{C}$ for clarity). Varieties are reduced and irreducible schemes. Points are closed points. Bundles are locally free sheaves. The symbol K_S denotes both the canonical class of divisors (in the additive notation) and the corresponding invertible sheaf (in the multiplicative notation).

B is the base variety in the general case, S in the surface case, and C in the curve case.

$\langle E|$ denotes the functor $\mathrm{Ext}^0_{\mathcal{O}_B}(E,*)$;

$|E\rangle$ denotes the functor $\mathrm{Ext}^0_{\mathcal{O}_B}(*,E)$, ${}^i\langle E\,|\,F\rangle$ denotes $\mathrm{Ext}^i_{\mathcal{O}_B}(E,F)$.

The diagrams

$$ {}^i\langle E\,|\,F\rangle \mathrel{\substack{\dashleftarrow \\ \longrightarrow}} {}^i\langle E'\,|\,F\rangle \tag{0.2} $$

$$ {}^i\langle E\,|\,F\rangle \mathrel{\substack{\dashrightarrow \\ \longrightarrow}} {}^i\langle E\,|\,F'\rangle \tag{0.2'} $$

denote, respectively, the "solid" homomorphisms

$$ \begin{aligned} \mathrm{Ext}^i_{\mathcal{O}_B}(E,F) &\longrightarrow \mathrm{Ext}^i_{\mathcal{O}_B}(E',F) \\ \mathrm{Ext}^i_{\mathcal{O}_B}(E,F) &\longrightarrow \mathrm{Ext}^i_{\mathcal{O}_B}(E,F') \end{aligned} $$

induced by a "broken" homomorphisms $E' \dashrightarrow E$ and $F \dashrightarrow F'$.

The symbols $\overset{(1.2.3)}{=\!=\!=}$ and $\overset{\mathrm{SD}}{=\!=\!=}$ mean "equals by virtue of formula (1.2.3)" and "by virtue of the Serre duality".

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CHAPTER 1
Symplectic structure

§ 1 The big lattice and hierarchy of moduli.

Let B be an irreducible, smooth and complete algebraic variety of dimension b. Our first attempt to describe the topology of the set of coherent sheaves on B is based on the following definition.

DEFINITION 1.1.1. Two sheaves F_1 and F_2 on B are called *close to each other* if there exist irreducible smooth scheme M, a coherent sheaf $\mathcal{F}$ on $B \times M$, flat over M, and points m_1, $m_2 \in M$ such that

$$\mathcal{F}|_{(m_i \times B)} = F_i. \tag{1.1.1}$$

The sheaf $\mathcal{F}$ on $B \times M$ is called a *flat family with base M*.

The chains of close sheaves generate an equivalence relation R. The class of F in this relation is denoted by $\{F\}$. Since the set of extensions

$$0 \longrightarrow F_1 \longrightarrow F \longrightarrow F_2 \longrightarrow 0$$

can be endowed with the structure of a flat family with base ${}^1\langle E \mid F \rangle$, we have

$$\{F\} = \{F_1 \oplus F_2\}. \tag{1.1.2}$$

The set $F(B)$ of classes $\{F\}$ of coherent sheaves on B is a semigroup with respect to the operation

$$\{F_1\} \oplus \{F_2\} = \{F_1 \oplus F_2\}, \tag{1.1.3}$$

and $F(B)$ generates a $\mathbb{Z}$-module K^0_{alg}. This definition is justified because, by virtue of (1.1.2), one can define an epimorphism

$$K^0(B) \xrightarrow{r} K^0_{\mathrm{alg}}(B) \tag{1.1.4}$$

of $\mathbb{Z}$-modules, where $K^0(B)$ is the Grothendieck group of B, and the Chern character gives rise commutative diagram

$$\begin{tikzcd}
K^0(B) \arrow[r, "r"] \arrow[d] & K^0_{\mathrm{alg}}(B) \arrow[d] \\
\tilde{R}(B) \arrow[r, "r"] \arrow[d, equal] & \tilde{A}(B) \arrow[d, equal] \\
\bigoplus_{\imath=0}^{b} R^\imath(B) & \bigoplus_{\imath=0}^{b} A^\imath(B)
\end{tikzcd} \tag{1.1.5}$$

where $R^\imath(B)$ $[A^\imath(B)]$ is the group of codimension $\imath$ cycles on B modulo rational [algebraic] equivalence. On the $\mathbb{Z}$-module $K^0_{\mathrm{alg}}(B)$ one can define an integral bilinear form

$$-\chi(F_1, F_2) = \sum_{\imath=0}^{b} (-1)^{\imath+1} \operatorname{rk} {}^i\big\langle F_2 \,\big|\, F_1 \big\rangle. \tag{1.1.6}$$

The group Pic B acts on $K^0_{\mathrm{alg}}(B)$: if $L \in \operatorname{Pic} B$, then

$$T_L(\{F\}) = \{F \otimes L\}, \tag{1.1.7}$$

and, since ${}^i\big\langle F_1 \,\big|\, F_2 \big\rangle = {}^i\big\langle F_1 \otimes L \,\big|\, F_2 \otimes L \big\rangle$, the operator T_L preserves the bilinear form $-\chi(1.1.6)$. Furthermore the operators $(\mathrm{Id} - T_L)$ are nilpotent:

$$(\mathrm{Id} - T_L)^N = 0 \tag{1.1.8}$$

(see [9], 8.7).

Since for a smooth B the Grothendieck group $K^0(B)$ is generated by bundles, one can define the "star" operator on $K^0_{\mathrm{alg}}(B)$: if E is a bundle, then

$$\{E\}^* = \{E^*\}, \tag{1.1.9}$$

where $E^* = \mathrm{Hom}(E, \mathcal{O}_B)$.

A subtler equivalence between sheaves is given by the following

Definition 1.1.2. Two sheaves F_1 and F_2 on B are called *modularly close* if there exist an irreducible smooth scheme M, a flat family of sheaves $\mathcal{F}$ on $B \times M$, and two points $m_1, m_2 \in M$ such that (1.1.1) holds, and for both points m_1 and m_2 the Kodaira–Spencer homomorphism

$$TM_{m_\imath} \xrightarrow{k} {}^1\big\langle F_\imath \,\big|\, F_\imath \big\rangle \tag{1.1.10}$$

is a monomorphism.

The chains of modularly close sheaves generate an equivalence relation R_m. The R_m-equivalence class of F will be denoted by $\{F\}_m$.

It is easy to see that

$$F_1 \overset{R_m}{\sim} F_2, \quad F_1' \overset{R_m}{\sim} F_2' \Rightarrow F_1 \oplus F_1' \overset{R_m}{\sim} F_2 \oplus F_2'.$$

Therefore the set $F_m(B)$ of the R_m-equivalence classes of sheaves on B is a semigroup with respect to the operation of direct sum, and $F_m(B)$ generates a $\mathbb{Z}$-module $V_Z(B)$, related to $K^0_{\mathrm{alg}}(B)$ via the epimorphism

$$V_Z(B) \xrightarrow{r_m} K^0_{\mathrm{alg}}(B). \tag{1.1.11}$$

This epimorphism induces a form $-\chi$ (1.1.6) on $V_Z(B)$, which is denoted by the old symbol.

Definition 1.1.3. The $\mathbb{Z}$-module $V_Z(B)$ with the integral bilinear form $-\chi$ (1.1.6) will be called the *big lattice* of the variety B.

The group Pic B acts on the big lattice $V_Z(B)$ by $-\chi$-isometries (1.1.7).

According to their variational properties, the sheaves on a complete smooth variety B form a complicated hierarchy, from which we select only three levels.

Definition 1.1.4.

1) A sheaf F on B is called *simple* if

$$^0\langle F \mid F\rangle = \mathbb{C}.$$

2) F is called *modular* if there exist a smooth scheme M, a point $m \in M$, and a flat family $\mathcal{F}$ on $B \times M$ such that

   a)

   $$F = \mathcal{F}|_{B\times m}; \tag{1.1.12}$$

   b) the Kodaira– Spencer homomorphism

   $$TM_m \xrightarrow{k} {}^1\langle F \mid F\rangle \tag{1.1.13}$$

   is an isomorphism.

3) F is called *compact modular* if there exist a complete smooth variety M, a point $m \in M$, and a flat family $\mathcal{F}$ on $B \times M$ such that (1.1.12) holds, for any points $m \in M$ the homomorphism (1.1.13) is an isomorphism, and for any two points $m_1, m_2 \in M$

$$\mathcal{F}|_{B\times m_1} \neq \mathcal{F}|_{B\times m_2}. \tag{1.1.14}$$

For simple sheaves one has

Theorem (Altman and Kleiman [2]). *For any simple sheaf F on a complete smooth B there exists a coarse moduli scheme* $\mathrm{Spl}(F_0)$ *of simple sheaves.*

If $[F] \in \mathrm{Spl}(F_0)$ *is the point corresponding to the sheaf* F*, then the Zariski tangent space at this point is of the form*

$$T_Z\mathrm{Spl}(F_0)_{[F]} = {}^1\langle F \mid F\rangle. \tag{1.1.15}$$

Examples show that the scheme $\mathrm{Spl}(F_0)$ need not be separated or reduced. However if F is a simple modular sheaf, then for the family from Definition 1.1.4, 2), the classifying morphism

$$M \xrightarrow{f} \mathrm{Spl}(F) \tag{1.1.16}$$

identifies an analytic neighborhood of $m \in M$ with an analytic neighborhood of $[F]$ in $\mathrm{Spl}(F)$ for each point $[F']$ of which

$$TM_{[F']} \overset{(1.1.13)}{=\!=\!=} {}^1\langle F' \mid F'\rangle \overset{(1.1.15)}{=\!=\!=} T_Z\mathrm{Spl}(F)_{[F']}.$$

Thus if $\mathrm{Spl}(F_0)$ contains a point $[F]$ corresponding to a simple modular sheaf, then $\mathrm{Spl}(F_0)$ is reduced.

Definition 1.1.5. If the scheme $\mathrm{Spl}(F_0)$ is of dimension s and reduced, we set

$$\begin{gathered} \mathrm{Sing}\,\mathrm{Spl}(F_0) = \{[F] \in \mathrm{Spl}(F_0) | \mathrm{rk}\,{}^1\langle F \mid F\rangle > s\}, \\ \mathrm{Spl}_0(F_0) = \mathrm{Spl}(F_0) - \mathrm{Sing}\,\mathrm{Spl}(F_0). \end{gathered} \tag{1.1.17}$$

Thus, for every simple modular sheaf F on B, $[F] \in \mathrm{Spl}_0(F)$.

At present we have only one criterion of modularity in terms of the cohomology of the sheaf itself.

The Mukai–Artamkin Criterion. *If* F *is a simple sheaf on a complete smooth* B *and the natural homomorphism*

$$j_F : H^0(K_B) \longrightarrow {}^0\langle F \mid F \otimes K_B\rangle \tag{1.1.18}$$

is an epimorphism, then F *is modular.*

This criterion was proved by Mukai for surfaces [11], and by Artamkin for an arbitrary smooth base [3].

Serre duality along with this criterion shows that on abelian, Del Pezzo, and K3 surfaces any simple sheaf is modular.

Proposition 1.1.1. *If the scheme* $\mathrm{Spl}(F_0)$ *is reduced, then the sheaf* F *corresponding to the point* $[F] \in \mathrm{Spl}_0(F_0)$*, is modular.*

Proof. For any smooth point $[F] \in \mathrm{Spl}_0(F_0)$the Kodaira–Spencer map (1.1.13) is an embedding and, therefore, an isomorphism. This means that for any vector $e \in {}^1\langle F \mid F\rangle$ the first obstruction $e \circ e \in {}^2\langle F \mid F\rangle$, where

$${}^1\langle F \mid F\rangle \circ {}^1\langle F \mid F\rangle \longrightarrow {}^2\langle F \mid F\rangle \tag{1.1.19}$$

is the Yoneda pairing, and all other obstruction vanish (see [1],§ 1). Similarly to [11], § 1, or [3], this implies the existence of a formal germ of the scheme M at the point m and a modular family $\mathcal{F}$. This germ algebraises and coincides with a neighborhood of $[F]$ in $\mathrm{Spl}_0(F_0)$. $\square$

DEFINITION 1.1.6. Suppose that a flat morphism $f : X \longrightarrow M$ makes a scheme X into an M-scheme. An $\mathcal{O}_M$-flat sheaf $\mathcal{F}$ is called *M-simple* if the natural homomorphism

$$\mathcal{O}_M \longrightarrow \mathcal{E}xt^0_{\mathcal{O}_M}(\mathcal{F}, \mathcal{F}) \tag{1.1.20}$$

is a isomorphism.

It was proved in [2] that for any scheme $\mathrm{Spl}_0(F_0)$ there exist a family of schemes $\{U_\alpha\}$ and sheaves $\mathcal{F}_\alpha$ on $B \times U_\alpha$ such that

1) $\mathcal{F}_\alpha$ is universal and U_α is a simple sheaf on $B \times U_\alpha$, and

2) the family of classifying morphisms

$$\left\{U_\alpha \xrightarrow{f_\alpha} \mathrm{Spl}_0(F_0)\right\} \tag{1.1.21}$$

is an etale covering of the scheme $\mathrm{Spl}_0(F_0)$:

$$\coprod f_\alpha : \coprod U_\alpha \longrightarrow \mathrm{Spl}_0(F_0). \tag{1.1.22}$$

In general the sheaves $\mathcal{F}_\alpha$ cannot be glued into a sheaf on $\mathrm{Spl}_0(F_0)$ (see, for example, [10]). However, we have the following

Proposition 1.1.2.

1) *For any simple bundle E on B the sheaves $\mathcal{E}xt^\imath_{U_\alpha}(\mathcal{F}_\alpha, \mathcal{F}_\alpha \otimes \pi_B^* E)$ glue into a coherent $\mathcal{O}_{\mathrm{Spl}_0(F_0)}$-sheaf, which we denote by*

$$\mathcal{E}xt^\imath_{\mathrm{Spl}_0(F_0)}(\mathcal{F}, \mathcal{F} \otimes \pi_B^* E).$$

2) *In particular, we have the isomorphisms*

$$\begin{gathered} \mathcal{O}_{\mathrm{Spl}_0(F_0)} \xrightarrow{\sim} \mathcal{E}xt^0_{\mathrm{Spl}_0(F_0)}(\mathcal{F}, \mathcal{F}) \\ T\mathrm{Spl}_0(F_0) \xrightarrow{\sim} \mathcal{E}xt^1_{\mathrm{Spl}_0(F_0)}(\mathcal{F}, \mathcal{F}) \\ T^*\mathrm{Spl}_0(F_0) \xrightarrow{\sim} \mathcal{E}xt^{b-1}_{\mathrm{Spl}_0(F_0)}(\mathcal{F}, \mathcal{F} \otimes \pi_B^* K_B) \end{gathered} \tag{1.1.23}$$

and the stalkwise duality between these sheaves is given by the relative Serre duality.

3) *The sheaves* (1.1.23) *are universal in the natural sense.*

PROOF. The right-hand sides of (1.1.23) coincide with the sheaves $W^\imath_{T,\mathcal{F}}$ from [11], § 2, Definition 2.1. Therefore the desired results follow from Propositions 2.2, 2.5, 2.6, and Corollary 2.7 of the same paper. □

Thus, for any simple modular sheaf F on B the moduli scheme $\mathrm{Spl}_0(F) \ni [F]$ is reduced and smooth, but possibly nonseparated. This does not stop us from correctly defining

1) tensor structures on moduli (for example, an algebraic symplectic structure, which is of interest to us), and

2) the birational type of the moduli variety

$$M(F) \overset{\text{bir}}{\sim} \mathrm{Spl}_0(F). \tag{1.1.24}$$

More precisely, by $M(F)$ we will understand any Zariski-dense subvariety of $\mathrm{Spl}_0(F)$ containing $[F]$.

It is easy to see that, for a compact modular sheaf (see Definition 1.1.4, 3)), M is a fine moduli variety and $\mathcal{F}$ is a universal family. In this case M has all the properties of the base B, $\mathcal{F}$ can be viewed as a family of sheaves on M, and B is often the variety of moduli of sheaves on M.

Since compact modular sheaves are simpler to work with, we will use them to illustrate basic principles.

The most important class of sheaves is given by the following

DEFINITION 1.1.7. A simple sheaf F on B is called *exceptional* if

$$^1\langle F \mid F\rangle = 0. \tag{1.1.25}$$

By Definition 1.1.2, an exceptional sheaf F is unique in its class

$$\{F\}_m \in F_m(B) \subset V_Z(B)$$

(but not in the class $\{F\} \in F(B) \subset K^0_{\mathrm{alg}}(B)$).

We denote the set of exceptional sheaves by

$$R(B) \subset F_m(B) \subset V_Z(B). \tag{1.1.26}$$

Clearly

$$h^{1,0}(B) \neq 0 \Rightarrow R(B) = \varnothing,\ h^{1,0}(B) = 0 \Rightarrow \mathrm{Pic}\, B \subset R(B). \tag{1.1.27}$$

In Mukai's interpretation, the big lattice $(V_Z(B), -\chi)$ is an analogue of the Picard lattice, $F_m(B)$ is an analogue of the semicone of effective divisors and $R(B)$ is an analogue of the set of exceptional divisors. We shall clarify this analogy in the next section when we investigate the integral bilinear form $-\chi$.

§ 2 The Mukai lattice and structure.

Associated with a complete smooth algebraic variety B of dimension b there are two graded rings

$$\tilde{H}(B,\mathbb{Z}) = \bigoplus_{i=0}^{b} H^{2i}(B,\mathbb{Z}), \quad \tilde{A}(B) = \bigoplus_{i=0}^{b} A^i(B) \tag{1.2.1}$$

of even-dimensional cohomology and cycles modulo algebraic equivalence, which are connected via a standard homomorphism $h : \tilde{A}(B) \longrightarrow \tilde{H}(B,\mathbb{Z})$. Let

$$\tilde{H}_a(B) = h(\tilde{A}(B)) \subset \tilde{H}(B,\mathbb{Z}) \tag{1.2.2}$$

be its image.

The involution $*$ acts componentwise:

$$\begin{gathered} *|_{H^{4i}(B,\mathbb{Z})} = \mathrm{Id}, \quad *|_{H^{4i+2}(B,\mathbb{Z})} = -\mathrm{Id}, \\ *|_{A^{2i}(B)} = \mathrm{Id}, \quad *|_{A^{2i+1}(B)} = -\mathrm{Id}. \end{gathered} \tag{1.2.3}$$

The components $H^0(B,\mathbb{Z}) = A^0(B)$ and $H^{2b}(B,\mathbb{Z}) = A^b(B)$ can be naturally identified with $\mathbb{Z}$.

For any $u \in \tilde{H}(B,\mathbb{Z})$ we denote the ith component of this element by $[u]_i$. Let K_b denote the canonical class of B. On the $\mathbb{Z}$-module $\tilde{H}(B,\mathbb{Z})$ one can define two bilinear forms

$$\begin{gathered} (u,v) = -[u^* \cdot v]_{2b} = (-1)^b [v^* \cdot u]_{2b}, \\ ((u,v)) = -\frac{1}{2} K_B [u^* \cdot v]_{2b-2}. \end{gathered} \tag{1.2.4}$$

The former is symmetric if $b \equiv 0 \pmod 2$ and skew-symmetric otherwise. The latter has inverse symmetry.

The Chern character composes the epimorphisms r_m (1.1.11) and $h \otimes \mathbb{Q}$ into the chain

$$V_Z(B) \xrightarrow{r_m} K^0_{\mathrm{alg}}(B) \xrightarrow{\mathrm{ch}} \tilde{A}(B) \otimes \mathbb{Q} \xrightarrow{h} \tilde{H}(B,\mathbb{Q}). \tag{1.2.5}$$

By Riemann–Roch (see, for example, [5]), the bilinear form (1.1.6) is defined by

$$\chi(F_1,F_2) = [\mathrm{ch}\, F_2 \cdot \mathrm{ch}\, F_1^* \cdot \mathrm{td}_B]_b, \tag{1.2.6}$$

where $\mathrm{td}_B \in \tilde{A}(B) \otimes \mathbb{Q}$ is the Todd class. This form decomposes into the sum of the symmetric and skew-symmetric parts:

$$\chi(F_1,F_2) = \chi_+(F_1,F_2) + \chi_-(F_1,F_2), \quad \chi_\pm(F_2,F_1) = \pm\chi(F_1,F_2). \tag{1.2.7}$$

A direct calculation shows that

$$\mathrm{td}_B^* = \mathrm{td}_B \cdot \mathrm{ch}\, K_B = \mathrm{td}_B \cdot e^{K_B}. \tag{1.2.8}$$

Set $2\,\mathrm{td}_B^{\pm} = \mathrm{td}_B \pm \mathrm{td}_B^* \overset{(1.2.8)}{=\!=\!=} \mathrm{td}_B\left(1 \pm e^{K_B}\right)$. Then

$$\chi_{\pm}(F_1,F_2) = \left[\mathrm{ch}\,F_2^* \cdot \mathrm{ch}\,F_1 \cdot \mathrm{td}_B^{\pm}\right]_b\,, \qquad b \equiv 0 \pmod 2)$$
$$\chi_{\pm}(F_1,F_2) = \left[\mathrm{ch}\,F_2^* \cdot \mathrm{ch}\,F_1 \cdot \mathrm{td}_B^{\pm}\right]_b\,, \qquad b \equiv 1 \pmod 2).$$

The conditions

$$\sigma_B^2 = \mathrm{td}_B^+, \quad [\sigma_B]_0 = 1, \quad \xi_B^2 = \mathrm{td}_B \frac{e^{K_B}-1}{K_B}, \quad [\xi_B]_0 = 1 \tag{1.2.9}$$

uniquely determine two elements σ_B, $\xi_B \in \tilde{A}(B) \otimes \mathbb{Q}$. Both elements are $*$ -symmetric: $\sigma_B^* = \sigma_B$, $\xi_B^* = \xi_B$ and for $B = B_1 \times B_2$

$$\sigma_{B_1\times B_2} = \pi_1^*\sigma_{B_1} \cdot \pi_2^*\sigma_{B_2}, \quad \xi_{B_1\times B_2} = \pi_1^*\xi_{B_1} \cdot \pi_2^*\xi_{B_2}. \tag{1.2.10}$$

These elements define two lattice homomorphisms

$$\begin{gathered} V_Z(B) \xrightarrow{r_m} K^0_{\mathrm{alg}}(B) \overset{w}{\underset{v}{\rightrightarrows}} \tilde{A}(B) \otimes \mathbb{Q} \xrightarrow{h} \mathrm{alg}\tilde{H}(B,\mathbb{Q}) \\ v(F) = \mathrm{ch}\,F \cdot \sigma_B \\ w(F) = \mathrm{ch}\,F \cdot \xi_B \end{gathered} \tag{1.2.11}$$

and the $\pm$-components (1.2.7) of the bilinear form (1.1.6) are as follows:

$$\begin{aligned} &\left.\begin{aligned}\chi_+(F_1,F_2) &= (v(F_1), v(F_2)) \\ \chi_-(F_1,F_2) &= ((\,w(F_1), w(F_2)\,))\end{aligned}\right\} \text{ when } b \equiv 0 \pmod 2) \\ &\left.\begin{aligned}\chi_+(F_1,F_2) &= ((\,w(F_1), w(F_2)\,)) \\ \chi_-(F_1,F_2) &= (v(F_1), v(F_2))\end{aligned}\right\} \text{ when } b \equiv 1 \pmod 2) \end{aligned} \tag{1.2.12}$$

where $(\,*\,,\,*\,)$ and $((\,*\,,\,*\,))$ are the forms (1.2.4).

The Hodge decomposition

$$H^{2i}(B,\mathbb{C}) = \bigoplus_{p+q=2i} H^{p,q}(B,\mathbb{C})$$

gives rise to a mixed Hodge structure on

$$\tilde{H}(B,\mathbb{C}) = \tilde{H}(B,\mathbb{Z}) \otimes \mathbb{C} = \bigoplus_{p+q\equiv 0 \pmod 2} H^{p,q}(B). \tag{1.2.13}$$

Definition 1.2.1. Let $b \equiv 0 \pmod 2)$.

1) The *Mukai lattice* $(M(B), (\,*\,,\,*\,)_M)$ is the minimal sublattice of $\tilde{H}(B,\mathbb{Q})$ containing $\tilde{H}(B,\mathbb{Z})$ and $v(V_Z(B))$ with integral inner product $(\,*\,,\,*\,)_M$, induced by $(\,*\,,\,*\,)$.

2) The orthogonal complement of $v(V_Z(B))$ with respect to $(\,*\,,\,*\,)_M$:

$$T(B) = v(V_Z(B))^{\perp} \subset M(B), \tag{1.2.14}$$

is called the *lattice of transcendental cycles*.

3) The Hodge decomposition (1.2.13) induces a mixed Hodge structure on $T(B) \otimes \mathbb{C}$, which we will call the *Mukai structure*.

It is easy to see that, for $B = B_1 \otimes B_2$

$$M(B_1 \otimes B_2) = M(B_1) \otimes M(B_2).$$

One has similar constructions in the case $b \equiv 1 \pmod 2$.

Examples. In concrete calculations it is convenient to use the standard polynomials in the Chern classes $(c_1, \cdots, c_b)$ of a sheaf F on B:

$$p_k(F) = \det \begin{pmatrix} c_1(F) & 1 & 0 & \dots & 0 \\ 2c_2(F) & c_1(F) & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ kc_k(F) & (k-1)C_{k-1} & 0 & \dots & 1 \end{pmatrix} \tag{1.2.15}$$

Then $\mathrm{ch}\,(F) = \sum \dfrac{p_k(F)}{k!}$. In particular,

$$p_0(F) = 1, \quad p_1 = c_1, \quad p_2 = c_1^2 - 2c_2, \quad p_3 = c_1^3 - 3c_1c_2 + 3c_3, \cdots \tag{1.2.16}$$

(see, for example, [5], 15.1.2).

1) Let $B = S$ be a smooth algebraic surface ($b = 2$) with invariants

$$\chi = \chi(\mathcal{O}_S) = p_g - q + 1, \quad e = e_S = \chi(S, \mathbb{Z}) \tag{1.2.17}$$

where e is the Euler–Poincaré characteristic. Then

$$\begin{aligned} &\tilde{H}(S,\mathbb{Z}) = \mathbb{Z} \oplus H^2(S,\mathbb{Z}) \oplus \mathbb{Z} \\ &\sigma_S = \left(1\,,\,0\,,\,\frac{\chi}{2}\right)\,, \quad \xi_S = \left(1\,,\,0\,,\,\frac{e}{24}\right) \\ &v(F) = \left(\mathrm{rk}F\,, c_1(F)\,,\,\frac{1}{2}\,p_2(F) + \frac{1}{2}\,\mathrm{rk}F \cdot \chi\right) \\ &w(F) = \left(\mathrm{rk}F\,,\,c_1(F)\,,\,\frac{1}{2}\,p_2(F) + \frac{1}{24}\,\mathrm{rk}F \cdot e\right) \\ &\chi_+(F_1, F_2) = \end{aligned} \tag{1.2.18}$$

$$= \frac{1}{2}\left(\mathrm{rk}F_1 \cdot p_2(F_2) + 2\,c_1(F_1) \cdot c_1(F_2) + \mathrm{rk}F_2 \cdot p_2(F_1)\right) - \mathrm{rk}F_1 \cdot \mathrm{rk}F_2 \cdot \chi$$

$$\chi_-(F_1, F_2) = -\frac{1}{2}\,K_B \cdot \det \begin{pmatrix} \mathrm{rk}F_1 & \mathrm{rk}F_2 \\ c_1(F_1) & c_1(F_2) \end{pmatrix}.$$

If the intersection form on Pic S is even and χ is an even number, then the Mukai lattice $M(S)$ coincides with $\tilde{H}(S,\mathbb{Z})$ endowed with inner product $(\,*\,,\,*\,)$ from (1.2.4). In particular, if S is a K3 surface, then
(1.2.19)

$$\begin{aligned}
&\chi(F_1,F_2)=\chi_+(F_1,F_2)\\
&\sigma_S=\xi_S=(1,0,1)\\
&v(F)=(\mathrm{rk}F\,,\,c_1(F)\,,\,g(c_1(F))-1-c_2(F)+\mathrm{rk}F)\in\tilde{H}(S,\mathbb{Z})=\mathbb{Z}^{24}\\
-&\chi(F_1,F_2)=\mathrm{rk}^1\big\langle F_2\,\big|\,F_1\big\rangle-\mathrm{rk}^0\big\langle F_2\,\big|\,F_1\big\rangle-\mathrm{rk}^0\big\langle F_1\,\big|\,F_2\big\rangle.
\end{aligned}$$

In the general case,

$$M(S)\subset\mathbb{Z}\oplus H^2(S,\mathbb{Z})\oplus\frac{1}{2}\mathbb{Z} \tag{1.2.20}$$

with form $2\,(\,*\,,\,*\,)$ (1.2.4), and for a pair of sheaves F_1 and F_2 on S

$$\begin{aligned}
-\chi(F_1,F_2)&=(v(F_1),v(F_2))+((v(F_1),v(F_2))),\\
v^2(F)&=\mathrm{rk}^1\langle F_2,F_1\rangle-\mathrm{rk}^0\langle F_2,F_1\rangle-\mathrm{rk}^2\langle F_2,F_1\rangle.
\end{aligned} \tag{1.2.21}$$

For any sheaf with Mukai vector $v(F)=(r,c_1,s)$

$$\chi(F)=h^0(F)-h^1(F)+h^2(F)=s-\frac{1}{2}c_1(F)K_B+r\chi/2. \tag{1.2.22}$$

2) Let B be a solid ($b=3$). Set

$$K=-c_1(B),\quad k=c_2(B). \tag{1.2.23}$$

Then

$$\begin{array}{c}
\tilde{H}(B,\mathbb{Z})=\mathbb{Z}\oplus H^2(B,\mathbb{Z})\oplus \underset{\displaystyle \overset{\|}{H^2(B,\mathbb{Z})^*}}{H^4(B,\mathbb{Z})}\oplus\mathbb{Z},
\end{array}$$

$$\begin{aligned}
\sigma_B&=\left(1,0,\frac{K^2+k}{2},0\right),\quad \xi_B=\left(1,0,\frac{k}{24},0\right),\\
v(F)&=\left(\mathrm{rk}F,c_1(F),\frac{1}{2}p_2(F)+r\frac{K^2+k}{24},\frac{c_1(K^2+k)}{24}+\frac{1}{6}p_3(F)\right),\\
w(F)&=\left(\mathrm{rk}F,c_1(F),\frac{1}{2}p_2(F)+r\frac{k}{24},\frac{c_1k}{24}+\frac{1}{6}p_3(F)\right),\\
-\chi_+(F_1,F_2)&=\frac{K}{2}\left[c_1(F_1)c_2(F_2)-\frac{1}{2}\mathrm{rk}(F_2)\cdot p_2(F_1)-\right.\\
&\left.-\frac{1}{2}\mathrm{rk}(F_1)\cdot p_2(F_2)-\mathrm{rk}(F_1)\cdot\mathrm{rk}(F_2)\cdot\frac{k}{12}\right],\\
-\chi_-(F_1,F_2)&=\frac{1}{6}\det\begin{pmatrix}\mathrm{rk}F_1 & \mathrm{rk}F_2\\ p_3(F_1) & p_3(F_2)\end{pmatrix}.
\end{aligned} \tag{1.2.24}$$

REMARK. If B is a Fano variety and $S \in |-K_B|$ is a smooth K3 surface, then $k|_S = e_S = K_S^2 + e_S$. Therefore, for bundles F_1 and F_2 on B, (1.2.18) and (1.2.24) give rise to the convenient equality

$$\chi_+(F_1, F_2) = \frac{1}{2}\chi(F_1|_S, F_2|_S). \tag{1.2.25}$$

Now let us come back to the general even-dimensional case. Let $B = B_1 \times B_2$, $\dim B_i = b_i \equiv 0 \pmod 2$, and let π_1 and π_2 be the projections onto the direct summands:

$$B_1 \xleftarrow{\pi_1} B = B_1 \times B_2 \xrightarrow{\pi_2} B_2. \tag{1.2.26}$$

Any vector bundle E on $B_1 \times B_2$ gives rise to the Mukai vector

$$v(E^*) \in M(B_1 \times B_2) = M(B_1) \otimes M(B_2) \tag{1.2.27}$$

and the homomorphism

$$\begin{array}{ccc} f_{E^*} : M(B_1) & \longrightarrow & M(B_2) \\ \cup & & \cup \\ m_1 & \longrightarrow & (\pi_2)_*(v(E^*)m_1) \end{array} \tag{1.2.28}$$

Similarly, the Mukai vector $v(E)$ gives rise to the homomorphism

$$\begin{array}{ccc} f_E : M(B_2) & \longrightarrow & M(B_1) \\ \cup & & \cup \\ m_2 & \longrightarrow & (\pi_1)_*(v(E^*)m_2) \end{array} \tag{1.2.29}$$

Proposition 1.2.1. $(m_1, f_E(m_2))_M = (f_{E^*}(m_1), m_2)_M$.

PROOF. Repetitively applying the projection formula, we have

$$(m_1, f_E(m_2)) \overset{(1.2.4)}{=\!=\!=} [m_1^*(\pi_1)_*(v(E)\cdot \pi_2^*(m_2))]_{b_1} =$$
$$= [(\pi_1)_*(\pi_1^*(m_1^*)\cdot \pi_2(m_2)(v(E))]_{b_1} = [\pi_1^*(m_1^*)\cdot \pi_2^*(m_2)(v(E)]_{b_1+b_2}\,.$$

Similarly,

$$(m_2, f_{E^*}(m_1)) \overset{(1.2.4)}{=\!=\!=} [m_2^*(\pi_2)_*(v(E^*)\cdot \pi_1^*(m_1))]_{b_2} =$$
$$= [\pi_2^*(m_2^*)\cdot \pi_1^*(m_1)\cdot v(E^*)]_{b_1+b_2}\,.$$

But on the even-dimensional variety $B_1 \times B_2$ the component $[\ \cdot\ ]_{b_1+b_2}$ is $*$-invariant (see (1.2.3)). This implies the desired equality. □

Clearly, for any point $b \in B_2$ the Mukai vector of the bundle $E|_{B_1 \times b} = E_1$ does not depend on $b \in B_2$.

Proposition 1.2.2. *For any $m \in M(B_1)$*

$$\big[f_{E^*}(m)\big]_0 = (m, v(E_1^*))_M.$$

Proof. By the definition of f_{E^*} and (1.2.10),

$$f_{E^*}(m) = (\pi_2)_*(\underbrace{\pi_1^*(m) \cdot \pi_1^* \sigma_{B_1}}_{\pi_1^*(m \cdot \sigma_{B_1})} \cdot \mathrm{ch}\, E^* \cdot \pi_2^* \sigma_{B_2}).$$

By the projection formula,

$$\big[f_{E^*}(m)\big]_0 = \big[m \cdot \mathrm{ch}\, E_1^* \cdot \sigma_{B_1}\big]_{b_1} = (m, v(E_1^*)).$$

$\square$

Since $v(E^*) \in \tilde{H}_a(B_1 \times B_2)$ (see (1.2.2)), the homomorphism f_{E^*} induces a homomorphism of the lattices of transcendental cycles (1.2.14):

$$f_{E^*}^{\tau} : T(B_1) \longrightarrow T(B_2) \tag{1.2.30}$$

as well as the homomorphism $f_{E^*}^{\tau} \otimes \mathbb{C}$ of the Hodge structures. In particular, the $(2,0)$-component

$$(f_{E^*}^{\tau} \otimes \mathbb{C})^{2,0} : H^{2,0}(B_1) \longrightarrow H^{2,0}(B_2) \tag{1.2.31}$$

of this homomorphism relates the symplectic structures on B_1 and B_2. This will be the main construction in the next chapter for the technically simplest case, when $B_1 = S$ is a smooth regular surface with $p_g > 0$, $B_2 = M$ is a component of the moduli variety, and E is a universal family.

§ 3 Symplectic structure and the local invariant.

Henceforth the base $B = S$ will be a smooth regular surface.

For any modular sheaf F on S we want to find an extension of the homomorphism

$$^1\big\langle F \,\big|\, E \big\rangle \underset{\overline{f}_s}{\overset{\mathrm{Id} \otimes s}{\rightrightarrows}} {}^1\big\langle F \,\big|\, E \otimes K_S \big\rangle$$

induced by homomorphism $\mathrm{Id}\otimes s$, where $s \in H^0(K_s)$, to an algebraic symplectic structure

$$\begin{gathered} T\mathrm{Spl}_0(F) \xrightarrow{\omega_s} T^*\mathrm{Spl}_0(F)\ , \\ \omega_s^* = -\omega_s\ , \quad \omega_s|_{[F]} = \overline{f}_s, \end{gathered} \tag{1.3.1}$$

and investigate the corresponding map

$$H^0(\Lambda^2 T^* S) \xrightarrow{\tau} H^0(\Lambda^2 T^* \mathrm{Spl}_0(F)) \,. \tag{1.3.2}$$

To simplify the notation we set $\mathrm{Spl}_0(F) = M$, and denote the system of locally universal sheaves $\{\mathcal{F}_\alpha\}$ in (1.1.21) and (1.1.22) by $\mathcal{F}$, keeping in mind that this symbol should only appear in the expressions of type (1.1.23) of Proposition 1.1.2. In particular,

$$TM = \mathcal{E}xt^1_{\mathcal{O}_M}(\mathcal{F},\mathcal{F}), \quad \Omega M = T^*M = \mathcal{E}xt^1_{\mathcal{O}_M}(\mathcal{F},\mathcal{F}\otimes\pi_S^* K_S), \tag{1.3.3}$$

where K_S is the canonical class of the surfaces S and π_S is the projection onto the direct summand.

Consider the $\mathcal{O}_M$-sheaf $\mathcal{E}xt^1_{\mathcal{O}_M}(\mathcal{F},\mathcal{F}\otimes\pi_s^* K_s)$. The contraction map (i.e., the Yoneda pairing) gives rise to a homomorphism of $\mathcal{O}_M$-sheaves

$$\begin{tikzcd}
\mathcal{E}xt^1_{\mathcal{O}_M}(\mathcal{F},\mathcal{F})\otimes\mathcal{E}xt^0_{\mathcal{O}_M}(\mathcal{F},\mathcal{F}\otimes\pi_S^* K_S) \arrow[r, "\bar\varphi"] \arrow[d, equal, "(1.3.3)"] & \mathcal{E}xt^1_{\mathcal{O}_M}(\mathcal{F},\mathcal{F}\otimes\pi_S^* K_S) \arrow[d, equal, "(1.3.3)"'] \\
TM & T^*M
\end{tikzcd}$$

Since the $\mathcal{O}_M$-sheaf $\mathcal{E}xt^1_{\mathcal{O}_M}(\mathcal{F},\mathcal{F}) = TM$ is locally free, the homomorphism $\bar\varphi$ can be interpreted as a homomorphism

$$\varphi : \mathcal{E}xt^0_{\mathcal{O}_M}(\mathcal{F},\mathcal{F}\otimes\pi_S^* K_S) \longrightarrow \mathrm{Hom}(TM, T^*M) = \Omega M^{\otimes 2}$$

of $\mathcal{O}_M$-sheaves, which induces a homomorphism of sections

$$\varphi^0 : H^0(M, \mathcal{E}xt^0_{\mathcal{O}_M}(\mathcal{F},\mathcal{F}\otimes\pi_S^* K_S)) \longrightarrow H^0(\Omega M^{\otimes 2}). \tag{1.3.4}$$

On the other hand, the homomorphism

$$\begin{tikzcd}
H^0(K_S) \otimes \mathcal{O}_M \arrow[r, "j_{\mathcal{F}}"] \arrow[d, equal] & \mathcal{E}xt^0_{\mathcal{O}_M}(\mathcal{F},\mathcal{F}\otimes\pi_S^* K_S) \\
H^0(\Lambda^2 T^* S) &
\end{tikzcd} \tag{1.3.5}$$

coincides stalkwise with the homomorphism

$$H^0(K_S) \xrightarrow{j_F} {}^0\langle F \,|\, E\rangle, \quad j_F(s) = \mathrm{Id}\otimes s, \tag{1.3.6}$$

which is just (1.1.18).

Finally, composing (1.3.5) and (1.3.4), we obtain a homomorphism

$$\tau_{\mathcal{F}} : H^0(\Lambda^2\Omega S) \longrightarrow H^0(\Omega M^{\otimes 2}) = H^0(\Lambda^2 T^* M) \oplus H^0(S^2 T^* M). \tag{1.3.7}$$

THEOREM 1.3.1. $\tau_{\mathcal{F}}(H^0(\Lambda^2\Omega S)) \subset H^0(\Lambda^2\Omega M)$, *i.e. we have a homomorphism*

$$\tau_{\mathcal{F}} : H^0(\Lambda^2\Omega S) \longrightarrow H^0(\Lambda^2\Omega M).$$

PROOF. Any section $f \in H^0(\mathcal{E}xt^0_{\mathcal{O}_M}(\mathcal{F}, \mathcal{F} \otimes \pi_S^* K_S))$, i.e. homomorphism

$$\mathcal{F} \xrightarrow{f} \mathcal{F} \otimes \pi_S^* K_S\,,$$

induces a homomorphism

$$\mathcal{E}xt^1_{\mathcal{O}_M}(\mathcal{F}, \mathcal{F}) \underset{\bar f}{\overset{f}{\dashrightarrow}} \mathcal{E}xt^1_{\mathcal{O}_M}(\mathcal{F}, \mathcal{F} \otimes \pi_S^* K_S)$$

Since the stalkwise duality of the bundles TM and T^*M is given by the relative Serre duality, the inclusion $\bar f \in H^0(\Lambda^2 T^* M)$ is equivalent to the following condition: for any points $m \in M$ and vector $e \in {}^1\langle F_m | F_m\rangle$ the contraction

$$e \circ (\bar f|_m(e)) \in {}^2\langle F_m | F_m \otimes K_S\rangle \tag{1.3.9}$$

where $\mathcal{F}_m = \mathcal{F}|_{S\times m}$, equals zero. If $f = f_s \in j_{\mathcal{F}}(s)$, where $s \in {}^0\langle \mathcal{O}_S, K_S\rangle$ (see (1.3.5)), then the homomorphism

$${}^1\big\langle F_m \,\big|\, F_m\big\rangle \underset{\bar f_m}{\overset{f_s = \mathrm{Id}\otimes s}{\dashrightarrow}} {}^1\big\langle F_m \,\big|\, F_m \otimes K_S\big\rangle$$

is induced by $\mathrm{Id} \otimes s$ and the element from ${}^2\big\langle F_m \,\big|\, F_m \otimes K_S\big\rangle$ in (1.3.9) can be represented as a composition of contractions

$$e \circ \bar f|_m(e) = e \circ e \circ \big(\mathrm{Id} \otimes s\big).$$

But $e \circ e \in {}^2\big\langle F_m \,\big|\, F_m\big\rangle$, being the obstruction to the modularity of F_m, equals zero (see (1.1.7)), i.e., $e \circ \bar f_m(e) = 0$ (1.3.9) and $\tau(j_{\mathcal{F}}(s)) \in H^0(\Lambda^2 \Omega M)$. □

Similarly one can define a homomorphism of Poisson structures on S and M. For any modular sheaf F on S, the homomorphism

$${}^1\big\langle F \,\big|\, F\big\rangle \underset{\bar f'_s}{\overset{\mathrm{Id}\otimes s'}{\rightleftarrows}} {}^1\big\langle F \,\big|\, F \otimes K_S\big\rangle = {}^1\big\langle F \otimes K_S^* \,\big|\, F\big\rangle$$

induced by $\mathrm{Id} \otimes s'$, where $s' \in H^0(K_s^*) = H^0(\Lambda TS)$, can be extended to the Poisson structure

$$T\mathrm{Spl}_0(F) \xleftarrow{\alpha_s} T^*\mathrm{Spl}_0(F) \ : \quad \alpha_s^* = -\alpha_s\,, \quad \alpha_s|_{[F]} = \bar f'_s\,. \tag{1.3.1$'$}$$

Indeed, for a modular family $\mathcal{F}$ on $S \times M$ the contraction map gives rise to a homomorphism

$$\begin{CD}
\mathcal{E}xt^1_{\mathcal{O}_M}(\mathcal{F}\otimes\pi_S^* K_S^*, \mathcal{F}) \otimes \mathcal{E}xt^0_{\mathcal{O}_M}(\mathcal{F}, \mathcal{F}\otimes\pi_S^* K_S^*) @>{\bar\varphi'}>> \mathcal{E}xt^1_{\mathcal{O}_M}(\mathcal{F}\otimes\pi_S K_S^*, \mathcal{F}\otimes\pi_S^* K_S^*)\\
@| @|\\
\mathcal{E}xt^1_{\mathcal{O}_M}(\mathcal{F}, \mathcal{F}\otimes\pi_S^* K_S) @. \mathcal{E}xt^1_{\mathcal{O}_M}(\mathcal{F}, \mathcal{F})\\
@| @|\\
T^*M @. TM
\end{CD}$$

of $\mathcal{O}_M$-sheaves, which induces a homomorphism

$$\varphi'^0 : H^0(M, \mathcal{E}xt^0_{\mathcal{O}_M}(\mathcal{F}, \mathcal{F} \otimes \pi_S^* K_S^*)) \longrightarrow H^0(TM^{\otimes 2}). \tag{1.3.4$'$}$$

On the other hand, we have the homomorphism

$$H^0(K_S^*) \otimes \mathcal{O}_M \xrightarrow{j'_{\mathcal{F}}} \mathcal{E}xt^0_{\mathcal{O}_M}(\mathcal{F}, \mathcal{F} \otimes \pi_S^* K_S^*), \tag{1.3.5$'$}$$

which coincides stalkwise with

$$H^0(K_S) \xrightarrow{0} \langle F | F \otimes K_S^* \rangle, \quad j'_F(s') = \mathrm{Id} \otimes s'. \tag{1.3.6$'$}$$

The composition of (1.3.5') and (1.3.4') yields a homomorphism

$$\tau'_{\mathcal{F}} : H^0(\Lambda^2 TS) \longrightarrow H^0(TM^{\otimes 2}). \tag{1.3.7$'$}$$

THEOREM 1.3.1'. $\tau'_{\mathcal{F}}(H^0(\Lambda^2 TS)) \subset H^0(\Lambda^2 TM)$.

The construction of homomorphisms $\tau_{\mathcal{F}}$ and $\tau'_{\mathcal{F}}$ is a globalisation of the constructions of the local invariant of a fixed sheaf F on S: the vector space homomorphisms

$$\begin{array}{ccc} {}^1\langle F \mid F \rangle \otimes H^0(K_S) & \xrightarrow{\tau_F} & {}^1\langle F \mid F \rangle^* \\ \Vert & & \Vert \\ T_F & & T_F^* \\ \Vert & & \Vert \\ {}^1\langle F \mid F \rangle & \xleftarrow{\tau'_F} & {}^1\langle F \mid F \rangle^* \otimes H^0(K_S^*) \end{array} \tag{1.3.11}$$

can be interpreted as skew-symmetric homomorphisms of sheaves on the projective spaces $|K_S|$ or $|-K_S|$:

$$\begin{aligned} T_F \otimes \mathcal{O}_{|K_S|} &\xrightarrow{\tau_F} T_F^* \otimes \mathcal{O}_{|K_S|}(1), \quad \tau_F^* = -\tau_F(1), \\ T_F^* \otimes \mathcal{O}_{|-K_S|} &\xrightarrow{\tau'_F} T_F \otimes \mathcal{O}_{|-K_S|}(1), \quad \tau_F'^* = -\tau'_F(1), \end{aligned} \tag{1.3.12}$$

i.e., as hypernets of skew-correlations of the vector space T_F (or T_F^*).

DEFINITION 1.3.1. The class τ_F (mod GL)(T_F) (or τ'_F (mod GL)(T_F)) of hypernets (1.3.12) is called the *local invariant of the sheaf F on S*.

Geometric equivalents of the local invariant are sufficiently informative only if the complete linear series $|K_s|$ or $|-K_s|$ is sufficiently ample. The constructive analogy between local invariants of symplectic and Poisson structures is obvious. However their geometric meanings are different in principle. We will illustrate this in the case when F is a simple bundle. Then

(1.3.13)

$$T_F = H^1(\mathrm{ad}\, F), \quad T_F^* = H^1(\mathrm{ad}\, F \otimes K_S), \quad F \otimes F^* = \mathrm{End}\, F = \mathcal{O}_S \oplus \mathrm{ad}\, F.$$

If C is a curve from the complete linear series $|K_s|$ or $|-K_s|$ then we have exact triples

$$(1.3.14)\qquad \begin{gathered} 0 \longrightarrow \operatorname{ad} F \longrightarrow \operatorname{ad} F \otimes K_s \longrightarrow \operatorname{ad} F|_C \otimes \mathcal{O}_C(K_S) \longrightarrow 0 \\ 0 \longrightarrow \operatorname{ad} F \otimes K_S \longrightarrow \operatorname{ad} F \longrightarrow \operatorname{ad} F|_C \longrightarrow 0 \end{gathered}$$

and their cohomology sequences
(1.3.15)

$$H^0(\operatorname{ad} F|_C \otimes \mathcal{O}_C(K_S)) \longrightarrow T_F \xrightarrow{\tau_F(C)} T_F^* \longrightarrow H^1(\operatorname{ad} F|_C \otimes \mathcal{O}_C(K_S)),$$

$$H^0(\operatorname{ad} F|_C) \longrightarrow T_F^* \xrightarrow{\tau'_F(C)} T_F \longrightarrow H^1(\operatorname{ad} F|_C).$$

The lower sequence shows that skew-symmetric form $\tau'_F(C)$ has a kernel if and only if $F|_C$ no longer simple.

The interpretation of the kernel of the skew-symmetric form $\tau_F(C)$ is more complicated. First of all, on C we have the theta-characteristic $\theta = \mathcal{O}_C(K_C)$, since by the adjunction formula

$$(1.3.16)\qquad \omega_C = \mathcal{O}_C(2K_S) = \theta^2.$$

A necessary (and sufficient, if F satisfies the Mukai–Artamkin criterion) condition for the degeneration of $\tau_F(C)$ is the vanishing of the analogue of the θ-constant for the bundle $\operatorname{ad} F|_C$ (see [14], (3.2.7)).

CHAPTER 2

Modular operations

§ 1 Special modular families.

The homomorphism τ_F is especially simple when S is a K3 surface, i.e., when the algebraic symplectic structure on S is everywhere nondegenerate.

THE MUKAI THEOREM [11]. *If S is a $K3$ surface, then for any locally modular family $\mathcal{F}$ on $S \times M$ the homomorphism τ_F is an isomorphism and the algebraic symplectic structure $\tau_F(\omega)$ is everywhere nondegenerate.*

Any coherent sheaf F on an arbitrary smooth surface S gives rise to an exact quadruple

$$0 \longrightarrow T(F) \xrightarrow{i} F \xrightarrow{\text{can}} F^{**} \xrightarrow{j} C(F) \longrightarrow 0, \tag{2.1.1}$$

where $T(F)$ is the torsion subsheaf, *can* is the canonical homomorphism from a sheaf to its $\mathcal{O}_S$-double dual, and $C(F)$ is a sheaf with zero-dimensional support.

Let us recall the terminology of [13].

DEFINITION 2.1.1. A zero-dimensional subscheme $\xi \subset S$ of length d is called a *thin cycle of degree d.*

Any thin cycle ξ can be given by its structure $\mathcal{O}_S$-sheaf $\mathcal{O}_\xi$ or by its ideal sheaf $J_\xi \subset \mathcal{O}_S$, which are related by the standard exact sequence

$$0 \longrightarrow J_\xi \xrightarrow{i} \mathcal{O}_S \xrightarrow{\text{res}} \mathcal{O}_\xi \longrightarrow 0 \tag{2.1.2}$$

The variety $\tilde{S}^{(d)}$ of all thin cycles of degree d is called the *Douady space.* It is a smooth irreducible variety of dimension $2d$. The formula

$$\xi \rightsquigarrow \sum n_i p_i, \quad \operatorname{supp}\xi = \bigcup p_i, \quad \sum n_i = d, \tag{2.1.3}$$

assigns a cycle to each thin cycle. In the ideal theoretic language this corresponds to assigning to an ideal $J \subset m_{p_i}$ of the local ring $\mathcal{O}_{p_i}$ an m_{p_i}-graded ideal. This correspondence gives rise to a birational morphism

$$\tilde{S}^{(d)} \longrightarrow S^{(d)}, \tag{2.1.4}$$

where $S^{(d)}$ is the dth symmetric power of S, which is biregular outside Sing $S^{(d)}$. Therefore the morphism (2.1.4) resolves the singularities of $S^{(d)}$ and establishes a birational isomorphism

$$\tilde{S}^{(d)} \xrightarrow{\text{bir}} S^{(d)}. \tag{2.1.5}$$

Lemma 2.1.1. *If F is a rank 1 torsion-free sheaf, then*

1) *F is simple,*
2) *the components of the vector $v(F) = (1, D, s)$ (1.2.18) are related by the formula*

$$s - \frac{1}{2}D^2 - \frac{\chi}{2} = d \tag{2.1.6}$$

where χ is defined by (1.2.17) *and d is a positive integer, and*

3) *F is compact modular and* $\mathrm{Spl}_0(F) = \tilde{S}^{(d)}$.

Proof. Since $T(F) = 0$, the quadruple (2.1.1) becomes a triple

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F & \xrightarrow{\text{can}} & F^{**} & \longrightarrow & C(F) & \longrightarrow & 0 \\ & & \| & & \| & & \| & & \\ 0 & \longrightarrow & J_\xi(D) & \longrightarrow & \mathcal{O}_S(D) & \longrightarrow & \mathcal{O}_\xi & \longrightarrow & 0. \end{array} \tag{2.1.7}$$

Since F^{**} is reflexive and, therefore, locally free ($\dim S = 2$), $F^{**} = \mathcal{O}_S(D)$, $D = c_1(F) \in \mathrm{Pic}\, S$. Hence $v(F) = (1, D, \frac{1}{2}D^2 + \chi/2 - d)$, where $d = \deg \xi$, and and assertion 2) follows .

The cohomology sequence of the lower triple of (2.1.7) yields an isomorphism

$$H^2(J_\xi(D)) = H^2(\mathcal{O}_S(D)). \tag{2.1.8}$$

The simplicity of $J_\xi(D)$ is equivalent to that of J_ξ, but

$$\begin{aligned} {}^0\langle J_\xi \mid J_\xi \rangle \subset {}^0\langle J_\xi \mid \mathcal{O}_S \rangle &\overset{\text{SD}}{=\!=} {}^2\langle \mathcal{O}_S \mid J_\xi \otimes K_S \rangle^* = \\ &= H^2(J_\xi \otimes K_S)^* \overset{(2.1.8)}{=\!=} H^2(K_S)^* \overset{\text{SD}}{=\!=} H^0(\mathcal{O}_S) = \mathbb{C}, \end{aligned}$$

and assertion 1) is proved.

The modularity of $J_\xi(D)$ is also equivalent to that of J_ξ. To check the latter we use the Mukai–Artamkin criterion:

$$\begin{aligned} {}^0\langle J_\xi \mid J_\xi \otimes K_S \rangle \subset {}^0\langle J_\xi \mid K_S \rangle &\overset{\text{SD}}{=\!=} {}^2\langle \mathcal{O}_S \mid J_\xi \rangle^* = \\ &= H^2(J_\xi)^* \overset{(2.1.8)}{=\!=} H^2(\mathcal{O}_S)^* \overset{\text{SD}}{=\!=} H^0(K_S)\ . \end{aligned}$$

Therefore, the homomorphism j_{J_ξ} (1.1.18) is an isomorphism. The modular family $\mathcal{F}$ on $S \times \tilde{S}^{(d)}$ is part of the canonical triple

$$(2.1.9)\qquad 0 \longrightarrow \overbrace{J_Z \otimes \pi_S^* \mathcal{O}_S(D)}^{\mathcal{F}} \longrightarrow \pi_S^* \mathcal{O}_S(D) \longrightarrow \mathcal{O}_Z \otimes \pi_S^*(D) \longrightarrow 0,$$

where $Z \subset S \times \tilde{S}^{(d)}$ is the universal Douady subscheme

$$(2.1.10)\qquad Z \cdot (S \times \xi) = \xi.$$

Assertion 3) follows. □

REMARK. For the $\mathcal{O}_{S\times\tilde{S}^{(d)}}$-sheaf $\mathcal{F}$ in (2.1.9) we have

$$(2.1.11)\qquad Z = c_2(\mathcal{F}),$$

where c_2 is the second Chern class.

Lemma 2.1.2. *For the modular family* $J_Z = \mathcal{F}$ (2.1.9)

1) *the homomorphism* (1.3.7) (resp., (1.3.7')) $$\tau_{J_Z} : H^0(\Lambda^2 T^* S) \longrightarrow H^0(\Lambda^2 T^* \tilde{S}^{(d)})$$ *is an isomorphism, and*

2) *each nonzero algebraic symplectic structure on* $\tilde{S}^{(d)}$ *(resp., Poisson structure) is nondegenerate.*

PROOF. For the family (2.1.9) we have

$$\begin{aligned}\mathcal{E}xt^0_{\tilde{S}^{(d)}}(J_Z, J_Z \otimes \pi_S^* K_S) \subset \mathcal{E}xt^0_{\tilde{S}^{(d)}}(J_Z, \pi_S^* K_S) = \\ = \mathcal{E}xt^0_{\tilde{S}^{(d)}}(J_Z, \mathcal{O}_{S\times\tilde{S}^{(d)}}) \otimes H^0(K_S) = H^0(K_S) \otimes \mathcal{O}_{\tilde{S}^{(d)}}\ .\end{aligned}$$

For any thin cycle $\xi_1 \in \tilde{S}^{(d-1)}$ of degree of $d-1$ one can define a regular embedding

$$(2.1.12)\qquad i : S \longrightarrow \tilde{S}^{(d)}, \quad J_{i(p)} = J_{\xi_1} \otimes J_p.$$

It is easy to see that for any $\omega \in H^0(\Lambda^2 T^* S)$

$$(2.1.13)\qquad i^*(\tau_{J_Z}(\omega)) = \omega.$$

Indeed, $J_Z|_{S\times i(S)} = J_\Delta$, where Δ is the diagonal in $S\times S$, and $TS = \mathcal{E}xt^i_{\mathcal{O}_S}(J_\Delta, J_\Delta)$. The homomorphism

$$(2.1.14)\qquad i^*\tau_{\tilde{S}^{(d)}} : TS \longrightarrow T^*S = TS \otimes K_S$$

is given by the multiplication by a section $s \in H^0(K_S)$. From this we get the assertion for symplectic structures. The argument for Poisson structures is similar. □

If $\xi = \sum p_i$, $p_i \neq p_j$, is a cycle without multiple points, then identifying the tangent spaces we have

$$TS^{(d)}_{\xi} = T_{J_\xi} = \overset{d}{\underset{i=1}{\oplus}} TS_{p_i};$$

hypernet of correlations of the local invariant (1.3.12) is the direct sum of hypernets of the form

$$TS_p \otimes \mathcal{O}_{|K_S|} \xrightarrow{u \otimes S_{p_i}} T^* S_{p_i} \otimes \mathcal{O}_{|K_S|}(1),$$

where u is a nondegenerate skew-symmetric isomorphism and $S_{p_i} \in H^0(\mathcal{O}_{|K_S|}(1))$, $(S_{p_i})_0 = |K_S - p_i|$. Thus the local invariant τ_{J_ξ} is the set of d hyperplanes $\bigcup |K_S - p_i|$ in $|K_S|$. The cycle ξ can be recovered from this set.

Let $|D|$ be a complete linear series on S such that

$$|D|_0 = \{C \in |D| \, | C \text{ reduced and irreducible}\} \tag{2.1.15}$$

is nonempty. For any curve $C \in |D|_0$ one can define a generalised Picard variety

$$\operatorname{Pic}_d C = \{\text{rank 1 torsionfree } \mathcal{O}_C\text{-sheaves of degree } d\}. \tag{2.1.16}$$

When the curve C varies in $|C|_0$ the generalized Picard varieties sweep a variety

$$\operatorname{Pic}_d |D|_0 = \bigcup_{C \in |D|_0} \operatorname{Pic}_d C \tag{2.1.17}$$

(see [2]). One can canonically make it into bundle

$$\operatorname{Pic}_d |D|_0 \xrightarrow{\pi} |D|_0 \tag{2.1.18}$$

with fiber (2.1.16).

Lemma 2.1.3. *If C is a curve from $|D|_0$ and the rank 1 torsion free $\mathcal{O}_C$-sheaf $\mathcal{O}_C(\xi)$ of degree d is viewed as an $\mathcal{O}_S$-sheaf then*

1) *$\mathcal{O}_C(\xi)$ is simple,*

2) *the components of the vector $v(\mathcal{O}_C(\xi)) = (0, D, s)$ are related by the formula*

$$s - \frac{1}{2} D^2 = d \tag{2.1.19}$$

where d is an integer, and

3) *$\mathcal{O}_C(\xi)$ is modular if $h^1(\mathcal{O}_S(D)) = 0$ and $\operatorname{Spl}_0(\mathcal{O}_C(\xi)) \supset \operatorname{Pic}_d |D|_0$ as a Zariski dense subset.*

Proof. Assume for simplicity that C is smooth. Then $\mathcal{O}_C(\xi)$ is invertible as an $\mathcal{O}_C$-sheaf, and ${}^0\langle \mathcal{O}_C(\xi) \,|\, \mathcal{O}_C(\xi)\rangle = \mathrm{Ext}^0_{\mathcal{O}_C}(\mathcal{O}_C(\xi), \mathcal{O}_C(\xi)) = \mathbb{C}$, i.e., the $\mathcal{O}_S$-sheaf $\mathcal{O}_C(\xi)$ is simple. Moreover, ${}^0\langle \mathcal{O}_C(\xi) \,|\, \mathcal{O}_C(\xi)\otimes K_S\rangle = H^0(\mathcal{O}_C(K_S))$. The exact triple

$$0 \longrightarrow \mathcal{O}_S(K_S - D) \longrightarrow K_S \longrightarrow \mathcal{O}_C(K_S) \longrightarrow 0$$

yields

$$\begin{aligned}(2.1.20)\quad 0 \longrightarrow H^0(\mathcal{O}_S(K_S - D)) \longrightarrow H^0(K_S) \longrightarrow H^0(\mathcal{O}_C(K_S)) \longrightarrow \\ \longrightarrow H^1(\mathcal{O}_S(K_S - D)) \overset{\text{SD}}{=\!=\!=} H^1(\mathcal{O}_S(D))^* .\end{aligned}$$

Hence $\mathcal{O}_C(\xi)$ viewed as an $\mathcal{O}_S$-sheaf satisfies the Mukai–Artamkin criterion if and only if $h^1(\mathcal{O}_S(D)) = 0$. Furthermore, we have the exact triple

$$(2.1.21)\qquad \begin{tikzcd} 0 \arrow[r] & H^1(\mathcal{O}_C) \arrow[r] \arrow[d, equal] & T_{\mathcal{O}_C(\xi)} \arrow[r] & H^0(\mathcal{O}_C(D)) \arrow[r] \arrow[d, equal] & 0 \\ & T\mathrm{Pic}_{\,d}C_\xi & & T|D|_C & \end{tikzcd}$$

whence assertion 3). Assertion 2) can be checked directly: $d = \deg \xi$. □

Lemma 2.1.4. *Let $p_g(S) > 0$, $K_S \neq 0$, $C \in |D|_0$ and*

$$1)\ \dim|D| \geqslant 2\,, \qquad 2)\ h^1(\mathcal{O}_S(D)) = h^2(\mathcal{O}_S(D)) = 0\,.$$

Then for the family $\mathrm{Spl}_0(\mathcal{O}_C(\xi))$ *(see Lemma* 2.1.3*) each symplectic structure from $\tau(H^0(\Lambda^2T^*S))$ is degenerate and the local invariant of $\mathcal{O}_C(\xi)$ depends only on curve $C \in |D|$.*

Proof. For any $s \in H^0(K_S)$ the homomorphism $\tau(s)|_{\mathcal{O}_C(\xi)}$ can be extended to a homomorphism

$$(2.1.22)\qquad \begin{tikzcd} & & & 0 \arrow[d] & \\ 0 \arrow[r] & H^1(\mathcal{O}_C) \arrow[r] \arrow[d, "\otimes s"'] & T_{\mathcal{O}_C(\xi)} \arrow[r] \arrow[d, "\tau(s)"'] & H^0(\mathcal{O}_C(D)) \arrow[r] \arrow[d, "\otimes s"'] & 0 \\ 0 \arrow[r] & H^1(\mathcal{O}_C(K_S)) \arrow[r] \arrow[d, equal, "\text{SD}"'] & T^*_{\mathcal{O}_C(\xi)} \arrow[r] & H^0(\mathcal{O}_C(K_S + D)) \arrow[r] \arrow[d, equal, "\text{SD}"'] & 0 \\ & H^0(\mathcal{O}_C(D))^* \arrow[d] & & H^1(\mathcal{O}_C)^* & \\ & 0 & & & \end{tikzcd}$$

of exact triples. Under conditions 1) and 2), $K_S \cdot D > 0$, and the vertical homomorphism on the left-hand side has a kernel. The local invariant is given by a hypernet of contraction homomorphisms

$$H^0(K_S) \otimes H^0(\mathcal{O}_C(D)) \xrightarrow{\mu} H^0(K_C) \tag{2.1.23}$$

modulo $\mathrm{GL}(H^0(\mathcal{O}_C(D))) \times \mathrm{GL}(H^0(K_C))$ and does not depend on the sheaf $\mathcal{O}_C(\xi) \in \mathrm{Pic}\ _dC$. □

Corollary. *If C is a smooth rigid curve, i.e. $H^0(\mathcal{O}_C(C)) = 0$, then any invertible sheaf $\mathcal{O}_C(\xi)$ is compact modular:*

$$\mathrm{Spl}_0(\mathcal{O}_C(\xi)) = \mathrm{Pic}\ _dC,$$

and the homomorphism τ (2.1.7) for this the family vanishes.

Thus the symplectic structures induced on the components of the variety of moduli of sheaves exhaust all the possibilities of Definition 0.1.

In our examples the sheaves $J_\xi(D)$ were torsion-free but not locally free, and the sheaves $\mathcal{O}_C(\xi)$ were of cohomological dimension 1. We will see in this chapter that modular operations make them into bundles and preserve all modular properties and local invariants.

§ 2 The universal extension operation.

Operations on bundles preserving properties of their moduli can be found in one form or another in almost all papers on bundles. They have natural descriptions in the derived category of sheaves (A. Gorodentsev [7]), in the category of quadratic algebras (A. Bondal), in the category of representations of quivers (A. Bondal), etc. In this respect one can recall the concluding remark from [1] about "the unexplained manifestation of the mysterious unity of all phenomena". However we will not need this generality. We will only split the Mukai "reflection" operation (see [12] (2.22) and (2.24)) into the operations of universal extension and universal division, and also define a composite variant of these operations.

Remark. The composite division operation was cleverly used by Mestrano (see [10]) to disprove the existence of the fine varieties of moduli. Because of this we call it the *Mestrano operation* (see the next section).

Definition 2.2.1.

1) If E is a sheaf on S, then

$$E' = E \otimes K_S^* \tag{2.2.1}$$

is called the *derived sheaf of E*.

2) A pair of sheaves F_1 and F_2 is called a *regular pair* if

$$ {}^1\langle F_1 \mid F_2 \rangle = {}^1\langle F_2 \mid F_1 \rangle = 0 \tag{2.2.2} $$

3) A regular pair of sheaves is called *independent* if

$$ {}^0\langle F_1 \mid F_2 \rangle = {}^0\langle F_2 \mid F_1 \rangle. $$

For a pair of sheaves E and F, the extensions of the form

$$ 0 \longrightarrow {}^1\langle E' \mid F \rangle \otimes E \longrightarrow \varepsilon_E(F) \longrightarrow F \longrightarrow 0 \tag{2.2.3} $$

are given by the vectors of the space

$$ \begin{tikzcd}[column sep=small] {}^1\langle F \mid {}^1\langle E' \mid F \rangle\rangle \otimes E = {}^1\langle E' \mid F \rangle \otimes {}^1\langle F \mid E \rangle \arrow[d, equal, "\mathrm{SD}"'] = \operatorname{End} {}^1\langle E' \mid F \rangle \\ {}^1\langle E' \mid F \rangle^* , \end{tikzcd} \tag{2.2.4} $$

i.e., by the endomorphisms of the space ${}^1\langle E' \mid F \rangle$.

Applying the functor $\langle E' |$ to the exact triple (2.2.3), we obtain the long exact sequence

$$ \begin{tikzcd} {}^1\langle E' \mid F \rangle \otimes {}^1\langle E' \mid E \rangle \arrow[r] & {}^1\langle E' \mid \varepsilon_E(F) \rangle \arrow[r] & {}^1\langle E' \mid F \rangle \arrow[r, "\delta"] & {} \\ \arrow[r, "\delta"] & {}^1\langle E' \mid F \rangle \otimes {}^2\langle E' \mid E \rangle \arrow[r] \arrow[d, equal, "\mathrm{SD}"'] & {}^2\langle E' \mid \varepsilon_E(F) \rangle \arrow[r] & {}^2\langle E' \mid F \rangle \\ & {}^0\langle E \mid E \rangle^* & \mathbb{C} , \arrow[l, hook', "\mathrm{tr}^*"'] & \end{tikzcd} \tag{2.2.5} $$

where ${}^0\langle E \mid E \rangle \xrightarrow{\mathrm{tr}} \mathbb{C}$ is the trace homomorphism. It is not difficult to see that the cocycle corresponding to the extensions (2.2.3) is proportional to the endomorphism

$$ {}^1\langle E' \mid F \rangle \xrightarrow{\delta} {}^1\langle E' \mid F \rangle \otimes \mathrm{tr}^*(1). \tag{2.2.6} $$

Definition 2.2.3. The sheaf $\varepsilon_E(F)$ in (2.2.3), given by the cocycle δ (2.2.6) which is an isomorphism, is called the *universal extension of the sheaf F by the sheaf E*.

Clearly the $\mathcal{O}_S$-sheaf $\varepsilon_E(F)$ does not depend on the choice of an automorphism of ${}^1\langle E' \mid F \rangle$.

Let E be an exceptional sheaf (see Definition 1.1.7) and

$$ R(S) \subset F_m(S) \subset V_Z(S) \tag{2.2.7} $$

the set of exceptional sheaves in the big lattice (1.1.26).

Lemma 2.2.1. *If $E \in R(S)$, then*

$$\begin{aligned} {}^1\langle E' \mid \varepsilon_E(F)\rangle &= 0\,, \\ {}^2\langle E' \mid \varepsilon_E(F)\rangle &= {}^2\langle E' \mid F\rangle\,. \end{aligned} \tag{2.2.8}$$

Proof. By ${}^1\langle E \mid E\rangle \overset{\text{SD}}{=\!=} {}^1\langle E' \mid E\rangle$ the first term in (2.2.5) vanishes, and by ${}^0\langle E \mid E\rangle = \mathbb{C}$ the coboundary homomorphism δ is an isomorphism. □

Lemma 2.2.2. *If $E \in R(S)$ and ${}^0\langle E \mid F\rangle = 0$, then there is an exact sequence*

$$0 \longrightarrow {}^0\langle F' \mid E\rangle \otimes {}^1\langle E' \mid F\rangle \longrightarrow {}^0\langle \varepsilon_E(F) \mid \varepsilon_E(F)\rangle \longrightarrow {}^0\langle F \mid F\rangle. \tag{2.2.9}$$

Proof. The equality ${}^0\langle E \mid F\rangle = 0$ means that each endomorphism of $\varepsilon_E(F)$ preserves the exact triple (2.2.3). Hence

$$\begin{array}{ccccccccc} 0 \longrightarrow & {}^0\langle \varepsilon_E(F) \mid E\rangle & \otimes & {}^1\langle E' \mid F\rangle & \longrightarrow & {}^0\langle \varepsilon_E(F) \mid \varepsilon_E(F)\rangle & \longrightarrow & {}^0\langle F \mid F\rangle \\ & {\scriptstyle\text{SD}}\Big\| & & & & & & \\ & {}^2\langle E' \mid \varepsilon_E(F)\rangle^* & \overset{(2.1.8)}{=\!=\!=} & {}^2\langle E \mid F\rangle^* & \overset{\text{SD}}{=\!=\!=\!=\!=} & {}^0\langle F \mid E\rangle. & & \end{array}$$

□

Corollary. *If $E \in R(S)$, F is simple and*

$${}^0\langle E \mid F\rangle = {}^0\langle F \mid E\rangle = 0, \tag{2.2.10}$$

then $\varepsilon_E(F)$ is simple.

Lemma 2.2.3. *If equalities* (2.2.10) *hold and ${}^1\langle E|\varepsilon_E(F) = 0$, i.e., $(E, \varepsilon_E(F))$ is a regular pair* (see (2.2.2))*, then there is a canonical isomorphism*

$$\underbrace{{}^1\langle \varepsilon_E(F) \mid \varepsilon_E(F)\rangle}_{\substack{\| \\ T_{\varepsilon_E(F)}}} = \underbrace{{}^1\langle F \mid F\rangle}_{\substack{\| \\ T_F}} \tag{2.2.11}$$

and the local invariants (1.3.12) *of F and $\varepsilon_E(F)$ coincide.*

Proof. Applying $\langle E|$ to the triple (2.2.3), we have

$$0 \longrightarrow {}^1\langle E' \mid F\rangle \otimes \underset{\substack{\| \\ \mathbb{C}}}{{}^0\langle E \mid E\rangle} \longrightarrow {}^0\langle E \mid \varepsilon_E(F)\rangle \longrightarrow {}^0\langle E \mid F\rangle$$

Therefore ${}^0\langle E \mid F\rangle = 0$ implies

$${}^0\langle E \mid \varepsilon_E(F)\rangle = {}^1\langle E' \mid F\rangle. \tag{2.2.12}$$

Applying $|\varepsilon_E(F)\rangle$ to (2.2.3) we have

$$\begin{array}{ccccccc} & & \mathrm{End}\,{}^1\langle E' \mid F\rangle & & & & \\ & & \Big\|{\scriptstyle(2.2.12)} & & & & \\ {}^0\langle \varepsilon_E(F) \mid \varepsilon_E(F)\rangle & \longrightarrow & {}^1\langle E' \mid F\rangle^* \otimes {}^0\langle E \mid \varepsilon_E(F)\rangle & \longrightarrow & {}^1\langle E \mid \varepsilon_E(F)\rangle & \longrightarrow & \end{array}$$

$$\longrightarrow {}^1\big\langle \varepsilon_E(F) \,\big|\, \varepsilon_E(F)\big\rangle \to {}^1\big\langle E' \,\big|\, F\big\rangle^* \otimes {}^1\big\langle E \,\big|\, \varepsilon_E(F)\big\rangle$$
$$\Vert$$
$$0$$

and hence an exact triple

$$0 \longrightarrow \operatorname{ad} {}^1\big\langle E' \,\big|\, F\big\rangle \longrightarrow {}^1\big\langle F \,\big|\, \varepsilon_E(F)\big\rangle \longrightarrow {}^1\big\langle \varepsilon_E(F) \,\big|\, \varepsilon_E(F)\big\rangle \longrightarrow 0. \tag{2.2.13}$$

On the other hand, applying $\langle F|$ to (2.2.3), we have

$$\begin{tikzcd}[column sep=small]
 & & \operatorname{End}^1\big\langle E' \,\big|\, F\big\rangle \arrow[d, equal, "\textsc{sd}"', "(2.2.12)"] & & \\
0 \arrow[r] & {}^0\big\langle F \,\big|\, F\big\rangle \arrow[r] & {}^1\big\langle E' \,\big|\, F\big\rangle \otimes {}^0\big\langle F \,\big|\, E\big\rangle \arrow[r] & {}^1\big\langle F \,\big|\, \varepsilon_E(F)\big\rangle \arrow[r] & {}
\end{tikzcd}$$

$$\begin{tikzcd}[column sep=small]
{} \arrow[r] & {}^1\big\langle F \,\big|\, F\big\rangle \arrow[r] & {}^1\big\langle E' \,\big|\, F\big\rangle \otimes {}^2\big\langle F \,\big|\, E\big\rangle \arrow[d, equal, "\textsc{sd}"] \\
 & & {}^0\big\langle E' \,\big|\, F\big\rangle^*
\end{tikzcd}$$

and hence an exact triple

$$0 \longrightarrow \operatorname{ad} {}^1\big\langle E' \,\big|\, F\big\rangle \longrightarrow {}^1\big\langle F \,\big|\, \varepsilon_E(F)\big\rangle \longrightarrow {}^1\big\langle F \,\big|\, F\big\rangle \longrightarrow 0. \tag{2.2.13$'$}$$

The monomorphisms of (2.2.13) and (2.2.13') coincide, and, therefore, their terms on the right coincide too. This yields (2.2.11). Shifting these constructions by K_S we prove the equality of the local invariants.

THEOREM 2.2.1. *Let $E \in R(S)$ be an exceptional bundle and F a simple modular sheaf. If*

$$ {}^0\big\langle E \,\big|\, F\big\rangle = {}^0\big\langle F \,\big|\, E\big\rangle = {}^0\big\langle E' \,\big|\, F\big\rangle = 0, \quad {}^1\big\langle E \,\big|\, \varepsilon_E(F)\big\rangle = 0, \tag{2.2.14}$$

then

1) *$\varepsilon_E(F)$ is a simple modular sheaf,*

2) *$\operatorname{Spl}_0(F)$ is birationally isomorphic to $\operatorname{Spl}_0(\varepsilon_E(F))$, and*

3) *in $K^0_{\mathrm{alg}}(S)$ (see (1.1.3))*

$$\{\varepsilon_E(F)\} = \{F\} - \chi(F, E) \cdot \{E\}, \tag{2.2.15}$$

where $-\chi$ is the form (1.1.6).

PROOF. By (2.2.14)

$$\mathrm{rk}\,{}^1\langle E' \,|\, F\rangle = -\chi(E', F) \overset{\text{D.S.}}{=\!=} -\chi(F, E), \tag{2.2.16}$$

and we obtain (2.2.15).

Let $\{U_\alpha\}$ be an etale covering of $\mathrm{Spl}_0(F)$ (1.1.22) and $\{\mathcal{F}_\alpha\}$ a family of universal sheaves on $\{S \times U_\alpha\}$. For any $[F] \in U_\alpha$ consider the $\mathcal{O}_{U_\alpha}$-sheaf $\mathcal{E}xt^1_{U_\alpha}(\pi_S^* E', \mathcal{F}_\alpha)$ and the extension

$$0 \longrightarrow \pi_{U_\alpha}^* \mathcal{E}xt^1_{U_\alpha}(\pi_S^* E', \mathcal{F}_\alpha) \otimes \pi_S^* E \longrightarrow \varepsilon_E(\mathcal{F}_\alpha) \longrightarrow \mathcal{F}_\alpha \longrightarrow 0 \tag{2.2.17}$$

of sheaves on $S \times U_\alpha$ given by the identity isomorphism

$$\mathrm{Id}: \quad \mathcal{E}xt^1_{U_\alpha}(\mathcal{F}_\alpha, \pi_S^* E) \longrightarrow \mathcal{E}xt^1_{U_\alpha}(\mathcal{F}_\alpha, \pi_S^* E)$$

(in the stalkwise sense (2.2.4)). By Lemma 2.2.2 and its corollary, $\varepsilon_E(\mathcal{F}_\alpha)$ is $\mathcal{O}_{U_\alpha}$-simple in a Zariski neighborhood U'_α of the sheaf $[\varepsilon_E(\mathcal{F}_\alpha)]$. By (2.2.11), we only have to check that in some Zariski neighborhood U''_α any sheaf F_1 can be recovered from $\varepsilon_E(F_1)$. To this end define $U''_\alpha \subset U'_\alpha$ by

$$U''_\alpha = \{[F_1] \in U'_\alpha | {}^0\langle E \,|\, F_1\rangle = 0\}.$$

Then for the canonical homomorphism

$${}^0\langle E \,|\, \varepsilon_E(F_1)\rangle \otimes E \xrightarrow{\text{can}} \varepsilon_E(F_1) \tag{2.2.18}$$

we have

$$F = \text{coker can} \tag{2.2.19}$$

Thus $\{U''_\alpha\}$ is an etale covering of $\mathrm{Spl}_0(\varepsilon_E(F)) \ni \varepsilon_E(F)$ and $\{\varepsilon_E(\mathcal{F}_\alpha)\}$ is the family of universal sheaves (1.1.22). $\square$

EXAMPLE. Let S be a K3 surface. Then $E' = E$, ${}^1\langle E \,|\, F\rangle \overset{\text{SD}}{=\!=\!=} {}^1\langle F \,|\, E\rangle^*$, the form $-\chi(F_1, F_2)$ (1.1.6) is symmetric, and

$$R(S) = \{\, F \mid {}^0\langle F \,|\, F\rangle = \mathbb{C}\,,\ -\chi(F, F) = -2\,\} \tag{2.2.20}$$

(see, for example, [12],§ 3).

Conditions (2.2.14) of Theorem 2.2.1 have the form

$${}^0\langle E \,|\, F\rangle = {}^2\langle E \,|\, F\rangle = 0, \tag{2.2.14$'$}$$

and the equality ${}^1\langle E \,|\, \varepsilon_E(F)\rangle = 0$ is automatically satisfied. Therefore, transformation (2.2.15) is the involution of the lattice with respect to a root vector. Furthermore for the sheaves $E = \mathcal{O}_S(C) \in R(S)$ and $F = J_\xi, \quad \xi \in \tilde{S}^{(d)}$ (see (2.1.4) and (2.1.5)) conditions (2.2.14') of Theorem 2.2.1 have the form

$$h^0(J_\xi(-C)) = h^2(J_\xi(-C)) \overset{(2.1.8)}{=\!=\!=} h^2(\mathcal{O}_S(-C)) \overset{\text{SD}}{=\!=\!=} h^2(\mathcal{O}_S(C)) = 0.$$

Therefore, if $h^0(\mathcal{O}_S(C))=0$, then for a general $\xi\in\tilde S^{(d)}$ with $d\geqslant h^0(\mathcal{O}_S(-C))+1$ we have

$$\mathrm{Spl}_0(\varepsilon_{\mathcal{O}_S(C)}(J_\xi))\overset{\mathrm{bir}}{\sim} S^{(d)}.$$

It is proved in [13] that for a general ξ the sheaf $\varepsilon_{\mathcal{O}_S(C)}(J_\xi)$ is simple and locally free, i.e., $\varepsilon_{\mathcal{O}_S(C)}(J_\xi)$ is a bundle.

It is not easy to check conditions (2.2.14) of Theorem 2.2.1 for an arbitrary regular surface S. Any reflexive sheaf on the surface is locally free. Thus to save space we will express the local freeness of a sheaf F by the equality $F^{**}=F$.

Lemma 2.2.4. *If $E\in R(S)$, $E^{**}=E$, $\varepsilon_E(F)^{**}=\varepsilon_E(F)$, and for of a curve $C\in|K_S|$*

$$^0\big\langle E'\,\big|\,_C, F|_C\big\rangle=0, \tag{2.2.21}$$

then $^1\big\langle E\,\big|\,\varepsilon_E(F)\big\rangle=0$.

Proof. Restricting the triple (2.2.3) to $C\in|K_S|$, we have

$$0\longrightarrow {}^1\big\langle E'\,\big|\,F\big\rangle\otimes E|_C\longrightarrow \varepsilon_E(F)|_C\longrightarrow F|_C\longrightarrow 0.$$

By (2.2.11),

$$^0\big\langle E'\,\big|\,_C|\varepsilon_E(F)|_C\big\rangle={}^0\langle E'|_C|^1\big\langle E'\,\big|\,F\big\rangle\otimes E|_C\rangle. \tag{2.2.22}$$

The cohomology sequence of the exact triple

$$0\longrightarrow E^*\otimes E\longrightarrow E'^*\otimes E\xrightarrow{\mathrm{res}}(E'^*\otimes E)|_C\longrightarrow 0$$

shows, that $^0\big\langle E'\,\big|\,E\big\rangle\xrightarrow{\mathrm{res}}{}^0\big\langle E'\,\big|\,_C|E|_C\big\rangle$ is an epimorphism ($^1\big\langle E\,\big|\,E\big\rangle=0$) and, therefore, the restriction homomorphism

$$\begin{array}{ccc} ^0\big\langle E'\,\big|\,\varepsilon_E(F)\big\rangle & \xrightarrow{\mathrm{res}} & ^0\big\langle E'\,\big|\,_C,\varepsilon_E(F)|_C\big\rangle \\ & & \|(2.2.22) \\ & & ^1\big\langle E'\,\big|\,F\big\rangle\otimes{}^0\big\langle E'\,\big|\,_C|E|_C\big\rangle. \end{array} \tag{2.2.23}$$

is also an epimorphism. But the cohomology sequence of the triple

$$0\longrightarrow E'\otimes\varepsilon_E(F)\longrightarrow E'^*\otimes\varepsilon_E(F)\longrightarrow (E'^*\otimes\varepsilon_E(F))|_C\longrightarrow 0$$

is of the form

$$\begin{aligned} 0\longrightarrow {}^0\big\langle E\,\big|\,\varepsilon_E(F)\big\rangle\longrightarrow {}^0\big\langle E'\,\big|\,\varepsilon_E(F)\big\rangle\xrightarrow{\mathrm{res}}{}^0\big\langle E'\,\big|\,_C|\varepsilon_E(F)|_C\big\rangle\longrightarrow \\ \longrightarrow {}^1\big\langle E\,\big|\,\varepsilon_E(F)\big\rangle\longrightarrow {}^1\big\langle E'\,\big|\,\varepsilon_E(F)\big\rangle\overset{(2.1.8)}{=\!=\!=} 0. \end{aligned}$$

Now the surjectivity of (2.2.23) implies $^1\big\langle E\,\big|\,\varepsilon_E(F)\big\rangle=0$. $\square$

We can now apply the construction from [13],§ 4, to a surface S of general type.

THEOREM 2.2.2. *Suppose that $|K_S|$ contains an irreducible curve C. Then for any divisor class $C \in \operatorname{Pic} S$ with*

$$h^0(\mathcal{O}_S(C)) = 0, \quad C \cdot K_S > K_S^2, \tag{2.2.24}$$

and any integer $d > h^0(\mathcal{O}_S(K_S - C))$

1) *the sheaf $\varepsilon_{\mathcal{O}_S(C)}(J_\xi)$ is locally free and simple for a general $\xi \in \tilde{S}^{(d)}$, and*

2) $\operatorname{Spl}_0(\varepsilon_{\mathcal{O}_S(C)}(J_\xi)) \overset{\text{bir}}{\sim} \tilde{S}^{(d)}$.

PROOF. Let us check that the pair $\mathcal{O}_S(C)$, J_ξ satisfies conditions (2.2.14) of Theorem 2.2.1:

$$h^0(J_\xi(-C)) = 0, \quad {}^0\langle J_\xi, \mathcal{O}_S(C)\rangle \overset{\text{SD}}{=\!=\!=} h^2(J_\xi(K_S - C))^* =$$
$$\overset{(2.1.8)}{=\!=\!=} h^2(\mathcal{O}_S(K_S - C))^* \overset{\text{SD}}{=\!=\!=} h^0(\mathcal{O}_S(C)) = 0.$$

Under these conditions, for a general ξ

$$d > h^0(\mathcal{O}_S(K_S - C)) \Rightarrow h^0(J_\xi(K_S - C)) = 0$$

and, according to Lemma 1.2 of [13],

$$\varepsilon_{\mathcal{O}(C)}(J_\xi)^{**} = \varepsilon_{\mathcal{O}(C)}(J_\xi).$$

For an irreducible curve $K \in |K_S|$ and a general ξ we have $J_\xi \otimes \mathcal{O}_K = \mathcal{O}_K$ and condition (2.2.21) of Lemma 2.2.4 is equivalent to $h^0(\mathcal{O}_K(K_S - C)) = 0$. But for an irreducible curve K

$$K_S(K_S - C) < 0 \Rightarrow h^0(\mathcal{O}_K(K_S - C)) = 0$$

and condition (2.2.21) is satisfied. Then, by Lemma 2.2.4, ${}^1\big\langle \mathcal{O}_S(C) \,\big|\, \varepsilon_{\mathcal{O}(C)}(J_\xi)\big\rangle = 0$ and the last condition (2.2.14) of Theorem 2.2.1 is satisfied. □

Clearly $\operatorname{Pic} S = \mathbb{Z} \cdot K_S$ and the class of $\mathcal{O}_S(C)$ satisfying conditions (2.2.24) is empty. However if the Picard number $\rho(S) > 1$ then the half-space

$$E_K = \{C \in \operatorname{Pic} S | C \cdot K_S > \text{const} = K_S^2\}$$

in the lattice $\operatorname{Pic} S$ cannot be contained in the convex semicone of effective divisors

$$C^+ = \{C \in \operatorname{Pic} S | h^0(\mathcal{O}_S(C)) > 0\},$$

and therefore any class

$$C \in E_K - E_K \bigcap C^+ \tag{2.2.25}$$

satisfies conditions (2.2.24).

Moreover, for any such class with $h^0(\mathcal{O}_S(K_S - C)) = 0$ and any d and $\xi \in \tilde{S}^{(d)}$

1) $\varepsilon_{\mathcal{O}(C)}(J_\xi)$ is a compact modular bundle, and

2) $\mathrm{Spl}_0(\varepsilon_{\mathcal{O}(C)}(J_\xi)) = \tilde{S}^{(d)}$.

Remark. In particular, if $d = 1$ we obtain an infinite family of moduli varieties of bundles, isomorphic to S itself.

It is very important that we obtained precisely these bundles, since if they are stable one can define a Hermite–Einstein metric on them and, integrating it, obtain a Weil–Petersson metric on their moduli varieties, i.e., on S itself. A method of Itô allows us to calculate the curvature of this metric.

Unlike the case of a K3 surface (see the example), for a surface of general type the form $-\chi$ (1.1.6) is not symmetric and the transformation (2.2.15) of the lattice $K^0_{\mathrm{alg}}(S)$ is not $-\chi$-orthogonal.

Lemma 2.2.5. *If $|K_S|$ contains an irreducible curve C, $K_S^2 > 0$, and $C \in \operatorname{Pic} S$ satisfies conditions* (2.2.24)*, then the $\mathbb{Z}$-linear transformation*

1) $\varepsilon_{\mathcal{O}(C)} : K^0_{\mathrm{alg}}(S) \longrightarrow K^0_{\mathrm{alg}}(S)$,

$$\varepsilon_{\mathcal{O}(C)}(f) = f - \chi(f, \{\mathcal{O}_S(C)\})\{\mathcal{O}_S(C)\}, \tag{2.2.26}$$

does not preserve the form $-\chi$ (1.1.6)*, and*

2) *if $d > h^0(\mathcal{O}_S(K_S - C))$, then*

$$\begin{gathered}-\chi(J_\xi, J_\xi) + \chi(\varepsilon_{\mathcal{O}(C)}(J_\xi), \varepsilon_{\mathcal{O}(C)}(J_\xi)) = \\ = \operatorname{rk} h^0(\operatorname{ad} \varepsilon_{\mathcal{O}(C)}(J_\xi) \otimes K_S) \geqslant r^2(p_g - 1) + K_S^2 r,\end{gathered} \tag{2.2.27}$$

where $r + 1 = \operatorname{rk} \varepsilon_{\mathcal{O}(S}(J_\xi)$.

Proof. Returning to§ 2 of Chapter 1 (1.2.17)–(1.2.22) we have

$$\begin{gathered}
v(J_\xi) = (1, 0, \chi/2 - d), \quad v(\mathcal{O}_S(C)) = \left(1, C, \tfrac{1}{2}C^2 + \chi/2\right), \\
-\chi(J_\xi, \mathcal{O}_S(C)) \overset{(1.2.21)}{=\!=\!=} \underbrace{(v(J_\xi), v(\mathcal{O}_S(C)))}_{\substack{\| \\ d - \frac{1}{2}C^2 - \chi}} + \underbrace{(v(J_\xi), v(\mathcal{O}_S(C)))}_{\substack{\| \\ \frac{1}{2}K_S \cdot C}} = \\
= d - \tfrac{1}{2}C^2 - \chi + \tfrac{1}{2}K_S \cdot C = r \geqslant 1, \\
-\chi(\mathcal{O}_S(C), J_\xi) = d - \tfrac{1}{2}C(C - K_S) - \chi, \\
v(\varepsilon_{\mathcal{O}(C)}(J_\xi)) = \left(r + 1, rC, \tfrac{r}{2}C^2 - d + (r+1)\tfrac{\chi}{2}\right), \\
v^2(\varepsilon_{\mathcal{O}(C)}(J_\xi)) = r^2C^2 - r(r+1)C^2 + 2d(r+1) - (r+1)^2\chi = \\
= \underbrace{2d - \chi}_{\substack{\| \\ v^2(J_\xi)}} + 2r\,\underbrace{(d - \frac{1}{2}C^2 - \chi}_{\substack{\| \\ r - \frac{1}{2}C \cdot K_S}} - r\tfrac{\chi}{2}) = v^2(J_\xi) - r^2(p_g - 1) - \tfrac{r}{2}C \cdot K_S.
\end{gathered}$$

On the other hand, ${}^i\langle J_\xi | J_\xi\rangle =^i \langle \varepsilon_{\mathcal{O}(C)}(J_\xi) | \varepsilon_{\mathcal{O}(C)}(J_\xi)\rangle$ when $i = 0, 1$. This gives us assertion 2). □

REMARK. Thus $\varepsilon_{\mathcal{O}(C)}(J_\xi)$ is a simple modular bundle which does not satisfy the Mukai–Artamkin modularity criterion.

That is why the concept of modularity is needed: modular operations preserve properties of moduli but invalidated the usual sufficient conditions like the Mukai–Artamkin criterion or stability.

Let us return to the beginning of the section. Any subspace $V \subset {}^1\langle E' \mid F\rangle$ gives rise to an extension

$$0 \longrightarrow ({}^1\langle E' \mid F\rangle/V)\otimes E \longrightarrow \varepsilon_E^V(F) \longrightarrow F \longrightarrow 0, \tag{2.2.3$'$}$$

given by a cocycle-epimorphism

$${}^1\langle E' \mid F\rangle \xrightarrow{\delta} ({}^1\langle E' \mid F\rangle/V)\otimes \mathrm{tr}^*(1), \quad \ker\delta = V. \tag{2.2.6$'$}$$

Let $F = \bigotimes_{i=1}^{d} F_i$ and $\mathrm{rk}\,{}^1\langle E' \mid F_i\rangle = n$. Then

$${}^1\langle E' \mid F\rangle = \bigotimes_{i=1}^{d} {}^1\langle E' \mid F_i\rangle\,, \qquad \mathrm{rk}^1\langle E' \mid F\rangle = dn$$

and the subgroup $\mathrm{GL}({}^1\langle E' \mid F_1\rangle)\times\cdots\times\mathrm{GL}({}^1\langle E' \mid F_d\rangle) \subset \mathrm{GL}({}^1\langle E' \mid F\rangle)$ acts on the Grassmannian $\mathrm{Gr}(n, {}^1\langle E' \mid F\rangle)$, where it has an open orbit U_0. It is easy to see that the sheaf $\varepsilon_E^V(F)$ does not depend on the choice of $V \in U_0$. We denote it by $\varepsilon_E^0\left(\overset{d}{\underset{i=1}{\oplus}} F_i\right)$.

Definition 2.2.3. The sheaf $\varepsilon_E^0\left(\overset{d}{\underset{i=1}{\oplus}} F_i\right)$ will be called the *composite universal extension.*

The proof of the next theorem is completely analogous to that of Theorem 2.2.1.

THEOREM 2.2.3. *Let E be an exceptional bundle and $(F_1,\dots,F_d)$ a set of modular bundles such that*

1) $\forall i \quad \mathrm{Spl}_0(F_1) = \mathrm{Spl}_0(F_i)$,

2) $\forall i \neq j \quad (F_i, F_j)$ *is an independent pair* (Definition (2.2.1, 3)), *and*

3) $\forall i {}^0\langle E \mid F_i\rangle = {}^0\langle F_i \mid E\rangle = 0,\ {}^1\left\langle E|\varepsilon_E^0\left(\overset{d}{\underset{i=1}{\oplus}} F_i\right)\right\rangle = 0.$

Then

1) $\varepsilon_E^0\left(\overset{d}{\underset{i=1}{\oplus}} F_i\right)$ *is a simple modular bundle,*

2) $\mathrm{Spl}_0\left(\varepsilon_E^0\left(\overset{d}{\underset{i=1}{\oplus}} F_i\right)\right) \overset{\mathrm{bir}}{\sim} (\mathrm{Spl}_0(F_1))^{(d)}$ *where (d) is the dth symplectic power,*

3) $T_{\varepsilon^0_E\left(\overset{d}{\underset{i=1}{\oplus}} F_i\right)} = \overset{d}{\underset{i=1}{\oplus}} T_{F_i}$,

4) *and for the hypernet of the local invariant* (1.3.12)*we have*

$$\tau_{\varepsilon^0_E\left(\overset{d}{\underset{i=1}{\oplus}} F_i\right)} = \overset{d}{\underset{i=1}{\oplus}} \tau_{F_i}.$$

§ 3 The universal division operation.

An ordered pair (E, F) of sheaves on a smooth variety gives rise to a pair of canonical homomorphisms

$$(2.3.1)\qquad {}^0\big\langle E \,\big|\, F\big\rangle \otimes E \xrightarrow{\text{can}} F, \quad E \xrightarrow{\text{can}^T} {}^0\big\langle E \,\big|\, F\big\rangle^* \otimes F,$$

which can be to extended to exact quadruples:
(2.3.2)

$$C(E,F):\quad 0 \longrightarrow \alpha_E(F) \longrightarrow {}^0\big\langle E \,\big|\, F\big\rangle \otimes E \xrightarrow{\text{can}} F \longrightarrow \gamma_E(F) \longrightarrow 0$$

$$C^T(E,F):\quad 0 \longrightarrow \alpha^T_E(F) \longrightarrow E \xrightarrow{\text{can}^T} {}^0\big\langle E \,\big|\, F\big\rangle^* \otimes F \longrightarrow \gamma^T_E(F) \longrightarrow 0$$

If all the sheaves in these sequences are locally free, then

$$(2.3.3)\qquad C^T(E,F) = C(F^*, E^*)^*.$$

Lemma 2.3.1. *If* $E \in R(B)$, *then* ${}^0\big\langle E \,\big|\, \alpha_E(F)\big\rangle = {}^1\big\langle E \,\big|\, \alpha_E(F)\big\rangle = 0$.

PROOF. Let $\operatorname{im}\ \text{can} = \beta_E(F) \subset F$. Then ${}^0\big\langle E \,\big|\, F\big\rangle = {}^0\big\langle E \,\big|\, \beta_E(F)\big\rangle$. Applying the functor $\langle E|$ to the exact triple

$$(2.3.4)\qquad 0 \longrightarrow \alpha_E(F) \longrightarrow {}^0\big\langle E \,\big|\, F\big\rangle \otimes E \xrightarrow{\text{can}} \beta_E(F) \longrightarrow 0$$

we obtain

$$\begin{array}{ccccccc}
0 \longrightarrow {}^0\big\langle E \,\big|\, \alpha_E(F)\big\rangle \longrightarrow & {}^0\big\langle E \,\big|\, F\big\rangle \otimes {}^0\big\langle E \,\big|\, E\big\rangle & \xrightarrow{\sim} & {}^0\big\langle E \,\big|\, \beta_E(F)\big\rangle \longrightarrow \\
& \| & & \| \\
& \mathbb{C} & & {}^0\big\langle E \,\big|\, F\big\rangle
\end{array}$$

$$\longrightarrow {}^0\big\langle E \,\big|\, \alpha_E(F)\big\rangle \longrightarrow {}^0\big\langle E \,\big|\, F\big\rangle \otimes {}^1\big\langle E \,\big|\, E\big\rangle = 0.$$

The lemma follows. □

Lemma 2.3.2. *If E is an exceptional bundle and (E,F) is a regular pair of bundles* (see (2.2.2)) *with $\gamma_E(F)=0$, then there exists an isomorphism*

$$ {}^0\big\langle E \,\big|\, F\big\rangle^* = {}^0\big\langle E^* \,\big|\, \alpha_E(F)^*\big\rangle, \tag{2.3.5}$$

with respect to which

$$ C(E,F)^* = C(E^*, \alpha_E(F)^*), \tag{2.3.6}$$

i.e.,

$$ \alpha_{E^*}(\alpha_E(F)^*) = F^*. \tag{2.3.7}$$

Proof. Applying ${}^0\langle E^*|$ to the triple $C(E,F)^*$, we have

$$\begin{tikzcd}
0 \arrow[r] & {}^0\big\langle E^* \,\big|\, F^*\big\rangle \arrow[r] & {}^0\big\langle E \,\big|\, F\big\rangle^* \otimes {}^0\big\langle E^* \,\big|\, E^*\big\rangle \arrow[r] \arrow[d, equal] & {}^0\big\langle E^* \,\big|\, \alpha_E(F)^*\big\rangle \arrow[r] & {}^1\big\langle E^* \,\big|\, F^*\big\rangle \arrow[d, equal] \\
 & & \mathbb{C} & & {}^1\big\langle E \,\big|\, F\big\rangle = 0.
\end{tikzcd}$$

We obtain the isomorphism (2.3.5) and prove (2.3.6) and (2.3.7) if we show that ${}^0\big\langle E^* \,\big|\, F^*\big\rangle = {}^0\big\langle F \,\big|\, E\big\rangle$. But if ${}^0\big\langle F \,\big|\, E\big\rangle \neq 0$ then, because $\gamma_E(F)=0$, we can find a nontrivial endomorphism $E \longrightarrow E$, since either $\operatorname{rk}{}^0\big\langle E \,\big|\, F\big\rangle > 1$ or $\operatorname{rk} E > \operatorname{rk} F$, which contradicts the simplicity of E. □

Lemma 2.3.3. *If (E,F) is a regular pair of bundles and $\alpha_E^T(F)=0$, then*

$$ \alpha_{F^*}(E^*) = \gamma_E^T(F)^*, \quad \alpha_F(\gamma_E^T(F)) = E. \tag{2.3.8}$$

Proof. If (E,F) is a regular pair then (F^*,E^*) is also a regular pair, and (2.3.3) together with (2.3.4) implies (2.3.8). □

Lemma 2.3.4. *If (E,F) is a regular pair of bundles and $\gamma_E(F)=0$ then*

$$ {}^0\big\langle E \,\big|\, F\big\rangle = {}^0\big\langle \alpha_E(F) \,\big|\, \alpha_E(F)\big\rangle. \tag{2.3.9}$$

In particular, if F is simple then $\alpha_E(F)$ is also simple.

Proof. Applying $\langle \alpha_E(F)|$ to $C(E,F)$ and $\langle F^*|$ to $C(E^*,\alpha_E(F)^*)$, we have

(2.3.10)

$$\begin{tikzcd}
0 \arrow[r] & {}^0\big\langle \alpha_E(F) \,\big|\, \alpha_E(F)\big\rangle \arrow[r] & {}^0\big\langle E \,\big|\, F\big\rangle \arrow[dr, equal] & \otimes\, {}^0\big\langle \alpha_E(F) \,\big|\, E\big\rangle \arrow[r, "\sigma_1"] \arrow[dl, equal] & {}^0\big\langle \alpha_E(F) \,\big|\, E\big\rangle \arrow[d, equal] \\
0 \arrow[r] & {}^0\big\langle F^* \,\big|\, F^*\big\rangle \arrow[r] & {}^0\big\langle E^* \,\big|\, \alpha_E(F)^*\big\rangle \otimes & {}^0\big\langle F^* \,\big|\, E^*\big\rangle \arrow[r, "\sigma_2"] & {}^0\big\langle F^* \,\big|\, \alpha_E(F)^*\big\rangle
\end{tikzcd}$$

It is easy to see that σ_i's are the standard contraction homomorphisms (Yoneda's pairing), and under the above identification $\sigma_1=\sigma_2=\sigma$. Hence

$$ {}^0\big\langle E \,\big|\, F\big\rangle = \ker\sigma = {}^0\big\langle \alpha_E(F) \,\big|\, \alpha_E(F)\big\rangle. $$

Continuing the sequences (2.3.10), we obtain

(2.3.11)
$$\begin{tikzcd}[column sep=small] & & {}^1\langle \alpha_E(F) \,|\, \alpha_E(F)\rangle \arrow[r] & {}^0\langle E \,|\, F\rangle \otimes {}^1\langle \alpha_E(F) \,|\, E\rangle \\ 0 \arrow[r] & \operatorname{coker}\sigma \arrow[ur] \arrow[dr, "\shortparallel"'] & & \\ & & {}^1\langle F^* \,|\, F^*\rangle \arrow[r] & {}^0\langle E^* \,|\, \alpha_E(F)^*\rangle \otimes^1 \langle F^* | E^*\rangle = 0 \end{tikzcd}$$

and the beginning of the upper sequence of (2.3.11) gives rise to a monomorphism

$$\begin{array}{ccc} d\alpha_E : {}^1\langle F \,|\, F\rangle & \longrightarrow & {}^1\langle \alpha_E(F) \,|\, \alpha_E(F)\rangle. \\ \| & & \| \\ T_F & & T_{\alpha_E(F)} \end{array} \tag{2.3.12}$$

If, in addition, the pair $(E, \alpha_E(F))$ is regular, then $d\alpha_E$ is an isomorphism.

It is not difficult to check that if $B = S$ is a regular surface then $d\alpha_E$ gives rise to a homomorphism of hypernets (1.3.12) of the local invariants of F and $\alpha_E(F)$, which is an equivalence of the invariants if $d\alpha_E$ is an isomorphism.

THEOREM 2.3.1. *Let E be an exceptional bundle, F a simple modular bundle, (E, F) a regular pair, $\gamma_E(F) = 0$, and $(E, \alpha_E(F))$ a regular pair. Then*

1) *$\alpha_E(F)$ is a simple modular bundle,*

2) *$\mathrm{Spl}_0(F)$ is birationally isomorphic to $\mathrm{Spl}_0(\alpha_E(F))$, and*

3) *if ${}^i\langle E \,|\, F\rangle = 0$ when $i > 0$, then in K^0_{alg} (see (1.1.3))*

$$\{\alpha_E(F)\} = -\{F\} + \chi(E, F)\{E\}. \tag{2.3.13}$$

PROOF. For a $U_\alpha \ni [F]$ from the covering (1.1.22) and the universal sheaf $\mathcal{F}_\alpha$ on $B \times U_\alpha$ the canonical homomorphism

$$\pi_U^* : \mathcal{E}xt^0_{U_\alpha}(\pi_B^* E, \mathcal{F}_\alpha) \otimes \pi_B^* E \xrightarrow{\text{can}} \mathcal{F}_\alpha$$

is an epimorphism on $B \times U'_\alpha$, where U'_α is a Zariski neighborhood of the point $[F]$. Moreover we can find a neighborhood $U''_\alpha \subset U_\alpha$ such that $\alpha_E(\mathcal{F}_\alpha) = \ker \operatorname{can}|_{B\times U_\alpha}$ is a U_α-simple flat sheaf. Formula (2.3.8) recovers the sheaf F_1, $[F_1] \in U''_\alpha$, from α, and, therefore, $\alpha_E(\mathcal{F}_\alpha)$ is a modular family. Assertions 1) and 2) now follow, and $C(E, F)$ yields assertion 3). □

EXAMPLE. Let us return to the example of the previous section: $B = S$ is a K3 surface. Then the conditions of Theorem 2.3.1 are as follows: ${}^1\langle E \,|\, F\rangle = 0$. By (2.2.20) the transformation α_E (2.3.13) of the $\mathbb{Z}$-module $K^0_{\mathrm{alg}}(S)$ is a reflection in the lattice with respect to the base vector $\{E\}$.

To obtain a sufficient criterion of the regularity of the pair $(E, \alpha_E(F))$ we restrict ourselves to the case when $B = S$ is a regular surface.

REMARK. The assertion "the complete linear series $|C|$ on S is base points free" will take the form $\gamma_{\mathcal{O}}(\mathcal{O}_S(C)) = 0$ in the notation (2.3.2).

Lemma 2.3.5. *If $E \in R(S)$, $E^{**} = E$, $\gamma_E(F) = 0$ and the contraction homomorphism*

$$\sigma' : {}^0\big\langle E' \,\big|\, E\big\rangle \otimes {}^0\big\langle E \,\big|\, F\big\rangle \longrightarrow {}^0\big\langle E' \,\big|\, F\big\rangle \tag{2.3.14}$$

is surjective, then the pair $(E, \alpha_E(F))$ is regular.

PROOF. ${}^1\big\langle \alpha_E(F) \,\big|\, E\big\rangle \overset{\text{SD}}{=\!=} {}^1\big\langle E' \,\big|\, \alpha_E(F)\big\rangle$. Applying ${}^0\langle E'|$ to $C(E, F)$ we have

$$\begin{aligned} 0 \longrightarrow {}^0\big\langle E' \,\big|\, \alpha_E(F)\big\rangle &\longrightarrow {}^0\big\langle E \,\big|\, F\big\rangle \otimes {}^0\big\langle E' \,\big|\, E\big\rangle \xrightarrow{\sigma'} {}^0\big\langle E' \,\big|\, F\big\rangle \longrightarrow \\ \longrightarrow {}^1\big\langle E' \,\big|\, \alpha_E(F)\big\rangle &\longrightarrow {}^0\big\langle E \,\big|\, F\big\rangle \otimes {}^1\big\langle E' \,\big|\, E\big\rangle \\ &\qquad\qquad \big\|_{\text{SD}} \\ &\qquad {}^1\big\langle E \,\big|\, E\big\rangle = 0. \end{aligned}$$

The surjectivity of σ' implies ${}^1\big\langle E' \,\big|\, \alpha_E(F)\big\rangle$. □

THEOREM 2.3.2. *Suppose that $\gamma_{\mathcal{O}}(K_S) = 0$, $\mathcal{O}_S(1) \in \operatorname{Pic} S$ is ample, $E \in R(S)$, and $E^{**} = E$. Then for any sheaf F we can find a number d_0 such that for any $d \geqslant d_0$*

1) *the pair $(E(-d), F)$ is regular and ${}^1\big\langle E(-d) \,\big|\, F\big\rangle = 0$,*

2) $\gamma_{E(-d)}(F) = 0$,

3) *pair $(E, \alpha_E(F))$ is regular.*

PROOF. By the theorem of Serre we can choose a d_0' such that for $d \geqslant d_0'$ condition 1) holds. Let us choose a $d_0'' \geqslant d_0'$ such that $\gamma_{\mathcal{O}}(E^* \otimes F(d)) = 0$ if $d \geqslant d_0''$. Tensoring the epimorphism

$$H^0(E^* \otimes F(d)) \otimes \mathcal{O}_S \xrightarrow{\text{can}} E^* \otimes F(d) \longrightarrow 0$$

by $E(-d)$, we have

$$\begin{tikzcd} {}^0\big\langle E(-d) \,\big|\, F\big\rangle \otimes E(-d) \arrow[r] \arrow[dr, "\text{can}"] & \operatorname{End} E \otimes F \arrow[r] \arrow[d] & 0 \\ & F \arrow[d] & \\ & 0 & \end{tikzcd} .$$

Assertion 2) now follows.

Now choose a $d_0 \geqslant d_0''$ so that $h^1(\alpha_{\mathcal{O}}(K_S) \otimes E^* \otimes F(d)) = 0$. Then the cohomology sequence of the triple $C(\mathcal{O}_S, K_S)$ tensored by $E^* \otimes F(d)$ yields an epimorphism

$$H^0(K_S) \otimes {}^0\big\langle E(-d) \,\big|\, F \big\rangle \longrightarrow {}^0\big\langle E'(-d) \,\big|\, F \big\rangle \longrightarrow 0.$$

But $H^0(K_S)$ is a direct summand of ${}^0\big\langle E'(-d) \,\big|\, E(-d) \big\rangle$, and the surjectivity on this summand implies the surjectivity of all of σ' (2.3.14). Assertion 3) follows from Lemma 2.3.5. □

Definition 2.3.1. The passage to the sheaf $\alpha_E(F)$ from the sheaf F is called the *operation of universal division (by E).*

Consider the modular family (2.1.17).

THEOREM 2.3.3. *Suppose that $\gamma_{\mathcal{O}}(K_S) = 0$ and $\mathcal{O}_S(1)$ is ample. Then there exists a number N_0 such that, if $N > N_0$, then*

1) *for a general $\mathcal{O}_C(\xi) \in \mathrm{Pic}_d|D|_0$ (2.1.17) the sheaf $\alpha_{\mathcal{O}(-N)}(\mathcal{O}_C(\xi))$ is locally free and simple*,

2) $\mathrm{Spl}_0(\alpha_{\mathcal{O}(-N)}(\mathcal{O}_C(\xi))) \overset{\mathrm{bir}}{\sim} \mathrm{Pic}_d|D|_0$, *and*

3) *the local invariants of $\mathcal{O}_C(\xi)$ and $\alpha_{\mathcal{O}(-N)}(\mathcal{O}_C(\xi))$ coincide.*

PROOF. If $C \in |D|_0$ is smooth then the stalks of $\mathcal{O}_C(\xi)$ have homological dimension 1 at all points. By the syzygy theorem, if $\gamma_{\mathcal{O}(-N)}(\mathcal{O}_C(\xi)) = 0$, then the local freeness of ${}^0\big\langle \mathcal{O}_S(-N) \,\big|\, \mathcal{O}_C(\xi) \big\rangle \otimes \mathcal{O}_S(-N)$ in the triple $C(\mathcal{O}_S(-N), \mathcal{O}_C(\xi))$ implies the local freeness of $\alpha_{\mathcal{O}_S(-N)}(\mathcal{O}_C(\xi))$. It is not difficult to check that the argument in the proof of Theorem 2.3.2 is also valid for $F = \mathcal{O}_C(\xi)$. We only need to replace (2.3.8) by the process of recovering $\mathcal{O}_C(\xi)$ from $\alpha_{\mathcal{O}(-N)}(\mathcal{O}_C(\xi))$. To this end, using

$$\mathcal{E}xt^1_{\mathcal{O}_S}(\mathcal{O}_C(\xi), \mathcal{O}_S) = \mathcal{O}_C(C^2 - \xi), \tag{2.3.15}$$

we invert the triple $C(\mathcal{O}_S(-N), \mathcal{O}_C(\xi))$:

$$0 \longrightarrow {}^0\big\langle \mathcal{O}_S(-N) \,\big|\, \mathcal{O}_C(\xi) \big\rangle^* \otimes \mathcal{O}_S(N) \longrightarrow \alpha_{\mathcal{O}(-N)}(\mathcal{O}_C(\xi))^* \longrightarrow$$

$$\longrightarrow \mathcal{O}_C(C^2 - \xi) \longrightarrow 0. \tag{2.3.16}$$

Choosing an N_0 such that ${}^0\big\langle \mathcal{O}_S(N) \,\big|\, \mathcal{O}_C(C^2 - \xi) \big\rangle = 0$ if $N \geqslant N_0$ and applying $\langle \mathcal{O}_S(N)|$ to the triple (2.3.16), we obtain

$$\begin{gathered} {}^0\big\langle \mathcal{O}_S(-N) \,\big|\, \mathcal{O}_C(\xi) \big\rangle^* = {}^0\langle \mathcal{O}_S(N) | \alpha_{\mathcal{O}(-N)}(\mathcal{O}_C(\xi))^* \rangle, \\ \mathcal{O}_C(C^2 - \xi) = \gamma_{\mathcal{O}(N)}(\alpha_{\mathcal{O}(-N)}(\mathcal{O}_C(\xi))^*). \end{gathered} \tag{2.3.17}$$

Thus, we obtain an infinite set of modular families with birationally equivalent moduli varieties. It is easy to see that in this case the map (2.3.13) is also not $-\chi$-orthogonal for a general S.

Let us go back to the beginning of this section. For any vector subspace $V \subset {}^0\langle E \,|\, F\rangle$ we can consider the restriction of the homomorphism can of (2.3.1) to $V \otimes E \subset {}^0\langle E \,|\, F\rangle \otimes E$ and obtain the exact quadruple

$$C(E,F)^V : 0 \longrightarrow \alpha_E^V(F) \longrightarrow V \otimes E \xrightarrow{\mathrm{can}_V} F \longrightarrow \gamma_E^V(F) \longrightarrow 0. \tag{2.3.18}$$

Let $F = \overset{d}{\underset{i=1}{\oplus}} F_i$ and $\mathrm{rk}\,{}^0\langle E \,|\, F_i\rangle = n$. Then ${}^0\langle E \,|\, F\rangle = \overset{d}{\underset{i=1}{\oplus}} {}^0\langle E \,|\, F_i\rangle$, $\mathrm{rk}\,{}^0\langle E \,|\, F\rangle = dn$, and the subgroup

$$\mathrm{GL}({}^0\langle E \,|\, F_1\rangle) \times \cdots \times \mathrm{GL}({}^0\langle E \,|\, F_d\rangle) \subset \mathrm{GL}({}^0\langle E \,|\, F\rangle),$$

acting on the Grassmannian $\mathrm{Gr}(n, {}^0\langle E \,|\, F\rangle)$, has an open orbit U. It is easy to check that the sheaves $\alpha_E^V(F)$ and $\gamma_E^V(F)$ in (2.3.18) do not depend on the choice of point $V \in U$. We denote them by the symbols

$$\alpha_E^0\left(\overset{d}{\underset{i=1}{\oplus}} F_i\right), \quad \gamma_E^0\left(\overset{d}{\underset{i=1}{\oplus}} F_i\right). \tag{2.3.19}$$

The proof of the next theorem breaks into a sequence of lemmas, and is similar to that of Theorem 2.3.1.

THEOREM 2.3.4. *Let E be an exceptional bundle and $(F_1, \cdots, F_d)$ a set of bundles such that*

1) *each F_i is simple modular, and* $\forall_i\ \mathrm{Spl}_0(F_1) = \mathrm{Spl}_0(F_i)$,

2) $\forall_{i \neq j}$ (F_i, F_j) *is independent* (Definition 2.2.1, 3)),

3) $\forall_i$ (E, F_i) *and* $\left(E, \alpha_E^0\left(\overset{d}{\underset{i=1}{\oplus}} F_i\right)\right)$ *is regular pairs, and*

4) $\gamma_E^0\left(\overset{d}{\underset{i=1}{\oplus}} F_i\right) = 0.$

Then:

1) $\alpha_E^0\left(\overset{d}{\underset{i=1}{\oplus}} F_i\right)$ *is simple modular*,

2) $\mathrm{Spl}_0\left(\alpha_E^0\left(\overset{d}{\underset{i=1}{\oplus}} F_i\right)\right) \overset{\mathrm{bir}}{\sim} (\mathrm{Spl}_0(F_1))^{(d)}$ *where (d) signifies the dth symmetric power,*

3) $T_{\alpha_E^0}\left(\overset{d}{\underset{i=1}{\oplus}} F_i\right) = \overset{d}{\underset{i=1}{\oplus}} T_{F_i}$,

4) *and for the hypernet of the local invariant* (1.3.12) *we have*

$$\tau_{\alpha_E^0}\left(\overset{d}{\underset{i=1}{\oplus}} F_i\right) = \overset{d}{\underset{i=1}{\oplus}} \tau_{F_i}.$$

Definition 2.3.2. The passage from the set $(F_1, \cdots, F_d)$ satisfying conditions 1)– 4) of Theorem 2.3.4 to the sheaf $\alpha_E^0\left(\overset{d}{\underset{i=1}{\oplus}} F_i\right)$ is called the *Mestrano operation.*

We now mention two known examples and a new one when this construction yields new modular the families.

EXAMPLE 1. Let $p_1, \cdots, p_d \in S$ be distinct points on the surfaces S. Setting $F_i = \mathcal{O}_{p_i}$ and $E = \mathcal{O}_S$, we have

$$\alpha_{\mathcal{O}}^0\left(\overset{d}{\underset{i=1}{\oplus}} \mathcal{O}_{p_i}\right) = J_\xi, \tag{2.3.20}$$

where $\xi = \sum p_i$ is the cycle constructed from the points p_i.

EXAMPLE 2. (MESTRANO [10]). Let $S \xrightarrow{\pi} \mathbb{P}^1$ be a bundle with genus g curves as fiber, i.e., a pencil of curves $|D| = \mathbb{P}^1$ without base points. Then in the family of sheaves $\mathcal{O}_C(\xi)$ (2.1.17) with base

$$\mathrm{Pic}_{g+1}|D| \xrightarrow{\pi} |D| = \mathbb{P}^1$$

the sheaves $(\mathcal{O}_{C_1}(\xi_1), \mathcal{O}_{C_2}(\xi_2))$ are independent if $C_1 \neq C_2$, where $C_i = \pi^{-1}(p_i)$ and $p_i \in \mathbb{P}^1$. Moreover $\alpha_{\mathcal{O}}^0(\mathcal{O}_{C_1}(\xi_1) \oplus \mathcal{O}_{C_2}(\xi_2))$ is a two-dimensional modular (and even stable) bundle. Its moduli variety is a bundle over $\mathbb{P}^2 = (\mathbb{P}^1)^{(2)}$:

$$\mathrm{Spl}_0(\alpha_{\mathcal{O}}^0(\mathcal{O}_{C_1}(\xi_1) \oplus \mathcal{O}_{C_2}(\xi_2))) \xrightarrow{\pi} \mathbb{P}^2 = (\mathbb{P}^1)^{(2)},$$
$$\pi^{-1}(p_1 + p_2) = J_{g+1}(C_1) \oplus J_{g+1}(C_2).$$

EXAMPLE 3. Let S be a K3 surface with polarization $\mathcal{O}_S(1)$, F a simple $\mathcal{O}_S(1)$-stable bundle with $v(E) = (r, C, s)$ (see (1.2.18)), and $v^2 = 0$. Then, by [12], § 4, the variety of moduli $M_{\mathcal{O}(1)}(v)$ of the bundle F is birationally equivalent to a K3 surface S' isogeneous to S. We also have

THEOREM 2.3.5. *For any distinct points $([F_1], \cdots, [F_d])$ on S' and exceptional bundle E there exists a number N_0 such that for $N \geqslant N_0$*

1) *the set of bundles $(F_1, \cdots, F_d)$ satisfies conditions 1) – 4) of Theorem 2.3.4,*

2) $\alpha_{E(-N)}^0\left(\overset{d}{\underset{i=1}{\oplus}} F_i\right)$ *is simple modular, and*

3) $\mathrm{Spl}_0\left(\alpha_{E(-N)}^0\left(\overset{d}{\underset{i=1}{\oplus}} F_i\right)\right) \overset{\mathrm{bir}}{\sim} (S')^{(d)}.$

Assertions 1) follows directly from calculations in § 4 of [12].

CHAPTER 3

Universality

§ 1 Constructive equivalence.

Coming back to §1 of Chapter I, we have the big lattice $V_Z(S)$ of a smooth regular surface S, the $\mathbb{Z}$-semicone $F_m(S) \subset V_Z(S)$ consisting of the classes of modularly close sheaves on S, the set $R(S)$ of exceptional sheaves on S (see (1.1.26)), and the diagram of $\mathbb{Z}$-modules

$$\begin{tikzcd} R(S) \arrow[d, "r_m"'] & \subset F_m(S) \subset \arrow[d, "r_m"'] & V_Z(S) \arrow[d, "r_m"', "(1.1.11)"] \\ r_m(R(S)) \subset & F(S) \subset & K^0_{\mathrm{alg}}(S) \end{tikzcd} \tag{3.1.1}$$

The group Pic S acts equivariantly on this diagram via χ-isometries (see (1.1.7).

Definition 3.1.1. Simple modular sheaves F_1 and F_2 will be called *constructively close* if for any $i \neq j$ F_i can be obtained from F_j by the universal extensions or universal division, under which T_{F_i} is identified with T_{F_j}, or, in other words, $F_i = L_{ij} \otimes F_j$, where $L_{ij} \in \operatorname{Pic} S$.

Remark. The conditions in Theorems 2.2.1 and 2.3.1 give criteria for constructive closeness but, as examples show, they are far from being necessary. Chains of the relations of constructive closeness and modular closeness R_m (see Definition 1.1.2) generate an equivalence relation R_K. We denote the equivalence class of F by $\{F\}_K$. Thus, $R(S)$ is the union of the R_K-equivalence classes

$$R(S) = \bigcup \{F_i\}_K. \tag{3.1.2}$$

Similarly, for the semicone $F_m(S)$:

$$F_m(S) = \bigcup \{F_j\}_K. \tag{3.1.3}$$

It is easy to see (see the proofs of Theorems 2.2.1 and 2.3.1), that modular sheaves from the same constructive class have birationally equivalent moduli variety:

$$F_1 \overset{R_K}{\sim} F_2 \Rightarrow \mathrm{Spl}_0(F_1) \overset{\mathrm{bir}}{\sim} \mathrm{Spl}_0(F_2). \tag{3.1.4}$$

The number of classes (3.1.2) is an interesting invariant of the surface S. Direct calculations show (see, for example, [12]) that if F_1 and F_2 are simples sheaves on a Del Pezzo or K3 surface, then

$$F_1 \overset{R_K}{\sim} F_2 \Rightarrow v^2(F_1) = v^2(F_2). \tag{3.1.5}$$

In particular,

$$F \in R(S) \Rightarrow v^2(F) = \begin{cases} -1, & S \quad \text{Del Pezzo} \\ -2, & S \quad \text{K3}. \end{cases} \tag{3.1.6}$$

Not much is known about the structure of $R(S)$ or the partition (3.1.2).

The Drezet–Le Potier Theorem [4].

$$R(\mathbb{P}^2) = \{\mathcal{O}_{\mathbb{P}^2}\}_K. \tag{3.1.7}$$

Recently Rudakov proved the "uniclassness" of the quadric

$$R(\mathbb{P}^1 \times \mathbb{P}^1) = \{\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}\}_K.$$

For a surface S of general type we can only show that the classes $\{F\}_K$ contain infinitely many elements modulo the action of $\mathrm{Pic}\, S$.

Lemma 3.1.1. *Let $\gamma_{\mathcal{O}}(K_S) = 0$ and $\mathcal{O}_S(1)$ be ample. Then for any pair $E_1, E_2 \in R(S)$ such that $E_i^{**} = E_i$ there exists a d_0 such that*

1) *for any $d \geqslant d_0$*

$$\alpha_{E_1}(E_2(d)) \in R(S),$$

2) *for any $d_1, d_2 \geqslant d_0$*

$$\alpha_{E_1}(E_2(d_1)) \neq \alpha_{E_1}(E_2(d_2)).$$

Proof. Assertion 1) follows from Lemma 2.3.2. Now recall formulas (1.2.18) – (1.2.21) from§ 2 of Chapter 1. Let $\mathcal{O}_S(1) = \mathcal{O}_S(H)$, $v(E_1) = (r_1, C_1, s_1)$, and $v(E_2) = (r_2, C_2, s_2)$. Then

$$v(E_2(d)) = \left(r_2, C_2 + r_2 dH, \frac{r_2}{2} H^2 d^2 + HC_2 d + s_2\right),$$
$$-\chi(E_1, E_2(d)) = r_1 r_2 \frac{H^2}{2} d^2 + r_1 r_2 H \left(\frac{C_2}{r_2} - \frac{C_1}{r_1} + K_S\right) d +$$
$$+ r_1 r_2 \left(\frac{s_1}{r_1} + \frac{s_2}{r_2} - \frac{C_1 C_2}{r_1 r_2} - \frac{C_1 \cdot K_S}{2r_1} - \frac{C_2 \cdot K_S}{2r_2}\right).$$

The exact triple $C(E_1, E_2(d))$ (2.3.2) shows that

$$\operatorname{rk}\alpha_{E_1}(E_2(d)) = Ad^2 + Bd + C,$$

where A, B and C are constants (depending on E_1 and E_2). From this assertion 2) follows. $\square$

COROLLARY. *For every class* $\{F_i\}_K \subset R(S)$ (3.1.2)

$$\#(\{F_i\}_K/\mathrm{Pic}S) = \infty.$$

Similar arguments prove

Lemma 3.1.2. *Under the conditions of Lemma* 3.1.1 *for any* $E \in R(S)$ *such that* $E^{**} = E$, *and any simple modular sheaf* F, *there exists a number* d_0 *such that*

1) *for any* $d \geqslant d_0$

$$\alpha_E(F(d)) \in \{F\}_K,$$

2) *for any* $d_1, d_2 \geqslant d_0$

$$\operatorname{rk}\alpha_E(F(d_1)) \neq \operatorname{rk}\alpha_E(F(d_2))$$

COROLLARY.*For each class* $\{F_j\}_K \subset F_m(S)$ (3.1.3)

$$\#(\{F_j\}_K/\mathrm{Pic}\,S) = \infty.$$

The next result illustrates the extreme importance of the structure of the "root" subset $R(B) \subset V_Z(B)$ of the big lattice.

THEOREM 3.1.1. *Let* M *be a smooth irreducible variety with*

$$h^0(TM) = h^0(\Omega M) = 0\ ,$$

F *a compact modular simple sheaf on a regular surface* S, $M = \mathrm{Spl}_0(F)$, *and* $\mathcal{F}$ *a universal sheaf on* $S \times M$ *(a Poincaré family). Then*

1) $\mathcal{F}$ *is an exceptional sheaf on* $S \times M$

$$\mathcal{F} \in R(S \times M), \tag{3.1.8}$$

2) $\operatorname{rk}{}^2\langle\mathcal{F}\,|\,\mathcal{F}\rangle \geqslant \operatorname{rk}\operatorname{im}\tau_{\mathcal{F}}$, *where* $\tau_{\mathcal{F}}$ *is homomorphism* (1.3.7)=(1.2.31).

PROOF. The relative variant of the spectral sequence 7.3 from Chapter II of [6] yields an exact sequence

$$0 \longrightarrow H^1(M, \underbrace{\mathcal{E}xt^0_M(\mathcal{F},\mathcal{F})}_{\substack{\|(1.1.23)\\ \mathcal{O}_M}}) \longrightarrow {}^1\langle\mathcal{F}\,|\,\mathcal{F}\rangle \longrightarrow H^0(M, \underbrace{\mathcal{E}xt^1_M(\mathcal{F},\mathcal{F})}_{\substack{\|(1.1.23)\\ TM}}) \longrightarrow$$
$$\longrightarrow H^2(M, \mathcal{E}xt^0_M(\mathcal{F},\mathcal{F})) \longrightarrow {}^2\langle\mathcal{F}\,|\,\mathcal{F}\rangle.$$

The end terms of the initial triple vanish, and $\mathcal{F}$ is an infinitesimally rigid sheaf. Since $\mathcal{F}$ is $\mathcal{O}_M$-simple, i.e., $\mathcal{O}_M = \mathcal{E}xt^0_M(\mathcal{F},\mathcal{F})$, $\mathcal{F}$ is simple. The monomorphism between the last two terms yields inequality 2). $\square$

Since

$$\pi_S^*(R(S)) \subset R(S \times M), \tag{3.1.9}$$

the restriction to $S \times [F]$ gives

$$\{\mathcal{F}\}_K|_{S\times[F]} \supset \{F\}_K. \tag{3.1.10}$$

Definition 3.1.2. The class $\{F\}_K \subset F_m(S)$ is said to be *generated by the class* $\{F_0\}_K$ if there exist sheaves $F \in \{F\}_K$ and $F \in \{F_0\}$ such that $F = \alpha_E^0\left(\overset{d}{\underset{i=1}{\oplus}} F_i\right)$ or $F = \varepsilon_F^0\left(\overset{d}{\underset{i=1}{\oplus}} F_i\right)$, where $[F_i] \in \operatorname{Spl}_0(F_0)$, α_E^0 is the Mestrano operation (see Definition 2.1.1), and ε_E^0 is the universal composite extension operation (Definition 2.2.3).

Example 1. $\{J_\xi\}_K$ is generated by the class of the skyscraper sheaf $\{\mathcal{O}_p\}_K$, $p \in S$.

Conjecture. *On a* $K3$ *surface, classes* (3.1.3) *are generated by classes* $\{F_i\}_K$*, where* F_i *is a simple modular sheaf with isotropy vector* $v(F_i) : v^2(F) = 0$.

The problem of describing the images of classes (3.1.2) and (3.1.3) under the projection r_m (3.1.1) (for example, in the case of a K3 surface) reduces to the problem of describing the fundamental domain of the group generated by reflections (2.3.15) and antireflections (2.3.13) in the ultrahyperbolic lattice $\left(K^0_{\mathrm{alg}}(S), -\chi\right)$ with respect to the vector from $R(S)$ (see (3.1.6)) (but not all the vectors of square -2).

To describe $R(S)$ for Del Pezzo surfaces, Rudakov and Gorodentsev introduced the concept of helixes in $R(S)$ (see [8]). A helix in $R(S)$ is a K_S-periodic set of exceptional bundles, parametrized by integers

$$\begin{gathered} H = (\cdots, E_i, E_{i+1}, E_{i+2}, \cdots), \\ {}^i\!\left\langle E_k \,\middle|\, E_{k+m}\right\rangle = 0, \quad i \geqslant 1, \quad m \geqslant 1, \quad E_{i+\rho+2} = E_i \otimes K_S, \end{gathered} \tag{3.1.11}$$

where ρ is the Picard number of S.

If we replace one of the bundles from the pair (E_i, E_{i+1}) by the bundle $\alpha_{E_i}(E_{i+1})$, $\alpha^T_{E_{i+1}}(E_i)$ or $\varepsilon_{E_{i+1}}(E_i)$, we obtain a new helix H', and the chains of such transformations gives rise to the relation R_K of constructive equivalence of helixes. The R_K-equivalence class of the helix H will be denoted by $\{H\}_K$.

Example 2. $S = \mathbb{P}^2$ [8]. The geometry of helixes in $R(\mathbb{P}^2)$ is as follows:

1) $R(S)$ is swept by helixes.

2) Each pair $E,\ E' \subset R(\mathbb{P}^2)$ lies in a unique helix.

3) All helixes are constructively equivalent and thus constructively equivalent to the unique helix of invertible sheaves $H_0 = (\cdots, \mathcal{O}_{\mathbb{P}^2}(i), \cdots)$.

Of course, this immediately implies (3.1.7).

Each helix H gives rise to the dual helix $\tilde{H} = (\cdots, \tilde{E}_i, \cdots)$ (see [8], § 3, for precise definition). A length $\rho(S)+2$ segment of the helix (3.1.11) (for example, $(E_1, \cdots, E_{\rho+2})$ is called a *coil* of H.

A coil of a helix and the corresponding coil $(\tilde{E}_1, \cdots, \tilde{E}_{\rho+2})$ of the dual helix give rise to a resolution of the diagonal $\Delta \subset S \times S$:

(3.1.12)
$$0 \longrightarrow E_{\rho+2} \boxtimes \tilde{E}_{\rho+2} \longrightarrow \cdots \longrightarrow E_1 \boxtimes \tilde{E}_1 \longrightarrow \mathcal{O}_{S\times S} \longrightarrow \mathcal{O}_\Delta \longrightarrow 0,$$

where $E \boxtimes \tilde{E} = \pi_1^* E \otimes \pi_2^* \tilde{E}$ and π_i is the projection onto the corresponding direct summand.

This resolution is an analogue of the Beilinson spectral sequence, the existence of which has the following consequences (well known for $S = \mathbb{P}^2$):

1) The classes $\{E_i\}$, $i = 1, \cdots, \rho+2$ of the bundles from the coil of the helix are generators of the lattice $(K^0_{\text{alg}}(S), -\chi)$.

2) For any sheaf F on S
$$\{F\} = \sum (-1)^i r_i \{E_i\}, \quad r_i = \operatorname{rk}{}^1\langle E_i \,|\, \mathcal{F}\rangle.$$

3) Any sheaf F on S is the cohomology of a complex built of direct sums of bundles E_i.

Later we will see that all these statements are no longer true if $p_g(S) > 0$.

§ 2 Universality.

Given an algebraic symplectic structure ω on a variety B_2, any morphism $f : B_1 \longrightarrow B_2$ induces algebraic symplectic structure $f^*\omega$ on B_1:

(3.2.1)
$$\begin{tikzcd} TB_1 \arrow[r, "df"] \arrow[d, no head] & f^*TB_2 \arrow[r, "\omega"] & f^*\Omega B_2 \arrow[r, "df^*"] & \Omega B_1 \\ {} \arrow[rrr, dotted, "f^*\omega"'] & & & {} \arrow[u] \end{tikzcd}$$

But an algebraic symplectic structure can also be induced when B_1 and B_2 are related to each other in a more complicated way.

Lemma 3.2.1. *Let*

(3.2.2)
$$\begin{tikzcd} & \tilde{B} \arrow[dl, "\pi"'] \arrow[dr, "f"] & \\ B_1 & & B_2 \end{tikzcd}$$

be a diagram of morphisms between smooth varieties, where $\pi : \tilde{B} \longrightarrow B_1$ *is a bundle with nondegenerate differential and complete smooth rational fibers. Then any algebraic symplectic structure* ω *on* B_2 *induces an algebraic symplectic structure* $(\pi, f)^*\omega$ *on* B_1.

Proof. Diagram (3.2.2) induces the following diagram of bundles on $\tilde{B}$:

(3.2.3)
$$\begin{tikzcd} 0 \arrow[r] & T\tilde{B}/\pi \arrow[r] \arrow[dr, "\varphi_1"'] & T\tilde{B} \arrow[r, "d\pi"] \arrow[d] & \pi^*TM \arrow[r] \arrow[d, "\varphi_2"] \arrow[dl, "f^*\omega"'] & 0 \\ 0 \arrow[r] & \Omega M \arrow[r, "d\pi^*"] & \Omega\tilde{B} \arrow[r] & \Omega\tilde{B}/\pi \arrow[r] & 0. \end{tikzcd}$$

Both the diagonal homomorphism φ_1 and the vertical homomorphism φ_2 in this diagram vanish on each fiber $\pi^{-1}(b)$, because

(3.2.4)
$$h^0(\Lambda^2\Omega\pi^{-1}(b)) = h^0(\Omega\pi^{-1}(b)) = 0.$$

This gives rise to the solid diagonal homomorphism $f^*\omega$.

Lemma 3.2.2. *The assertion of Lemma* 3.2.1 *remains true if the map* f *in diagram* (3.2.2) *is a rational map such that for the closed subset* Σ *of its indeterminacy points and any point* $b \in B_1$

(3.2.5)
$$\operatorname{codim}_{\pi^{-1}(b)}\Sigma \cap \pi^{-1}(b) \geqslant 2.$$

Proof. Although diagram (3.2.3) is defined only on the open set $\tilde{B}_0 = \tilde{B} - \Sigma$, the restrictions of the broken homomorphism φ_1 and φ_2 of (3.2.3) to the fiber $\pi^{-1}(b) - \Sigma \cap \pi^{-1}(b)$ vanish for each point $b \in B_1$, because, by Hartogs' theorem, equalities (3.2.4) hold. Therefore, for any point of $\tilde{B}_0$ the image of the homomorphism $f^*\omega$ lies in the subbundle $\Omega M \subset \Omega\tilde{B}$. □

Definition 3.2.1. Diagram (3.2.2) with a rational map f satisfying condition (3.2.5) of Lemma 3.2.2 is called a *rational correspondence* and denoted by (π, f).

Remark. Rational correspondence is not symmetric. The term "rational" is related to the map f and the variety $\pi^{-1}(b)$.

If $\mathcal{F}$ is a flat family of torsionfree sheaves on a regular surface S with base M, then the main construction of [13] gives rise to a rational correspondence $(\pi, \varphi^S_{\mathcal{F}})$ of the variety M with the Douady space $\tilde{S}^{(d)}$ (see (2.1.4)) such that the triangle

(3.2.6)
$$\begin{tikzcd} H^0(\Lambda^2\Omega S) \arrow[rr, "\tau_{\mathcal{F}}", "(1.3.7)"'] \arrow[dr, "\tau_{J_Z}", "(2.1.11)"'] & & H^0(\Lambda^2\Omega M) \\ & H^0(\Lambda^2\tilde{S}^{(d)}) \arrow[ur, "(\pi,\varphi^S_{\mathcal{F}})^*"'] & \end{tikzcd}$$

is commutative.

Let F be a rank $r+1$ torsion free sheaf on S. An embedding of sheaves $V \otimes K_S \xrightarrow{S} F$ modulo $\mathrm{GL}(V)$, where $V = \mathbb{C}^r$, which can be extended to an exact triple

$$(3.2.7) \qquad 0 \longrightarrow V \otimes K_S \xrightarrow{S} F \longrightarrow J_{\xi(S)}(c_1(F) - rK_S) \longrightarrow 0,$$

is called a *regular K-block* of F, and thin cycle $\xi(S) \in \tilde{S}^{(d)}$, where $d = c_2(F')$, is called a *degeneration cycle* of the K-block S. (See [13], Definition 1.2.2 – 1.2.4 and Lemma 1.2.1]). A sheaf F is called *regular* if

$$h^1(F) = 0, \quad h^2(F) = h^2(\mathcal{O}_S(c_1(F) - rK_S)).$$

In this case the second coboundary homomorphism in the exact cohomology sequence of the triple (3.2.7) yields an isomorphism

$$V = {}^1\langle \mathcal{O}_S | J_{\xi(S)}(c_1(F') + K_S)\rangle$$

(see Lemma 2.2 of [13]), and thus F becomes the universal extension

$$(3.2.8) \qquad F = \varepsilon_{K_S}(J_{\xi(S)}(c_1(F') + K_S))$$

(see Definition 2.2.2 of this paper). By Lemma 2.3 of [13], if F is a simple sheaf then any regular K-block S is uniquely determined by its thin degeneration cycle $\xi(S)$. Furthermore, for any regular torsionfree sheaf F the variety of regular blocks $B(F)$ either is empty or contains a Zariski open dense subset of the Grassmannian $G(r, H^0(F'))$ (see [13],§ 3).

It is not difficult to prove the following generalization of "Serre's theorem":

Proposition 3.2.1. *Let $\mathcal{O}_S(1)$ be an ample invertible sheaf on a smooth regular surface S. Then the following assertions are true*:

1) *For any torsionfree sheaf F of rank $\geqslant 2$ and any sufficiently large N*

$$(3.2.9) \qquad h^1(F(N)) = h^2(F(N)) = 0, \quad B(F) \neq \varnothing;$$

2) *For any flat family $\mathcal{F}$ of torsionfree sheaves of rank $\geqslant 2$ on $S \times M$ and any sufficiently large N*

   a) $\forall\, m \in M$, *$F_m(N)$ satisfies condition* (3.2.9), *and*

   b) $R^0\pi_M(\mathcal{F}' \otimes \pi_S^*\mathcal{O}_S(N))$ *is a locally free $\mathcal{O}_M$-sheaf.*

A family $\mathcal{F}$ on $S \times M$ such that, for all $m \in M$, F_m satisfies (3.2.9) and $R^0\pi_M\mathcal{F}' = \mathcal{H}(\mathcal{F})$ is locally free is called *regular*. A regular family $\mathcal{F}$ on $S \times M$ gives rise to the bundle

$$(3.2.10) \qquad G(r, \mathcal{H}(\mathcal{F})) \xrightarrow{\pi} M, \quad \pi^{-1}(m) = G(r, H^0(F'_m))$$

which is the Grassmannization of $\mathcal{H}(\mathcal{F})$, and the open subset

$$(3.2.11) \qquad B(\mathcal{F}) \subset B(r, \mathcal{H}(\mathcal{F}))$$

of regular blocks. Assigning to each regular block its degeneration cycle gives rise to the map $\varphi^S_{\mathcal{F}}$ of the diagram

$$\begin{tikzcd}[column sep=small] & G(r,\mathcal{H}(\mathcal{F})) \\ & \cup \\ & B(\mathcal{F}) \arrow[dl, "\pi"'] \arrow[dr, "\varphi^S_{\mathcal{F}}"] \\ M & & \tilde{S}^{(d)} \end{tikzcd} \qquad \begin{aligned} & d = c_2(F'_m)\ , \\ & \varphi^S_{F_m}\ :\ B(F_m) \longrightarrow \tilde{S}^{(d)}\ , \\ & \varphi^S_{F_m}(S) = \xi(S) \end{aligned} \tag{3.2.12}$$

(see [13],§ 2, (3.2)). The map $\varphi^S_{\mathcal{F}}$ is regular on $B(\mathcal{F})$ and rational on $G(r,\mathcal{H}(\mathcal{F}))$.

THEOREM 3.2.1. *For any* $\omega \in H^0(\Lambda^2\Omega S)$

$$\tau_{\mathcal{F}}(\omega) = (\pi, \varphi^S_{\mathcal{F}})^*(\tau_{J_Z}(\omega)),$$

i.e., diagram (3.2.1) *is commutative.*

PROOF. Twisting triple (3.2.7) by K^*_S, we get

$$0 \longrightarrow V \otimes \mathcal{O}_S \xrightarrow{S} F'_m \xrightarrow{j} J_\xi(D) \longrightarrow 0\ , \tag{3.2.7$'$}$$

where $D = c_1(F'_m)$, $\xi = \xi(S)$, $d = c_2(F'_m)$. Applying $|J_\xi(D)\rangle$ to the triple (3.2.7'), we have

$$\begin{tikzcd} {}^0\big\langle V \otimes \mathcal{O}_S \,\big|\, J_\xi(D)\big\rangle \arrow[r, "\delta"] \arrow[d, equal] & {}^1\big\langle J_\xi(D) \,\big|\, J_\xi(D)\big\rangle \arrow[d, equal] \arrow[r, shift left, dotted, leftarrow, "j"] \arrow[r, shift right] & {}^1\big\langle F'_m \,\big|\, J_\xi(D)\big\rangle \\ V^* \otimes H^0(J_\xi(D)) & TJ_\xi = T\tilde{S}^{(d)}_\xi & . \end{tikzcd} \tag{3.2.13}$$

Since $h^1(\mathcal{O}_S) = 0$, triple (3.2.7') yields another triple:

$$0 \longrightarrow V \longrightarrow H^0(F'_m) \longrightarrow H^0(J_\xi(D)) \longrightarrow 0.$$

Hence

$$V^* \otimes H^0(J_\xi(D)) = TG(r, H^0(F'_m))_V = TB(F_m)_V = (TB(\mathcal{F})/\pi)(m,S)$$

and the coboundary homomorphism δ in (3.2.13) decomposes as follows:

$$\begin{tikzcd} V^* \otimes H^0(J_\xi(D)) \arrow[r, "\delta"] \arrow[d, equal] & {}^1\big\langle J_\xi(D) \,\big|\, J_\xi(D)\big\rangle \\ (TB(\mathcal{F})/\pi)_{(m,S)} \arrow[r, "i"] & TB(\mathcal{F})_{(m,S)} \arrow[u, "d\varphi^S_{\mathcal{F}}"'] \quad . \end{tikzcd} \tag{3.2.14}$$

On the other hand, applying $\langle F'_m|$ to (3.2.7'), we have

$$\begin{tikzcd} {}^1\big\langle F'_m \,\big|\, J_\xi(D)\big\rangle & {}^1\big\langle F'_m \,\big|\, F'_m\big\rangle \arrow[l] \arrow[d, equal] & {}^1\big\langle F'_m \,\big|\, V \otimes \mathcal{O}_S\big\rangle \arrow[l, "S"'] \arrow[d, equal, "\begin{smallmatrix}\text{Serre}\\ \text{duality}\end{smallmatrix}"] \\ & TM_m \arrow[ul, dotted, "j"] & V^* \otimes H^1(F_m)^* = 0. \end{tikzcd} \tag{3.2.15}$$

Let $s \in H^0(K_S)$ and $F \xrightarrow{\otimes s} F \otimes K_S$ be the sheaf homomorphism defined by s. The theorem now follows from the commutativity of the diagram

$$
\begin{tikzcd}
T\tilde{S}^{(d)}_{\xi} \arrow[d, equal] & & TM_m \arrow[dl, dotted, "j"'] \arrow[d, equal] \\
{}^1\langle J_{\xi}(D) \mid J_{\xi}(D) \rangle \arrow[r] \arrow[d] \arrow[d, dashed, shift left=2, "\otimes S"] & {}^1\langle F'_m \mid J_{\xi}(D) \rangle \arrow[ul, dotted, "j"'] \arrow[d] \arrow[d, dashed, shift left=2, "\otimes S"] & {}^1\langle F'_m \mid F'_m \rangle \arrow[l] \arrow[d] \arrow[d, dashed, shift left=2, "\otimes S"] \\
{}^1\langle J_{\xi}(D) \mid J_{\xi}(D) \otimes K_S \rangle \arrow[r] \arrow[d, equal] & {}^1\langle F'_m \mid J_{\xi}(D) \otimes K_S \rangle \arrow[dl, dotted, "j"] & {}^1\langle F'_m \mid F'_m \rangle \arrow[l] \arrow[d, equal] \\
\Omega\tilde{S}^{(d)}_{\xi} & & \Omega M_m \arrow[ul, dotted, "j"]
\end{tikzcd}
\tag{3.2.16}
$$

To illustrate this let us turn to the compact modular case: let M be a complete smooth variety of moduli of bundles on S and $\mathcal{F}$ a universal sheaf on $S \times M$. Consider the algebraic cohomology class

$$c_2(\mathcal{F}) \in H^4(S \times M, \mathbb{Z}),$$

which is the second Chern class of $\mathcal{F}$, and its $(2,2)$-component in the Künneth decomposition

$$c_2(\mathcal{F})_{(2,2)} \in H^2(S, \mathbb{Z}) \otimes H^2(M, \mathbb{Z}). \tag{3.2.17}$$

This component is also an algebraic cycle ($H^1(S,\mathbb{Z}) = H^3(S,\mathbb{Z}) = 0$), and it defines a homomorphism

$$\mathbf{\Phi'} : \mathbf{H^2}(\mathbf{S}, \mathbb{Z})^* \longrightarrow \mathbf{H^2}(\mathbf{M}, \mathbb{Z}).$$

The Poincaré duality gives rise to a lattice isomorphism $H^2(S,\mathbb{Z})^* = H^2(S,\mathbb{Z})$, and, therefore, to a homomorphism

$$\mathbf{\Phi} : \mathbf{H^2}(\mathbf{S}, \mathbb{Z}) \longrightarrow \mathbf{H^2}(\mathbf{M}, \mathbb{Z}). \tag{3.2.18}$$

Since $c^2(\mathcal{F})_{(2,2)}$ is a type $(2,2)$ Hodge cocycle, the map $\mathbf{\Phi} \otimes \mathbb{C}$ is a homomorphism of Hodge structures and, in particular, we have the homomorphism

$$(\mathbf{\Phi} \otimes \mathbb{C})^{\mathbf{2,0}} : \mathbf{H^{2,0}}(\mathbf{S}) \longrightarrow \mathbf{H^{2,0}}(\mathbf{M}). \tag{3.2.19}$$

Lemma 3.2.3. $(\mathbf{\Phi} \otimes \mathbb{C})^{\mathbf{2,0}} = \tau_{\mathcal{F}} = -(\mathbf{f}^{\tau}_{\mathcal{F}*} \otimes \mathbb{C})^{\mathbf{2,0}}$ (1.2.31).

Proof. The homomorphism (3.2.19) does not change when $\mathcal{F}$ is twisted by elements of $\pi_S^*(\operatorname{Pic} S)$, and we may assume that $\mathcal{F}$ is regular. Then for any point $(F_m, S) \in B(\mathcal{F})$ (3.2.11) the triple (3.2.7') yields the equality

$$c_2(F'_m) = c_2(J_{\xi(S)}(c_1(F'_m))) = c_2(J_{\xi}) + \text{const}.$$

Thus, by Theorem 3.2.1, it suffices to check the desired equality for the family J_Z (2.1.9) with base $\tilde{S}^{(d)}$. But in this case the result follows from (2.1.9), (2.1.11), and (2.1.13). $\square$

§ 3 The image of the moduli variety in $K^0(S)$.

If S is a smooth regular surface let, as usual,

$$CH^2(S) = \frac{\text{the free abelian group of points on } S}{\text{cycles rationally equivalent to zero}} \tag{3.3.1}$$

be the Chow group of cycles on S, $CH^2(S) \xrightarrow{\deg} \mathbb{Z} \longrightarrow 0$ the "degree" epimorphism, and $CH_0^2(S)$ its kernel, i.e., the group of classes of degree of 0 cycles. Fixing a point $p_0 \in S$, we obtain an inductive system of maps $\{r_d\}$:

$$S^{(d)} \xrightarrow{r_d} CH_0^2(S), \quad r_d\left(\sum p_i\right) = \text{class}\left(\sum_{i=1}^{d} p_i - dp_0\right), \tag{3.3.2}$$

and therefore, a system of maps

$$\tilde{S}^{(d)} \xrightarrow{(2.1.4)} S^{(d)} \xrightarrow{r_d} CH_0^2(S) \tag{3.3.3}$$

between smooth varieties.

The irreducible unirational variety

$$\tilde{S}_{p_0}^{(d)} = \{\xi \in \tilde{S}^{(d)} | \mathrm{supp}\xi = p_0\} \tag{3.3.4}$$

gives rise to the rational correspondence (π, f_n) (see(3.2.2)):

$$\begin{tikzcd} & \tilde{S}^{(d)} \times \tilde{S}_{p_0}^{(n)} \arrow[dl, "\pi"'] \arrow[dr, "f_n"] & \\ \tilde{S}^{(d)} & & \tilde{S}^{(d+n)} \end{tikzcd}, \tag{3.3.5}$$

$$J_{f_n(\xi,\eta)} = J_\xi \otimes J_\eta.$$

The inductive system of rational correspondences $\{(\pi, f_n)\}$ allows us to define the following things:

(1) The inductive limit topology on $CH_0^2(S)$;

(2) The notion of a morphism φ from any smooth variety M into $CH_0^2(S)$ as a rational correspondence (3.2.2) between M and $\tilde{S}^{(d)}$ for some d:

$$\varphi : M \longrightarrow CH_0^2(S). \tag{3.3.6}$$

(3) The notion of a type $(2,0)$ form on $CH_0^2(S)$ (see Lemma 3.2.1) and, therefore, the space

$$H^{2,0}(CH_0^2(S))$$

(the forms are invariant under translations in $CH_0^2(S)$).

(4) The isomorphism

$$\tau : H^{2,0}(S) \longrightarrow H^{2,0}(CH_0^2(S)) \tag{3.3.7}$$

and any morphism $\varphi : M \longrightarrow CH_0^2(S)$ induce a homomorphism

$$\varphi^* : H^{2,0}(S) = H^{2,0}(CH_0^2(S)) \longrightarrow H^{2,0}(M) \tag{3.3.8}$$

(Lemma 3.2.1).

Now we can interpret the results of the previous section as follows:

1) For any family $\mathcal{F}$ on $S \times M$ the rational correspondence of degeneration blocks (3.2.12) determines uniquely, up to a translation, a morphism

$$\varphi_B : M \longrightarrow CH_0^2(S). \tag{3.3.9}$$

2) The homomorphism (1.3.7)

$$\tau_{\mathcal{F}} = \varphi_B^*. \tag{3.3.10}$$

It is easy to see that for a smooth regular surface S

$$K^0(S) \otimes \mathbb{Q} = K^0_{\mathrm{alg}}(S) \otimes \mathbb{Q} \oplus CH_0^2(S),$$

where $K^0(S)$ is the Grothendieck group of S. Therefore, we can interpret the morphism φ_B (3.3.9) as map of the variety into $K^0(S)$.

THEOREM 3.3.1. *Let F be a torsionfree simple modular sheaf on S for which there exists a section $s \in H^0(K_S)$ such that $\tau_F(S)$ (1.3.11) is nondegenerate, i.e.,*

$$\tau_F(S) \; : \; {}^1\langle F \,|\, F \rangle \xrightarrow{\otimes s} {}^1\langle F \,|\, F \otimes K_S \rangle \tag{3.3.11}$$

is an isomorphism.

Then for any analytic neighborhood U of the point $[F]$ in $\mathrm{Spl}_0(F)$ and any representation of the block morphism

$$U \xrightarrow{\varphi_B} CH_0^2(S)$$

as a composition

$$\begin{tikzcd} U \arrow[rr, "\varphi_B"] \arrow[dr, "\psi"'] & & CH_0^2(B) \\ & M' \arrow[ur, "\varphi'"'] & \end{tikzcd} \tag{3.3.12}$$

of morphism, where M' is a germ of an algebraic variety, we have

$$\ker d\psi_{[F]} = 0, \tag{3.3.13}$$

i.e., ψ is an immersion of $[F]$.

PROOF. By (3.3.10), $\ker d\psi_{[F]} \subset \ker \tau_F(S)$ (3.3.11). Hence, $\ker \tau_F(S) = 0$ implies (3.3.13). □

Corollary 1. *Under the assumptions of the theorem,*

$$\dim M' \geqslant \dim \mathrm{Spl}_0(F) = \mathrm{rk}^{\,1}\langle F|F\rangle. \tag{3.3.14}$$

Corollary 2. *If S is a $K3$ surface and F is simple sheaf, then the decomposition* (3.3.12) *implies equality* (3.3.13).

Indeed, it follows from [11] that F is modular and (3.3.11) is an isomorphism.

Theorem 3.3.1 make precise the following intuitive observation: the moduli variety of bundles on S with nondegenerate induced symplectic structure maps into $K_0(S)$ with no loss in dimension.

Thus, the principles of classification of bundles on a regular surface with $p_g > 0$ are entirely different from those in the case of a rational surfaces (for example, $\mathbb{P}^2$):

1) A bundle cannot be represented as the cohomologies of a standard monad.
2) There exists no resolution of the diagonal in $S \times S$ of the form (3.1.12);
3) The theory of helixes (3.1.11) in $R(S)$ is not applicable, etc.

As a matter of fact, any bundle (and all of its constructive class) is almost uniquely determined by its second Chern class in $CH^2(S)$, and the varieties of moduli of bundles are maximal finite-dimensional algebraic subvarieties of the "monster" $CH^2_0(S)$.

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The moduli spaces of vector bundles on threefolds, surfaces and curves I

Erlangen preprint, 1990.

Introduction.

The aim of this talk[1] is to bring together some results and constructions relating the geometric structure of moduli spaces of stable vector bundles on a flag of varieties of type:

$$X \supset S \supset C, \tag{0.1}$$

where X is a Fano threefold, $S \in |-K_X|$ is a K3-surface and C is a curve on C.

Let M_X, M_S, M_C be the components of the moduli spaces of stable vector bundles with fixed first Chern class c_1 on X, S and C such that

$$E \in M_X \implies E|_S \in M_S \implies E|_C \in M_C. \tag{0.2}$$

Then the restrictions provide maps

$$M_X \xrightarrow{\mathrm{res}_S} M_S \xrightarrow{\mathrm{res}_C} M_C. \tag{0.3}$$

M_X being regular means that for any $E \in M_X$

$$H^2(\operatorname{ad} E) = 0, \tag{0.4}$$

where, as usually, $\operatorname{ad} E \oplus \mathcal{O}_X = \operatorname{End} E = E^\vee \otimes E$. Then the long exact sequence of the short exact sequence

$$0 \longrightarrow \operatorname{ad} E \otimes K_X \longrightarrow \operatorname{ad} E \longrightarrow \operatorname{ad} E|_S \longrightarrow 0, \tag{0.5}$$

is short, too:

$$0 \longrightarrow H^1(\operatorname{ad} E) \xrightarrow{d\,\mathrm{res}} H^1(\operatorname{ad} E|_S) \longrightarrow H^2(\operatorname{ad} E \otimes K_X) \longrightarrow 0, \tag{0.6}$$

[1] This is the extended version of my talk on the Conference "Complex Abelian Varieties" in Bayreuth, 2 – 6 April 1990. I would like to express my thanks to K. Hulek, T. Peternell, M. Schneider and F.-O. Schreyer for the invitation. I would like to express my gratitude to Mathematisches Institut der Universität Erlangen–Nürnberg and personally to Herbert Lange for support and hospitality.

because $H^1(\operatorname{ad} E \otimes K_X) = H^2(\operatorname{ad} E) = 0$ by Serre-duality on X.

Now, by standard theory variations we can interpret the vector spaces of (0.6) as the fibres of concrete vector bundles on M_X and M_S:

$$H^1(\operatorname{ad} E) = (TM_X)_E \tag{0.7}$$

is the tangent space to M_X in the point $E \in M_X$. Similarly

$$H^1(\operatorname{ad} E|_S) = (TM_S)_{E|S}, \tag{0.8}$$

and we can interpret the monomorphism from (0.6) as the differential of the restriction map (0.3).

Remark. In the local setup, the condition (0.4) implies the existence and smoothness of the local moduli space. Moreover the first part of long exact sequence (0.5) is

$$0 \longrightarrow H^0(\operatorname{ad} E \otimes K_X) \longrightarrow H^0(\operatorname{ad} E) \longrightarrow H^0(\operatorname{ad} E|_S) \longrightarrow 0.$$

Hence, the simpleness ($h^0(\operatorname{ad} E) = 0$) implies the simpleness of the restriction $E|_S$ and from this the existence and smoothness of the local moduli space for $E|_S$ (see [15]).

Now, on M_S there is holomorphic symplectic structure, that is skew-symmetric homomorphism

$$\omega : TM_S \mapsto T^\vee M_S, \quad \omega^\vee = -\omega, \tag{0.9}$$

which over a point $E|_S$ is defined by Serre-duality

$$(TM_S)_{E|_S} = H^1(\operatorname{ad} E|_S) \cong H^1(\operatorname{ad} E|_S)^\vee = (T^\vee M_S)_{E|_S}.$$

The restrictions map res_S (0.3) induces a symplectic structure on M_X by the diagram

$$\begin{tikzcd} 0 \arrow[d] & 0 \\ TM_X \arrow[r, "\operatorname{res}_S^*\omega"] \arrow[d, "d\operatorname{res}_S"] & T^\vee M_X \arrow[u] \\ TM_S \arrow[r, "\omega"] & T^\vee M_S, \arrow[u, "d\operatorname{res}_S^\vee"'] \end{tikzcd} \tag{0.10}$$

where the verticals are the beginning of (0.6) and the end of $(0.6)^\vee$. But by Serre-duality

$$H^2(\operatorname{ad} E \otimes K_X) = H^1(\operatorname{ad} E)^\vee = T^\vee M_X, \tag{0.11}$$

and we can extend (0.10) to

$$\begin{tikzcd} 0 \arrow[r] & TM_X \arrow[r, "d\operatorname{res}_S"] \arrow[d, "\operatorname{res}_S^*\omega"'] & TM_S \arrow[r] \arrow[d, "\omega"] & T^\vee M_X \arrow[r] & 0 \\ 0 & T^\vee M_X \arrow[l] & T^\vee M_S \arrow[l, "d\operatorname{res}_S^\vee"'] & TM_X \arrow[l] & 0, \arrow[l] \end{tikzcd} \tag{0.12}$$

where the beginning of (0.12) is the dual of (0.10) and the horizontals are (0.6) and $(0.6)^\vee$.

Now, by functoriality of Serre-duality we can see, that

$$\omega(TM_X) = TM_X \implies \mathrm{res}_S^* \omega = 0.$$

Hence we proved

Proposition 0.1. *The image* $\mathrm{Im}(M_X) \subset M_S$ *is a Lagrangian subvariety of* M_S *and*

$$\dim M_S = 2 \dim M_X.$$

Let us go to the second part of the chain (0.3): in this situation together with (0.6) we consider also the sequence

$$\longrightarrow \mathrm{ad}\, E|_S \longrightarrow \mathrm{ad}\, E|_S(C) \longrightarrow \mathrm{ad}\, E|_C \otimes K_C \longrightarrow 0, \tag{0.13}$$

where $\mathrm{ad}\, E|_S(C) = \mathrm{ad}\, E|_S \otimes \mathcal{O}_S(C)$ and $K_C = \mathcal{O}_S(C)|_C$ is the canonical class of the curve C.

$$H^0(\mathrm{ad}\, E|_C \otimes K_C) \longrightarrow H^1(\mathrm{ad}\, E|_S) \longrightarrow H^1(\mathrm{ad}\, E|_S(C)) \longrightarrow 0. \tag{0.14}$$

A vector of the space $H^0(\mathrm{ad}\, E|_C \otimes K_C)$ can be interpreted as a homomorphism

$$\phi : E|_C \longrightarrow E|_C \otimes K_C, \tag{0.15}$$

which is called a *Higgs field* (on C) and by Serre-duality

$$H^0(\mathrm{ad}\, E|_C \otimes K_C) = H^1(\mathrm{ad}\, E|_C)^\vee = T^\vee M_C \tag{0.16}$$

is the fibre of the cotangent bundle on M_C.

Consider now the homomorphism

$$\omega^{-1} : T^\vee M_S \longrightarrow TM_S \tag{0.17}$$

as the Poisson structure on M_S (see [18]). The map $\mathrm{res}_C : M_S \longrightarrow M_C$ defines a Poisson structure on the image $\mathrm{res}_C(M_S) \subset M_C$ by the diagram:

$$\begin{tikzcd} T^\vee M_C \arrow[r, "d\,\mathrm{res}_C^\vee"] & T^\vee M_S \arrow[d, "\omega^{-1}"] \\ TM_C \arrow[u, "\mathrm{res}_C^* \omega^{-1}"] & TM_S, \arrow[l, "d\,\mathrm{res}_C"'] \end{tikzcd} \tag{0.18}$$

which can be extended to

(0.19)

$$\begin{tikzcd} 0 \arrow[r] & H^0(\mathrm{ad}E|_S(C)) \arrow[r] & T^\vee M_C \arrow[r, "d\,\mathrm{res}_C^\vee"] \arrow[d, "\mathrm{res}_C^*\omega^{-1}"'] & T^\vee M_S \arrow[r] \arrow[d, "\omega^{-1}"] & H^1(\mathrm{ad}E|_S(C)) \arrow[r] & 0 \\ 0 & H^2(\mathrm{ad}E|_S(-C)) \arrow[l] \arrow[d, equal] & TM_C \arrow[l] & TM_S \arrow[l, "d\,\mathrm{res}_C"'] & H^1(\mathrm{ad}E(-C)) \arrow[l] \arrow[d, equal] & 0 \arrow[l] \\ & H^0(\mathrm{ad}E|_S(C))^\vee & & & H^1(\mathrm{ad}E|_S(C))^\vee & \end{tikzcd}$$

where the lower sequence is induced by the analogue of (0.6) and the upper sequence is induced by (0.13).

Remark. The left hand zero of the upper sequence is provided by the simpleness of $E|_S$, and the right hand one is provided by the simpleness of $E|_C$. In which cases does the stability of E on S provides the stability of $E|_C$? This very important question will be investigated below in detail.

Now, we give the following

Definition 0.1. *The Higgs field* (0.15) *is called extendible iff it is the restriction of a homomorphism* $\tilde{\phi} : E_S \longrightarrow E_S(C)$ *of vector bundles on* S.

There exist two important partial cases of our situation:

I. All Higgs fields on C are unextendible.

II.

$$h^1(\operatorname{ad} E|_S(C)) = 0. \tag{0.20}$$

In the first case the restriction map $\operatorname{res}_C : M_S \longrightarrow M_C$ is a surjection and $M_X \longrightarrow M_C$ is a Lagrangian projection

$$\begin{tikzcd} M_X \arrow[r, "\operatorname{res}_S"] \arrow[dr] & M_S \arrow[d, "\operatorname{res}_C"] \\ & M_C \end{tikzcd} \tag{0.21}$$

In the second case $\operatorname{res}_C : M_S \longrightarrow M_C$ is a local embedding. There exists a regular map

$$f_0 : M_C \longrightarrow \mathbb{P}^{2g(C)-1} \tag{0.22}$$

into the projective space of conformal blocks (see the exact description in § 3 and [11], [4]), which is almost independent of C (see [11]). We provide the diagram

$$\begin{tikzcd} M_X \arrow[r, "\operatorname{res}_S"] \arrow[dr] & M_S \arrow[d, "\operatorname{res}_C"] \arrow[dr, "f"] & \\ & M_C \arrow[r, "f_0"] & \mathbb{P}^{2^{g(C)}-1} \end{tikzcd} \tag{0.23}$$

We would like to prove that f does not depend on the choice of the smooth curve C in the linear equivalence class of C (see § 3) and we make the first step to prove this conjecture.

Our aim is to investigate all parts of this diagram in detail and in a general situation, and to apply the information about the whole construction to the problem of describing the moduli space of the mathematical instantons on $\mathbb{P}^3$.

Below we consider the right hand part of the diagram (0.23) in the second case (0.20), that is the questions about relations between the moduli spaces of stable vector bundles on a surface S and on a curve $C \subset S$.

§ 1 Polarisations. Embedding theorem.

Let S be a smooth complete regular surface over $\mathbb{C}$ and H be a divisor class on S. Let us recall that a divisor class H is called a *polarization* if $H^2 > 0$ and $HC > 0$ for every effective curve C on S.

Then by Serre's theorem

1. For every divisor class $\mathcal{D}$ on S there exists a number d_0 such that for every $d \geqslant d_0$

$$\mathcal{D} + dH \text{ is a polarization.} \tag{1.1}$$

2. For every coherent sheaf F on S there exists a number d_0 such that for every $d \geqslant d_0$

$$H^i(F \otimes \mathcal{O}_S(dH)) = 0, \quad i \neq 0, \tag{1.2}$$

and for $i = 0$ the canonical homomorphism

$$H^0(F \otimes \mathcal{O}_S(dH)) \otimes \mathcal{O}_S \xrightarrow{\text{can}} F \otimes \mathcal{O}_S(dH) \tag{1.3}$$

is an epimorphism.

Hence, if we consider the lattice Pic S of divisor classes on S, then the subset of polarizations is a convex halfcone

$$V^+(S) \subset \text{Pic } S. \tag{1.4}$$

Moreover, for every homogenous integer-valued polynomial γ on Pic S the restriction

$$\gamma|_{V^+} \text{ determines } \gamma \text{ on Pic } S. \tag{1.5}$$

The last technical detail is the semicontinuity of the function

$$h(b) = \text{rk} H^i(F|_b), \quad b \in B \tag{1.6}$$

on the base B of a flat family of sheaves on S in the Zariski topology.

Let $M_H(2, c_1, c_2)$ be the moduli space of H-stable rk 2 vector bundles on S with Chern-classes $c_1 \in$ Pic S and $c_2 \in \mathbb{Z}$.

Let $C \in |dH|$ be a smooth curve and $M_C(2, \{c_1\})$ be the moduli space of stable rk 2 vector bundles on C with fixed determinant $\{c_1\}$. Multiplying if necessary the vector bundles by $L \in$ Pic S we may assume that

$$\deg\{c_1\} = \begin{cases} 0, & \text{if } \deg c_1 \cdot C \text{ is even} \\ -1, & \text{if } \deg C_1 \cdot C \text{ is odd.} \end{cases} \tag{1.7}$$

Theorem 1.1. *There exists a number $d_0(k)$ such that for a generic smooth curve $C \in |dH|$ and $d \geqslant d_0(k)$ the restriction map*

$$\mathrm{res}_C : M_H(2, c_1, k') \longrightarrow M_C(2, \{c_1\}) \tag{1.8}$$

is an embedding for all $k' \leqslant k$.

Proof. First of all, it is easy to see that the set of irreducible components of the union

$$\bigcup_{k' \leqslant k} M_H(2, c_1, k')$$

is finite by Bogomolov's inequality for stable of vector bundles

$$c_1^2 \leqslant 4k' \leqslant 4k. \tag{1.9}$$

Let M_H be any component of $\bigcup M_H(2, c_1, k'), k' \leqslant k$.

Consider on $M_H \times M_H$ the function

$$h^1(E_1, E_2, d) = \mathrm{rk} H^1(E_1^\vee \otimes E_2 \otimes \mathcal{O}_S(-dH)). \tag{1.10}$$

Let $\operatorname{supp} h^1(E_1, E_2, d)$ denote the support of this function. Then there exists a number $d_0(M_H)$ such that

$$\operatorname{supp} h^1(E_1, E_2, d) = \emptyset, \quad \forall d \geqslant d_0(M_H) \tag{1.11}$$

by (1.2) and (1.6) applyied to the family $E^\vee \otimes E_1 \otimes K_S$ with base $M_H \times M_H$ and Serre-duality. Consider

$$d_0^I = \max_{M_H \in \cup M_H(2, c_1, k')} d_0(M_H). \tag{1.12}$$

Then for every $C \in |dH|, d \geqslant d_0^I$ and for every pair E_1, E_2, of vector bundles of $\bigcup M_H(2, c_1, k')$

$$E_1|_C \cong E_2|_C \quad \Longrightarrow \quad E_1 \cong E_2. \tag{1.13}$$

Indeed, multiplying by $E_1^\vee \otimes E_2$ the short exact the sequence

$$0 \longrightarrow \mathcal{O}_S(-dH) \longrightarrow \mathcal{O}_S \longrightarrow \mathcal{O}_C \longrightarrow 0, \tag{1.14}$$

provides the short exact sequence

$$0 \longrightarrow E_1^\vee \otimes E_2 \otimes \mathcal{O}_S(-dH) \longrightarrow E_1^\vee \otimes E_2 \longrightarrow E_1^\vee|_C \otimes E_2|_C \longrightarrow 0. \tag{1.15}$$

By (1.11) the long exact sequence of (1.15) provides epimorphism

$$\begin{array}{ccccc} H^0(E_1^\vee \otimes E_2) & \longrightarrow & H^0(E_1^\vee|_C \otimes E_2|_C) & \longrightarrow & 0 \\ \| & & \| & & \\ \mathrm{Hom}(E_1, E_2) & & \mathrm{Hom}(E_1|_C, E_2|_C) & & \end{array} \tag{1.16}$$

Now an isomorphism $E_1|_C \xrightarrow{\phi} E_2|_C$ comes from a homomorphism

$$E_1 \xrightarrow{\tilde{\phi}} E_2 ,$$

which is an isomorphism over a general point of S. Hence

$$\Lambda^2\tilde{\phi} : \mathcal{O}_S(c_1) \longrightarrow \mathcal{O}_S(c_1)$$

is not zero and $\tilde{\phi}$ is an isomorphism.

Let us go to the question of stability of $E|_C$. For every vector bundle E on the curve C consider the number

$$l(E) = \max_{L_1 \subset E} \deg L_1, \tag{1.17}$$

where L_1 is a line subbundle. Thus, if we normalized $\deg E$ as in (1.7), then

$$l(E) \geqslant 0 \quad \Longleftrightarrow \quad E \quad \text{is not stable.} \tag{1.18}$$

Now consider again the set $\{M_H\}$ of components of $\bigcup\limits_{k' \leqslant k} M_H(2, c_1, k')$, and let the number

$$m_0 = \max_{M_H \subset \bigcup M_H(2,c_1,k)} \dim M_H \tag{1.19}$$

be the maximum of dimensions of the components of the union.

For any component M_H consider the direct product

$$M_H \times |dH|_0, \tag{1.20}$$

where $|dH|_0 \subset \mathbb{P}H^0(\mathcal{O}_S(dH))$ denotes open set of reduced curves of the complete linear system of curves.

Consider the subvariety of the direct product:

$$\begin{gathered} \mathcal{D}^d \subset M_H \times |dH|_0, \\ \mathcal{D}^d = \{(E, C) | l(E|_C) \geqslant 0\}, \end{gathered} \tag{1.21}$$

that is the subset of a pair $(E, C), E \in M_H, C \in |dH|_0$ such that $E|_C$ is not stable.

The projections pr_1 and pr_2 of the direct product $M_H \times |dH|$ onto its components define the projections pr_M and pr_C of $\mathcal{D}^d$:

$$\begin{tikzcd} & M_H \times |dH|_0 \arrow[dl, "\mathrm{pr}_1"'] \arrow[dr, "\mathrm{pr}_2"] \arrow[d, phantom, "\bigcup"] & \\ M_H & \mathcal{D}^d \arrow[l, "\mathrm{pr}_M"'] \arrow[r, "\mathrm{pr}_C"] & |dH|_0 \end{tikzcd} \qquad \begin{aligned} \mathrm{pr}_M &= \mathrm{pr}_1|_{\mathcal{D}^d} \\ \mathrm{pr}_C &= \mathrm{pr}_2|_{\mathcal{D}^d} \end{aligned} \tag{1.22}$$

For every $E \in M_H$ we define the subvariety of the projective space $|dH|$:

$$\mathcal{D}^d(E) = \mathrm{pr}_M^{-1}(E) \subset |dH|. \tag{1.23}$$

It is the set of curves in $|dH|$ for which the restriction of E is not stable.

Let us prove that there exists a number d_0^{II} such that for every $d \geqslant d_0^{II}$

$$\dim \mathcal{D}^d < \dim |dH|. \tag{1.24}$$

But

$$\begin{gathered} \dim \mathcal{D}^d = \dim M_H + \dim \mathcal{D}^d(E_{\text{gen}}) \\ \dim M_H \leqslant m_0 \text{ (see (1.19))}. \end{gathered} \tag{1.25}$$

Hence, it suffices to show that there exists a number d_0^{II} such that for *every* $E \in M_H$ and for every $d \geqslant d_0^{II}$

$$\operatorname{codim}_{|dH|} \mathcal{D}^d(E) \geqslant m_0 + 1. \tag{1.26}$$

To prove that, let us consider the subvariety of reducible curves in the projective space:

$$S^d|H| = \{C \in |dH| \quad | \quad C = c_1 \cup C_2 \cup \cdots \cup C_d, C_i \in |H|\}. \tag{1.27}$$

(We can assume that for $\mathcal{O}_S(H) = F$ the condition (1.3) is true passing if necessary to a multiple of H.)

Now, let

$$\mathcal{D}^d_{\text{red}}(E) = \mathcal{D}^d(E) \bigcap S^d|H| \tag{1.28}$$

be the set of the reducible curves of nonstability for E. Then

$$\operatorname{codim}_{|dH|} \mathcal{D}^d \geqslant \operatorname{codim}_{S^d|H|} \mathcal{D}^d_{\text{red}}(E). \tag{1.29}$$

At last, let

$$\sigma : |H|^d \longrightarrow S^d|H| \tag{1.30}$$

be the standard $d!$-sheeted covering and

$$\tilde{\mathcal{D}}^d_{\text{red}}(E) = \sigma^{-1}(\mathcal{D}^d_{\text{red}}(E)). \tag{1.31}$$

Let us remark that

$$\operatorname{codim}_{S^d|H|} \mathcal{D}^d_{\text{red}}(E) = \operatorname{codim}_{|H|^d} \tilde{\mathcal{D}}^d_{\text{red}}(E). \tag{1.32}$$

Now it suffices to show that for every $E \in M_H$ there exists a number $d_0^{II}(E)$ such that for every $d \geqslant d_0^{II}(E)$

$$\operatorname{codim}_{|H|^d} \tilde{\mathcal{D}}^d_{\text{red}}(E) \geqslant m_0 + 1 \tag{1.33}$$

(see (1.26), (1.29) and (1.31)).

But if this assertion were wrong, then there would exist a number N_0 such that for $d \geqslant N_0$ we have

$$
(1.34) \qquad \begin{array}{ccccc}
\tilde{\mathcal{D}}^d_{\mathrm{red}}(E) & = & \tilde{\mathcal{D}}^{N_0}_{\mathrm{red}}(E) & \times & |H|^{d-N_0} \\
\bigcap & & \bigcap & & \| \\
|H|^d & = & |H|^{N_0} & \times & |H|^{d-N_0},
\end{array}
$$

that is

$$
(1.35) \qquad l(E|_{C_1 \cup \dots \cup C_{N_0}}) \geqslant 0 \implies l(E|_{C_1 \cup \dots \cup C_{N_0} \cup C_{N_0+1} \cup \dots C_d}) \geqslant 0
$$

for every collection $C_{N_0+1}, \dots, C_d$ of curves of $|H|$. But this would give a contradiction.

Indeed, for every pair of curves C and C' on S

$$
(1.36) \qquad l(E|_{C \cup C'}) \leqslant l(E|_C) + l(E|_{C'}),
$$

because every line subbundle $L \subset E|_{C \cup C'}$ defines two line subbundles

$$
(1.37) \qquad L_1 = L|_C \subset E|_C \quad \text{and} \quad L_2 = L|_{C'} \subset E|_{C'}
$$

with an equality

$$
(1.38) \qquad \deg L = \deg L_1 + \deg L_2.
$$

Now, if $\mathcal{D}^1(E) \neq |H|$, then

$$
(1.39) \qquad C \notin \mathcal{D}^1(E) \Longrightarrow l(E|_C) \leqslant -1.
$$

Consider the number

$$
(1.40) \qquad \lambda(E) = \max_{(C_1 \cup \dots \cup C_{N_0}) \in |H|_0^{N_0}} l(E|_{C_1 \cup \dots \cup C_{N_0}})
$$

and a number d such that

$$
(1.41) \qquad d - N_0 \geqslant \lambda(E) + 1.
$$

Choosing a collection $(C_{N_0+1} \cup \dots \cup C_d)$ of curves from $|H|$ such that

$$
(1.42) \qquad \forall i \quad C_{N_0+i} \notin \mathcal{D}^1(E),
$$

we have

$$
(1.43) \qquad l(E|_{C_1 \cup \dots \cup C_{N_0} \cup C_{N_0+1} \cup \dots \cup C_d}) \leqslant \underbrace{l(E|_{C_1 \cup \dots \cup C_{N_0}}}_{\leqslant \lambda(E)} + \underbrace{\sum_{i=1}^{d-N_0} (E|_{C_{N_0+i}})}_{\leqslant -(d-N_0)} \leqslant -1
$$

by(1.39) – (1.42). This contradicts (1.34) and (1.35).

So the proof of the existence of the number $d_0^{II}(E)$ (1.33) is now completed by observing that we can assume

$$\mathcal{D}^1(E) \neq |H| \tag{1.44}$$

(see (1.42)) by the theorem of Mehta–Ramanathan–Flenner (see [13], [7]) passing if necessary to a multiple of H.

Now, there exists a number

$$d_0^{II}(M_H) = \max_{E \in M_H} d_0^{II}(E) \tag{1.45}$$

by (1.6) and the number

$$d_0^{II} = \max_{M_H \subset \bigcup M_H(2,c_1,k')} d_0^{II}(M_H). \tag{1.46}$$

The number of Theorem 1.1 is

$$d_0(k) = \max(d_0^I, d_0^{II}), \tag{1.47}$$

where d_0^I is a number (1.12), and we are done.

Remark. The proof of this theorem for the moduli space $M_H(r, c_1, c_2)$ of H-stable vector bundles of rank r with Chern classes $c_1 \in \operatorname{Pic} S$ and $c_2 \in \mathbb{Z}$ is similar except that in the general case the number $l(E)$ in (1.17) is replaced by the minimal slope of the nonzero torsion free quotient of E.

§ 2 Compactification. Extension of the restriction map.

The moduli space $M_H(2, c_1, k)$ of H-stable vector bundles on a regular algebraic surface S may not be complete, but it has a natural compactification $\overline{M}_H(2, c_1, k)$ constructed by Gieseker [9]. The geometric points of the corona (boundary of the closure)

$$G(M_H(2, c_1, k)) = \overline{M}_H(2, c_1, k) - M_H(2, c_1, k) \tag{2.1}$$

represent classes of torsion-free sheaves semistable in the sense of Giezeker (see [9]).

Let us recall this construction. First of all, replacing E by $E \otimes \mathcal{O}_S(dH)$, we can pass from our family of vector bundles to a family of vector bundles such that for all $E \in M_H(2, c_1, k)$ we have the conditions (1.2) and (1.3), that is $H^i(E) = 0$, $i > 0$, and E is generated by its global sections. Hence

$$\operatorname{rk} H^0(E) = \chi(E) \tag{2.2}$$

does not depend on $E \in M_H(2, c_1, k)$.

Let V be a vector space with $\operatorname{rk} V = \operatorname{rk} H^0(E)$. Choose a linear isomorphism

$$f : H^0(E) \longrightarrow V. \tag{2.3}$$

Let

$$H^0(\mathcal{O}_S(c_1)) = W \tag{2.4}$$

be the space of global sections of

$$\det E = \Lambda^2 E = \mathcal{O}_S(c_1). \tag{2.5}$$

Now, consider the homomorphism

$$\begin{gathered} t_{(E,f)} : \Lambda^2 V = \Lambda^2 H^0(E) \longrightarrow H^0(\det E) = W \\ t_{(E,f)}(s_1, s_2) = s_1 \wedge s_2. \end{gathered} \tag{2.6}$$

Thus, we can consider $t_{(E,f)}$ as a tensor from $\Lambda^2 V^* \otimes W$ and up to $\mathbb{C}^*$ as a point of the projective space $\mathbb{P}\Lambda^2 V^* \otimes W$.

The group $SL(V)$ acts on $\mathbb{P}\Lambda^2 V^* \otimes W$. According to geometric invariant theory the semistable orbits of this action fill a subvariety

$$P_{ss} \subset \mathbb{P}\Lambda^2 V^* \otimes W \tag{2.7}$$

and the algebraic variety

$$\begin{gathered} \mathcal{P}(n, n') = P_{ss}/SL(V) \\ n = \operatorname{rk} V, \quad n' = \operatorname{rk} W \end{gathered} \tag{2.8}$$

is complete. This variety can be embedded into a projective space by homogeneous invariant forms of sufficiently large degree:

$$\mathcal{P}(n, n') \xrightarrow{i} \mathbb{P}^N \tag{2.9}$$

and this embedding induces the linear vector bundle on $\mathcal{P}$

$$i^* \mathcal{O}_{\mathbb{P}^N}(1) \in \operatorname{Pic} \mathcal{P}(n, n'). \tag{2.10}$$

The orbit of the tensor $\mathbb{P}t_{(E,f)}$ is independent of f and depends on the vector bundle E only. Then we have a map

$$M_H(2, c_1, c_2) \xrightarrow{G} \mathcal{P}(n, n'), \tag{2.11}$$

which provides a line bundle

$$L = G^* \cdot i^* \mathcal{O}_{\mathbb{P}^N}(1). \tag{2.12}$$

In the paper [9] Gieseker proved that

1) G is an embedding. He gives the exact construction for recovering E from the tensor $t_{(E,f)}$ (2.6).
2) This construction leads to the interpretation of semi-stable tensors of $\mathbb{P}V^* \otimes W$ which are contained in the closure of $G(M_H(2, c_1, c_2))$ as semi-stable torsion-free sheaves on S.

The variety $\overline{M}_H(2, c_1, c_2)$ can be singular and it is natural to consider the group of formal divisor classes over $\mathbb{Q}$:

$$\text{Pic}_{\mathbb{Q}}\overline{M}_H(2, c_1, c_2) = \text{Pic}\ \overline{M}_H(2, c_1, c_2) \otimes \mathbb{Q}.$$

Now, if our surface S is simply connected and the intersection form q_S on $H^2(S, \mathbb{Z})$ is even, then there exists a line bundle $K_S^{\frac{1}{2}}$ on$S$ such that

$$(2.13) \qquad (K_S^{\frac{1}{2}})^{\otimes 2} = K_S \in \text{Pic}\ S,$$

where K_S is the canonical vector bundle on S. If intersection form q_S is odd, then we consider a virtual vector bundle

$$(2.13') \qquad K_S^{\frac{1}{2}} \in \text{Pic}\ S \otimes \mathbb{Q}, \quad (K_S^{\frac{1}{2}})^{\otimes 2} = K_S \in \text{Pic}\ S.$$

Now, we can construct a special line bundle $L_H \in \text{Pic}_{\mathbb{Q}}\overline{M}_H(2, c_1, c_2)$ by describing each fibre over $E \in \overline{M}_H(2, c_1, c_2)$ as line
(2.14)

$$(L_H)_E = \bigotimes_{i=0}^{2}(\Lambda^{\max} H^i(E \otimes K_S^{\frac{1}{2}} \otimes H)^{(-1)^i}) \otimes (\Lambda^{\max} H^i(E^\vee \otimes K_S^{\frac{1}{2}} \otimes H^\vee)^{(-1)^{i+1}}).$$

It is not hard to see that there exists a rational number $\alpha(c_1, c_2)$ such that in $\text{Pic}_{\mathbb{Q}} M_H(2, c_1, c_2)$

$$(2.15) \qquad L = L_H^{\alpha(c_1, c_2)}$$

where L is the line bundle (2.12).

Indeed, if $E \otimes K_S^{\frac{1}{2}} \otimes H$ satisfies the conditions (1.2) – (1.3), then

$$H^i(E \otimes K_S^{\frac{1}{2}} \otimes H) = 0, \qquad i \neq 0$$
$$H^i(E^\vee \otimes K_S^{\frac{1}{2}} \otimes H^\vee) = 0, \quad i \neq 2,$$

and the space $H^0(E \otimes K_S^{\frac{1}{2}} \otimes H)$ is dual to $H^2(E^\vee \otimes K_S^{\frac{1}{2}} \otimes H^\vee)$ by Serre-duality. Hence in that case

$$(2.16) \qquad (L_H)_E = (\Lambda^{\max} H^0(E \otimes K_S^{\frac{1}{2}} \otimes H))^2,$$

but by the invariant theory the homogenous invariant forms defining (2.9) are forms from $\det V = \det H^0(E \otimes K_S^{\frac{1}{2}} \otimes H)$ and we have (2.15), where L is the line bundle(2.12).

In analogy with these constructions we have the construction of the compactification of the moduli space of stable vector bundles with fixed determinant on the curve C.

There are two classes (1.7):

1) If $\{c_1\} = -1$, the $M_C(2,-1)$ is a complete, smooth, rational variety of dimension $3(g(C)-1)$, where $g(C)$ is a genus of the curve C.

2) If $\{c_1\} = 0$, the dimension is the same, we may assume that $\mathcal{O}_C(\{c_1\}) = \mathcal{O}_C$, and $M_C(2,0)$ may be compactified by the family of 2-vector bundles of type

$$\begin{gathered} E = L \oplus L^\vee \\ L \in J(C), \quad \deg L = 0, \end{gathered} \tag{2.17}$$

where $J(C)$ is an Jacobian of C. Hence

$$G(M_C(2,0)) = \overline{M}_C(2,0) - M_C(2,0) = K = J(C)/\{\pm \mathrm{id}\} \tag{2.18}$$

is the Kummer variety of the Jacobian $J(C)$. If $g(C) > 2$, then K is a singular set

$$K = \operatorname{Sing} \overline{M}_C(2,0). \tag{2.19}$$

If $g(2) = 2$, then

$$\overline{M}_C(2,0) = \mathbb{P}^3. \tag{2.20}$$

Remark. Let us remark that in this case $\overline{M}_H(2,0)$ as an algebraic variety does not depend on the moduli of curve C.

We will assume from now to the end of this paragraph that $C \in |dH|$ is even, that is $\mathcal{O}_S(\frac{d}{2}H) \in \operatorname{Pic} S$. In that case, the expression $\mathcal{O}_S(\frac{1}{2}C)$ makes sense. Let us return to the restrictions map (1.8)

$$\begin{array}{ccc} M_H(2,c_1,k) & \xrightarrow{\mathrm{res}_C} & M_C(2,0) \\ \bigcap & & \bigcap \\ \overline{M}_H(2,c_1,k) & & \overline{M}_C(2,0). \end{array} \tag{2.21}$$

Theorem 2.1.

(1) *The restriction map* (2.21) *extends to a map*

$$\overline{\mathrm{res}}_C : \overline{M}_H(2,c_1,k) \longrightarrow \overline{M}_C(2,0). \tag{2.22}$$

(2) *For every* $F \in G(M_H(2,c_1,k))$

$$\overline{\mathrm{res}}_C(F) = F^{\vee\vee}|_C, \tag{2.23}$$

where $F^{\vee\vee}$ *is the reflexive envelope of* F.

We would like to sketch the proof at the end of the paragraph.

Let us recall the notion of a reflexive envelope. Consider for a sheaf F on S the dual sheaf $F^\vee = \mathrm{Hom}_{\mathcal{O}_S}(F, \mathcal{O}_S)$. Then the sheaf $F^{\vee\vee}$ is called the *reflexive envelope* of F, because there exists the canonical homomorphism $F \xrightarrow{\text{can}} F^{\vee\vee}$ which can be completed to an exact sequence

$$0 \longrightarrow T(F) \longrightarrow F \xrightarrow{\text{can}} F^{\vee\vee} \longrightarrow C(F) \longrightarrow 0, \tag{2.24}$$

where $T(F)$ is a torsion of F, and if F is torsion-free, then $T(F) = 0$ and F is subsheaf $F^{\vee\vee}$.

The sheaf $C(F)$ is called the *cotorsion sheaf* of F. If $C(F) = 0$, then a torsion-free sheaf is called a *reflexive sheaf*. It is easy to see that $F^{\vee\vee}$ is a reflexive sheaf and for any smooth surface

$$F = F^{\vee\vee} \Longleftrightarrow F \text{ is locally free.} \tag{2.25}$$

The cotorsion sheaf $C(F)$ is an *Artinian sheaf*, that is a sheaf with 0-dimensional support. Moreover,

(1) F is H-stable $\Longleftrightarrow F^{\vee\vee}$ is H-stable;

(2)
$$c_1(F) = c_1(F^{\vee\vee}); \tag{2.26}$$

(3) $c_2(F) = c_2(F^{\vee\vee}) + \mathrm{rk}\, H^0(C(F))$.

For the investigation of the extension of the restriction map it is very useful to divide the geometrical points of the corona (2.1) into two parts:

$$\begin{gathered} G^s(M_H(2,c_1,k)) = \{F \in G(M_H(2,c_1,k)) \mid F \text{ is } H\text{-stable}\}\,, \\ G^{ss}(M_H(2,c_1,k)) = \\ = \{F \in G(M_H(2,c_1,k)) \mid F \text{ is } H\text{-semistable but not stable}\}\,, \\ G^s(M_H(2,c_1,k)) \cup G^{ss}(M_H(2,c_1,k)) = G(M_H(2,c_1,k))\,. \end{gathered} \tag{2.27}$$

We have two simple corollaries of Theorem 2.1:

Proposition 2.1.

$$\overline{\mathrm{res}}_C(G^s(M_H(2,c_1,k))) \subset \bigcup_{k'<k} \mathrm{res}_C(M_H(2,c_1,k')) \subset M_C(2,0).$$

We can see that the boundary of the image of the stable part of the corona is stratified by the subset of $\mathrm{res}_C(M_H(2,c_1,k'))$, $k' < k$, and by Theorem 1.1 the restriction map is an embedding for such $M_H(2,c_1,k')$.

Proposition 2.2. $\overline{\mathrm{res}}_C(G^{ss}(M_H(2,c_1,k)))$ *is a finite subset of the Kummer variety* (2.18).

Proof. By (2.27) we have

$$F \in G^{ss}(M_H(2, c_1, k)) \iff F^{\vee\vee} \in G^{ss}(M_H(2, c_1, c_2(F^{\vee\vee}))).$$

Furthermore, let $\mathcal{O}_S(\mathcal{D}) \subset F^{\vee\vee}$ be a line subbundle with intersection number $\mathcal{D} \cdot H = 0$. Then we have the exact sequence

$$0 \longrightarrow \mathcal{O}_S(\mathcal{D}) \longrightarrow F^{\vee\vee} \longrightarrow J_\xi \otimes \mathcal{O}_S(c_1 - \mathcal{D}) \longrightarrow 0,$$

where J_ξ is an ideal sheaf of the 0-dimensional subscheme ξ of S. From this

$$k' = c_2(F^{\vee\vee}) = \deg \xi + c_1 \cdot \mathcal{D} - \mathcal{D}^2.$$

Hence

$$k \geqslant k' \geqslant c_1 \cdot \mathcal{D} - \mathcal{D}^2.$$

Now, $\mathcal{D} \subset H^\perp \subset \operatorname{Pic}_S$, where $H^\perp$ is the orthogonal sublattice to H of Pic S. By the Hodge index theorem the intersection form on $H^\perp$ is negative definite. From this it is easy to see that there exists a finite number such of such divisor classes. Now, by Theorem 2.1 (2.23)

$$\overline{\operatorname{res}}_C(F) = \mathcal{O}_C(\mathcal{D}.C) \oplus \mathcal{O}_S(-\mathcal{D}.C), \tag{2.28}$$

and we are done. □

Furthermore, we can see that $\overline{\operatorname{res}}_C$ is not an embedding on the corona, because one forgets the cotorsion sheaf and the canonical epimorphism of its reflexive envelope to the cotorsion sheaf.

Let us return to Proposition 2.1. In $M_C(2, \{c_1\})$ consider the configuration

$$M_{C\subset S}(2, c_1, k) = \bigcup_{k'<k} \operatorname{res}_C M_H(2, c_1, k') \subset M_C(2, \{c_1\}). \tag{2.29}$$

This is a complete subvariety, perhaps apart from a set of points. Consider the numbers

$$\mu_{C\subset S}(2, c_1, k) = \text{number of irreducible components of } M_{C\subset S}(2, c_1, k). \tag{2.30}$$

This number and the whole configuration are very interesting invariants of the pair $C \subset S$.

Now, let

$$\begin{aligned} &M^{\mathrm{irr}}_{C\subset S}(2, c_1, k) \quad \text{be} \quad \left\{ \begin{matrix} \text{irreducible components of } M_{C\subset S}(2, c_1, k) \\ \text{containing } \operatorname{res}(M_H(2, c_1, k)) \end{matrix} \right\} \\ &M^{\mathrm{con}}_{C\subset S}(2, c_1, k) \quad \text{be} \quad \left\{ \begin{matrix} \text{connected components of } M_{C\subset S}(2, c_1, k) \\ \text{containing } \operatorname{res}(M_H(2, c_1, k)) \end{matrix} \right\} \\ &\mu^{\mathrm{irr}}_{C\subset S}(2, c_1, k) \quad = \quad \text{a number irreducible components } M^{\mathrm{irr}}_{C\subset S}, \end{aligned} \tag{2.31}$$

and so on.

Consider some partial cases of the general situation:

Example. Artamkin component. Let us recall that torsion-free sheaf F is a *quasibundle* if

$$C(F) = \mathcal{O}_\xi = \overset{d}{\underset{i=1}{\oplus}} \mathcal{O}_{p_i}, \tag{2.32}$$

is the structure sheaf of a 0-dimensional subscheme without nilpotents. A quasibundle F quasitrivial, if

$$F^{\vee\vee} = V \otimes \mathcal{O}_S, \quad V = \mathbb{C}^2. \tag{2.33}$$

The canonical epimorphism (see (2.24))

$$F^{\vee\vee} = V \otimes \mathcal{O}_S \xrightarrow{\phi} \mathcal{O}_\xi = \overset{d}{\underset{i=1}{\oplus}} \mathcal{D}_{p_i} = C(F) \tag{2.34}$$

is the sum of the local epimorphisms $V \otimes \mathcal{O}_S \xrightarrow{\phi_i} \mathcal{O}_{p_i}$ defined uniquely by the line $\ker \phi_i$. Hence a quasitrivial F is defined uniquely by a cycle $\tilde{\xi} = \sum_{i=1}^{d} \tilde{p}_i$ in the direct product $S \times \mathbb{P}(V)$ where $\tilde{p}_i = (p_i, \ker \phi_i)$ up to the action $\mathrm{Aut}\mathbb{P}(V) = \mathbb{P}\mathrm{GL}(V)$. Thus the moduli space of quasitrivial sheaves with $c_2 = d$ is

$$MQT(d) = \Sigma_d \backslash (S \times \mathbb{P}(V))^d / \mathbb{P}\mathrm{GL}(V), \tag{2.35}$$

where Σ_d is the symmetric group permuting the factors of the direct product.

There is

Artamkin's Theorem.[1]

(1) *For a general quasitrivial sheaf F the local universal deformation exists if $c_2(F) > \max(3, 3p_g)$.*

(2) *The general sheaf of the universal deformation of a general quasitrivial sheaf of* rk 2 *with $c_2 > 3p_g(S)$ is locally free ($p_g(S) = h^{2,0}(S)$ is the geometric genus of S).*

(3) *The general sheaf in the numerical deformation of a general quasitrivial sheaf F is H-stable with respect to any polarization H.*

From this we have the natural

Definition. A component

$$M_H^A(2,0,k) \subset M_H(2,0,k)\ ,$$

whose corona $GM_H^A(2,0,k)$ contains a general quasitrivial sheaf, is called an *Artamkin component*.

The component $M_H^A(2,0,k)$ is unique, because for a general quasitrivial F

$$\mathrm{Ext}^2(F,F) = \mathrm{Ext}^2(\mathcal{O}) = p_g$$

and F is a smooth point of $\overline{M}_H(2,0,k)$, so F cannot lie in the intersection of 2 such components.

$$\dim\, M_H^A(2,0,k) = 4c_2 - 3(p_g+1). \tag{2.36}$$

Furthemore, the Artamkin component $M_H^A(2,0,k)$ has a nice property

Proposition 2.3. [1]

(1) *For any polarization $H' \in V^+(S)$ (1.4) there exists a Zariski open set $M_{H,H'}^A$ of $M_H^A(2,0,k)$, the corresponding bundles of which are H'-stable bundles.*

(2) *$M_{H,H'}^A$ is invariant with respect to the action of the group of birational automorphism of S.*

Thus, a general vector bundle of the Artamkin component is absolutely stable and we have the very important

Question. *Does the intersection*

$$\bigcap_{H' \in V^+(S)} M_{H,H'}^A$$

containing an open set of $M_H^A(2,0,k)$ in Zariski topology (in the complex topology)?

It is easy to see that in the situation of Theorem 1.1 for the Artamkin component $M^A(2,0,k)$ we have

$$\overline{\mathrm{res}}_C(G(M_H^A(2,0,k))) \supset \mathrm{res}_C(M_H^A(2,0,k-1)). \tag{2.37}$$

Hence, if for our surface S

$$M_H(2,0,k) = M_H^A(2,0,k) \tag{2.38}$$

(it is true, for example, for the projective plane $\mathbb{P}^2$), then the number (2.30)

$$\mu_{C\subset S}(2,0,k) = 1$$

and we have the simplest configuration for $M_{C\subset S}(2,0,k) \subset M_C(2,0)$ (see (2.29)):

$$\begin{aligned}
\mathrm{Sing}\ \overline{\mathrm{res}}_C(\overline{M}_H^A(2,0,k)) &= \overline{\mathrm{res}}_C(\overline{M}_H^A(2,0,k-1))\ , \\
\mathrm{Sing}\ \mathrm{Sing}\,\overline{\mathrm{res}}_C(\overline{M}_H^A(2,0,k)) &= \mathrm{Sing}\ \overline{\mathrm{res}}_C(\overline{M}_H^A(2,0,k-1)) = \\
&= \overline{\mathrm{res}}_C(\overline{M}_H^A(2,0,k-2))\ ,
\end{aligned} \tag{2.39}$$

where Sing M is a set of singular points of M. From (2.36) we can see that

$$\dim\, M_H^A(2,0,k) - \dim\, M_H^A(2,0,k-1) = 4 \tag{2.40}$$

and we have Sing-filtration

$$\text{(2.41)}\quad \operatorname{Sing}\overline{\operatorname{res}}_C(\overline{M}_H^A(2,0,k)) \supset \operatorname{Sing}\overline{\operatorname{res}}_C(\overline{M}_H^A(2,0,k-1)) \supset \\ \supset \operatorname{Sing}\overline{\operatorname{res}}_C(\overline{M}_H^A(2,0,k-2)) \supset \cdots$$

of relative codimension 4. This filtration gives us the sequence of numbers

$$\text{(2.42)}\qquad \nu_{C\subset S}(k') = \begin{pmatrix} \text{the multiplicity of a general singular point of} \\ \operatorname{Sing}\overline{\operatorname{res}}_C\left(\overline{M}_H^A(2,0,k+1)\right) \end{pmatrix}$$

for $k' \leqslant k-1$.

Question. *Compute this sequence for* $\mathbb{P}^2$.

In the general case we have only the $M^{\mathrm{irr}}_{C\subset S}(2,0,k)$ which it is another compactification of $M_H(2,0,k)$ and we shall call it a *compactification of dense packing*. It is an analogue of the Satake-compactification of the moduli spaces of abelian varieties.

Sketch of the Proof of Theorem 2.1. We prove the assertion 1) by comparing the construction of Gieseker's compactification (2.2) - (2.11) with the construction of the theory of deformations of pairs (C,E) of Gieseker–Morrison (see [10]), where C is a curve and E a vector bundle on C. We can identify the tensors of the first (see (2.6)) and second construction (see [10]), because we can assume that $H^0(E) = H^0(E|_C)$ passing if necessary to a multiple of C.

To prove the assertion 2) of the Theorem 2.1 we need some homological algebra from § 2 of [1] to describe the relations between functor Tor (for restrictions) and functor Ext (for the operation $^{\vee\vee}$ on the sheaves).

§ 3 The projective space of conformal blocks.

The structure of the moduli space $M_C(2,0)$ of stable vector bundles of rk 2 with trivial determinant has been intensively studied. First of all, for every $E \in M_C(2,0)$ there exists a representation of the fundamental group $\pi_1(C)$

$$\text{(3.1)}\qquad \rho : \pi_1(C) \longrightarrow (2,\mathbb{C})$$

such that

$$\text{(3.2)}\qquad E = U \times \mathbb{C}^2/\pi_1(C),$$

where U is the universal covering of C and for $v \in \mathbb{C}$, $z \in U$ and $g \in \pi_1(C)$

$$\text{(3.3)}\qquad g(z,v) = (g(z), \rho(g)(v)).$$

Moreover, there exists a unique conjugacy class of unitary irreducible representations (see [16]). Thus we can construct a topological model of $M_C(2,0)$.

Let $g(C)$ be the genus of C, $SU(2)^{2g}$ be the product of $2g$ copies of the unitary group $SU(2)$ and the map

$$
\begin{aligned}
&f_g : SU(2)^{2g} \longrightarrow SU(2) \\
(3.4)\qquad &f_g(A_1,B_1,\dots,A_g,B_g) = \prod_{i=1}^{g(c)} [A_i,B_i], \quad A_i,B_i \in SU(2),
\end{aligned}
$$

where $[A,B]=ABA^{-1}B^{-1}$, is the "product of commutators" map. The group $SU(2)$ acts on $SU(2)^{2g}$ conjugating the components: for $g \in SU(2)$

$$
(3.5)\qquad g(A_1,B_1,\dots,A_g,B_g) = (gA_1g^{-1}, gB_1g^{-1},\dots,gA_gg^{-1}, gB_gg^{-1}),
$$

and the center of $SU(2)$ acts trivially.

Now, consider the open subset $SU(2)_0^{2g}$ of $SU(2)^{2g}$containing a collection of matrices

$$
(A_1,B_1,\ \dots\ ,A_g,B_g)
$$

without a common eigenvector. Then

$$
(3.6)\qquad M_C(2,0) \stackrel{\text{top}}{=} M(2,0) = (SU(2)_0^{2g} \cap f_g^{-1}(\mathrm{id}))/\mathbb{P}U(2),
$$

and

$$
\begin{aligned}
&(SU(2)^{2g} - SU(2)_0^{2g}) \subset f_g^{-1}(\mathrm{id})\ , \\
(3.7)\qquad &(SU(2)^{2g} - SU(2)_0^{2g})/\mathbb{P}U(2) = \mathrm{Hom}(\pi_1(C), U(1))/\{\pm\mathrm{Id}\} = \\
&\qquad = (U(1))^{2g}/\{\pm \mathrm{id}\} = K
\end{aligned}
$$

is the Kummer surface of $J(C)$.

Hence, topologically the compactification (2.18) is

$$
(3.8)\qquad \overline{M}(2,0) = f_g^{-1}(\mathrm{id})/\mathbb{P}U(2).
$$

It is easy to see (see for example [17]) that

$$
(3.9)\qquad H^2(M(2,0),\mathbb{Z}) = \mathbb{Z}.
$$

The holomorphic structure on C defines on $M(2,0)$ the structure of an algebraic variety $M_C(2,0)$ with Pic $M_C(2,0)=\mathbb{Z}$ and compactification (2.18). About the geometry of $\overline{M}_C(2,0)$ we do not know so much:

(1) $\overline{M}_C(2,0)$ is a Gorenstein variety satisfying the condition (2.19);

(2)

$$
(3.10)\qquad \mathrm{Pic}\ \overline{M}_C(2,0) = \mathbb{Z};
$$

(3) if $g \geqslant 3$, then $\overline{M}_C(2,0)$ determines the curve C uniquely.

Remark. For $\{c_1\}$ odd, $M_C(2,\{c_1\})$ is a complete and smooth algebraic variety.

We can compute a positive generator L_0 of Pic $M_C(2,0)$ in different ways. We need all of them.

1) Let $J_{g-1}(C)$ be the variety of divisor classes of degree $g-1$ on C and $K_C^{\frac{1}{2}} \in J_{g-1}(C)$ such that $(K_C^{\frac{1}{2}})^{\otimes 2} = K_C$ is the canonical class and consider the subset of $\overline{M}_C(2,0)$

$$\Delta = \{E \in \overline{M}_C(2,0) | h^0(E \otimes K_C^{\frac{1}{2}}) > 0\}. \tag{3.11}$$

Then

$$\mathcal{O}_{\overline{M}_C(2,0)}(\Delta) = L_0. \tag{3.12}$$

2) The line bundle $K_C^{\frac{1}{2}}$ defines a spinor structure on C (see [2]). The holomorphic structure on $E \in \overline{M}_C(2,0)$ defines a unique gauge class of unitary antiautodual connections on E_{top}. Let ∂_E be the selfadjoint Dirac operator acting on the sections of $E \otimes K_C^{\frac{1}{2}}$. Then the index of ∂_E is zero and we can define the line bundle L_0 on $\overline{M}_C(2,0)$ with the fibre over $E \in \overline{M}_C(2,0)$

$$(L_0)_E = (\det \operatorname{ind} \partial_E)^{-1} = \Lambda^{\max} H^0(E \otimes K_C^{\frac{1}{2}}) \otimes (\Lambda^{\max} H^1(E \otimes K_C^{\frac{1}{2}})^{-1}) \tag{3.13}$$

(recall that $E^\vee = E$ in our case).

3) Consider the fantastic situation (not real life).

Suppose that on $C \times M_C(2,0)$ there exists a universal family, that is a vector bundle $\mathfrak{E}$ on $C \times M_C(2,0)$ such that for every $[E] \in M_C(2,0)$

$$\mathfrak{E}|_{C \times [E]} = E. \tag{3.14}$$

Consider the second Chern class

$$c_2(\mathfrak{E}) \in H^4(C \times M_C(2,0), \mathbb{Z}) \tag{3.15}$$

and the (2,2) Künneth component $c_2^{(2,2)}(\mathfrak{E})$ of $c_2(\mathfrak{E})$. Then it is easy to see that for every line bundle L on $M_C(2,0)$

$$c_2^{(2,2)}(\mathfrak{E} \otimes \mathrm{pr}_M^* L) = c_2^{(2,2)}(\mathfrak{E}). \tag{3.16}$$

Indeed, $c_2(\mathfrak{E} \otimes \mathrm{pr}_M^* L) = c_2(\mathfrak{E}) + c_1(\mathfrak{E}) \cdot c_1(L) + c_1^2(L)$. But $c_1(\mathfrak{E}) = \mathrm{pr}_M^* L_1$, $L_1 \in \operatorname{Pic} M_C(2,0)$ because $\det E$ is trivial. Hence $c_1(\mathfrak{E}) \cdot c_1(L) + c_1^2(L)$ has Künneth type (0,4).

Now,

$$c_2^{(2,2)}(\mathfrak{E}) = [C] \otimes c_1(L_0), \tag{3.17}$$

where $[C]$ is the fundamental class of C (as 2-manifold). However, in real life there is no universal bundle on the direct product $C \times M_C(2,0)$, and one must use the following "covering trick". There exists a double cover $\phi : \tilde{M}_C(2,0) \longrightarrow M_C(2,0)$ such that on $C \times \tilde{M}_C(2,0)$ a "universal family" $\tilde{\mathfrak{E}}$ exists and

$$c_2^{(2,2)}(\tilde{\mathfrak{E}}) = [C] \otimes \phi^*(c_1(L_0)). \tag{3.17$'$}$$

Let us use the following notation for the positive generator of Pic $M_C(2,0)$: L_0 as line bundle and Δ as divisor (see (3.11)). Consider the complete linear system $|\Delta| = \mathbb{P}H^0(L_0)$ on $\tilde{M}_C(2,0)$.

The space of the global sections of L_0 is called the space of conformal blocks (see [11]). Define

$$\mathcal{H}_C = H^0(L_0). \tag{3.18}$$

The direct construction gives us an isomorphism

$$\mathcal{H}_C = H^0(J(C), \mathcal{O}(2\Theta)), \tag{3.19}$$

where Θ is the theta-divisor of the Jacobian $J(C)$. Hence

$$\operatorname{rk} \mathcal{H}_C = 2^{g(C)}. \tag{3.20}$$

Beauville's Theorem. [4] *The linear system $|\Delta|$ on $\overline{M}_C(2,0)$ is base point free and defines a morphism*

$$f_0 : \overline{M}_C(2,0) \longrightarrow \mathbb{P}\mathcal{H}_C, \tag{3.21}$$

which is finite and $\deg f_0 = 1$ *if C is not hyperelliptic and* 2 *otherwise.*

We see that the situation for the pair $(\overline{M}_C(2,0), \Delta)$ is very similar to the theory of theta-functions of second order on $J(C)$ and what is more, we can "extend" theta-functions of second order to $\overline{M}_C(2,0)$.

It is useful to consider instead of the moduli space $M_C(2,0)$ of vector bundles with trivial determinant the moduli space $M_C(2,K_C)$ of vector bundles with the canonical class K_C as determinant. Of course, as variety $\overline{M}_C(2,0)$ is isomorphic to $\overline{M}_C(2,K_C)$ by tensor-multiplying the vector bundles by $K_S^{\frac{1}{2}}$.

For any vector bundle $E \in \overline{M}_C(2,K_C)$ consider the following subvariety of the Jacobian $J(C)$:

$$\delta(E) = \{L \in J(C) | h^0(E \otimes L) > 0\}. \tag{3.22}$$

It is easy to see that $\delta(E)$ is a divisor of the complete linear system $|2\Theta|$ on $J(C)$, where Θ is the theta-divisor defining the principal polarization of $J(C)$.

Moreover, if $E = L \oplus L^\vee \in \operatorname{Sing} \overline{M}_C(2,K_C) = K$ (see (2.18)), then

$$\delta(L \oplus L^\vee) = \Theta_L + \Theta_{L^\vee} \in |2\Theta|, \tag{3.23}$$

where

$$\Theta_L = \{L' \in J(C) | h^0(K_C^{\frac{1}{2}} \otimes L \otimes L') > 0\}. \tag{3.24}$$

Thus we have a map

$$\begin{tikzcd} \delta : J(C) \arrow[rr] \arrow[dr] & & {|2\Theta|,} \\ & K \arrow[ur] & \end{tikzcd}$$

which identifies a point with (-point) on $J(C)$. This map extends to a map

$$\delta : \overline{M}_C(2, K_C) \longrightarrow |2\Theta|.$$

But the complete linear system $|2\Theta|$ defines the standard map

$$f_{|2\Theta|} : J(C) \longrightarrow |2\Theta|^\vee, \tag{3.25}$$

and there exists a canonical isomorphism $\beta : |2\Theta| \longrightarrow |2\Theta|^\vee$ such that the diagram

$$\begin{tikzcd} & & {|2\Theta|} \arrow[dd, "\beta"] \\ J(C) \arrow[r, "\phi"] & K \arrow[ur, "\delta"] \arrow[dr, "f_{|2\Theta|}"'] & \\ & & {|2\Theta|^\vee} \end{tikzcd} \tag{3.26}$$

is commutative (see [14]), which can be extended to the diagram

$$\begin{tikzcd} & & & {|2\Theta|} \arrow[dd, "\beta"] \\ J(C) \arrow[r, "\phi"] & K = \operatorname{Sing} \overline{M}_C(2,0) \arrow[r, "i"] & \overline{M}_C(2,0) \arrow[ur, "\delta"] \arrow[dr, "f_{|2\Theta|}"'] & \\ & & & {|2\Theta|^\vee} \end{tikzcd} \tag{3.27}$$

where ϕ is standard (2-1) covering (2.18).

Now let us return to § 2 and restrict the morphism δ:

$$\delta \circ \beta = f_0 \tag{3.28}$$

is the morphism (3.20) defined by the complete linear system $|\Delta|$ on $\overline{M}_C(2, 0)$.

We have the finite map

$$f_0 : M_{C\subset S}(2, c_1, k) \longrightarrow \mathbb{P}\mathcal{H}_C^\vee \tag{3.29}$$

(by Beauville's Theorem) and in particular

$$f_0 : M^{\mathrm{irr}}_{C\subset S}(2, c_1, k) \longrightarrow \mathbb{P}\mathcal{H}^\vee_C. \tag{3.30}$$

We have a morphism of degree 1 of the Gieseker-compactification $\overline{M}_H(2, c_1, k)$ (see (2.1) – (2.12)) to $\mathbb{P}\mathcal{H}^\vee_C$:

$$\begin{gathered} \overline{M}_H(2, c_1, k) \xrightarrow{\overline{\mathrm{res}}_C} M^{\mathrm{irr}}_{C\subset S}(2, c_1, k) \xrightarrow{f_0} \mathbb{P}\mathcal{H}^\vee_C \\ \overline{M}_H(2, c_1, k) \xrightarrow{\quad f_{C\subset S}\quad} \mathbb{P}\mathcal{H}^\vee_C . \end{gathered} \tag{3.31}$$

Now, we can compare the line bundles

$$L, \quad L_H, \quad f^*_{C\subset S}\mathcal{O}_{\mathbb{P}\mathcal{H}^\vee_C}(1) \tag{3.32}$$

(see (2.12) – (2.15)) on $M_H(2, c_1, k)$.

Proposition 3.1. *There exists a rational number $\beta \in \mathbb{Q}$ such that*

$$L^\beta_H = \phi^*_{C\subset S}\mathcal{O}_{\mathbb{P}\mathcal{H}^\vee_C}(1). \tag{3.33}$$

Proof. In the setup of § 2 we can assume in the formulas (2.13) – (2.15) that for $C \in |2H|$ we have (1.8), passing if necessary to a multiple of H. We consider the case $c_1(E) = 0$ only — the other cases can be considered by a similar method.

Then by the adjunction formula

$$K_S^{\frac{1}{2}} \otimes H|_C = K_C^{\frac{1}{2}}, \tag{3.34}$$

because $C \in |2H|$ and

$$K_S \otimes \mathcal{O}_S(C)|_C = K_C.$$

For every vector bundle $E \in M_H(2, c_1, k)$ we tensor the short exact sequence

$$0 \longrightarrow K^{\frac{1}{2}} \otimes H^\vee \longrightarrow K_S^{\frac{1}{2}} \otimes H \longrightarrow K_C^{\frac{1}{2}} \longrightarrow 0$$

by E to give

$$0 \longrightarrow E \otimes K^{\frac{1}{2}} \otimes H^\vee \longrightarrow E \otimes K_S^{\frac{1}{2}} \otimes H \longrightarrow E|_C \otimes K_C^{\frac{1}{2}} \longrightarrow 0. \tag{3.35}$$

The corresponding long exact sequence is

$$\begin{gathered} 0 \longrightarrow H^0(E \otimes K^{\frac{1}{2}} \otimes H^\vee) \longrightarrow H^0(E \otimes K_S^{\frac{1}{2}} \otimes H) \longrightarrow \\ \longrightarrow H^0(E|_C \otimes K_C^{\frac{1}{2}}) \longrightarrow \ \dots \end{gathered}$$

If we consider each cohomology space of this sequence as the fibre over the point $E \in M_H(2, 0, k)$ of a vector bundle on $M_H(2, 0, k)$, then the multiplicativity of determinants of vector bundles in an exact sequence gives the equation

$$L_0|_{M_H(2,0,k)} = L_H, \tag{3.36}$$

where L_0 is the vector bundle (3.12), L_H is the vector bundle (2.14), and we are done. □

Let us return to the finite morphism (3.30) and the space of conformal blocks (3.17) – (3.26). The name comes from conformal quantum field theory (see, for example, [20]). One of the main results of this theory is that the projectivization of this space is almost independent of the curve C. More precisely:

THEOREM 3.1. *On the vector bundle $\mathfrak{E}$ over the Teichmüller space T_g of marked curves with fibre*

$$\mathfrak{E}|_C = \mathcal{H}_C \tag{3.37}$$

over $C \in T_g$ there exists a canonical projective flat connection.

This means that we have a canonical way of identifying the projective spaces $\mathbb{P}\mathcal{H}_C^\vee$ and $\mathbb{P}\mathcal{H}_{C'}^\vee$, if the curve C' is in a small neighborhood of C.

There are at least three methods to prove this theorem. A more geometrical way is Hitchin's method from his preprint "Flat connection and geometric quantization" (see [11]). At the heart of his construction lies the "heat equation" relating the linear variation of the holomorphic structure on C with the quadratic form on the cotangent bundles $T^\vee M_C(2,0)$ that is the symmetric tensor

$$G \in H^0(S^2 T M_C(2,0), \tag{3.38}$$

which is defined over a point $E \in M_C(2,0)$ by the usual cup-product:

$$\begin{array}{ccccc} H^1(T) & \otimes H^0(\operatorname{ad} E \otimes K_C) & \longrightarrow & H^1(\operatorname{ad} E) \\ \| & \| & & \| \\ H^1(K_C^\vee) & T_E^\vee M_C(2,0) & & T_E M_C(2,0) \end{array} \tag{3.39}$$

(see [11]).

Now, if in the projective space $|C|$ we consider a little open ball B with the center C_0 not containing singular curves, then for every $C \subset B$ the projective spaces $\mathbb{P}\mathcal{H}_C$ and $\mathbb{P}\mathcal{H}_{C_0}$ can be identified by the projective connection.

Thus, we have the family of morphisms (3.29)

$$f_{C\subset S} : \overline{M}_H(2, c_1, k) \longrightarrow \mathbb{P}\mathcal{H}^\vee, \quad C \in B \tag{3.40}$$

in the same projective space of dimension $2^{g(C)} - 1$.

CONJECTURE 3.1. $f_{C\subset S}(\overline{M}_H(2, c_1, k))$ does not depend on $C \in B$.

Of course this conjecture is true if the restriction map

$$H^0(\overline{M}_C(2,0), f_0^* \mathcal{O}_{\mathbb{P}\mathcal{H}^\vee}(1)) \xrightarrow{\operatorname{res}_C} H^0(\overline{M}_H(2, c_1, k), f_{C\subset S}^* \mathcal{O}_{\mathbb{P}\mathcal{H}^\vee}(1)) \tag{3.41}$$

is an isomorphism.

But a priori we may obtain a kernel of rk k and cokernel of rk c and a nontrivial map of the complete linear system

$$|C| \xrightarrow{\phi_{|C|}} \mathrm{Gr}\left(k, 2^{g(C)} - |k-c|\right) \times \mathrm{Gr}\left(c, 2^{g(C)}\right) \tag{3.42}$$

into the direct product of Grassmanians.

But even if the restriction map (3.41) is not an isomorphism, the Conjecture may be true. Indeed, this is a question about the flat projective connection on the vector bundle $\mathfrak{E}$ over the Teichmüller space T_g (see (3.36)) restricted to the image of the family $|\tilde{C}|$ of marked curves from $|C|$ on S. This connection may be reducible and the invariant subspace is the image the restriction map (3.41).

This is a question about the geometrical structure of the tensor G (3.38) and at the end of this paragraph we will give a sketch of the geometric constructions.

Now, even if res_C (3.41) is not an isomorphism, but the dimension of $M_H(2,0,K)$ is "right", that is

$$\dim\, M_H(2,0,k) = 4k - 3(p_g+1)$$

(see (2.36)), then we have the constant

$$\gamma_S^{d(k)}(C) = \deg f_{C\subset S}(\overline{M}_H(2,0,k)), \tag{3.43}$$

which is independent of $C \in |C|$ of course. (We will give the explanation for the strange notation $d(k) = 4k - 3(p_g+1)$ in the next paragraph.)

What can we say about this constant? A priori, if we do not know the configuration $M_{C\subset S}(2,0,k)$, then we can say nothing. But if the configuration is simplest as in the case (2.37) and the restriction map (3.41) is an isomorphism for all $k' \leqslant k$, then we have the inequality

$$\gamma_S^{d(k)}(C) \geqslant \gamma_S^{d(k')} \cdot \nu_{C\subset S}(k') \tag{3.44}$$

for every $k' < k$, where $\nu_{C\subset S}$ is the multiplicity of a general singular point of

$$\mathrm{Sing}\ \overline{\mathrm{res}}_C(\overline{M}_H(2,0,k'+1))(2.42)\ .$$

After that we have the trivial inequality

$$\begin{array}{ccccc} \gamma_S^{d(k)}(C) & + & 4k-3(p_g+1) & \geqslant & 2^{g(C)} \\ \| & & \| & & \| \\ \deg f_{C\subset S}(\overline{M}) & & \dim f_{C\subset S}(\overline{M}) & & \dim \mathbb{P}\mathcal{H}^\vee + 1 \end{array} \tag{3.45}$$

Before the end of this paragraph, we consider the question of restricting the projective connection on $\mathfrak{E}$ over T_g (3.36) to the family $|\tilde{C}|$ of marked curves from $|C|$ and the conditions of the "heat equation" (3.38) on $|\tilde{C}|$ and E of the type $E|_C$, where E is a vector bundle on S.

For the variation of curve C in the complete linear system $|C|$ the tangent space to $|C| = \mathbb{P}^n$ in a point $C \in \mathbb{P}^n$ is

$$T_C|C| = H^0(N_{C\subset S}) = H^0(\mathcal{O}_C(C^2)), \tag{3.46}$$

where $N_{C\subset S}$ is the normal bundle of C in S.

The differential of the moduli map for curves on S is the coboundary homomorphism

$$H^0(N_{C\subset S}) \xrightarrow{\ \delta\ } H^1(TC) = H^1(K_C^\vee) \tag{3.47}$$

of the short exact sequence

$$0 \longrightarrow TC \longrightarrow TS|_C \longrightarrow N_{C\subset S} \longrightarrow 0. \tag{3.48}$$

This extension is defined by a cocycle

$$e \in H^1(N_{C\subset S}^\vee \otimes TC) = H^1(K_C^\vee \otimes \mathcal{O}_C(-C^2)), \tag{3.49}$$

and the coboundary homomorphism δ (3.47) is defined by the cup-product

$$H^1(K_C^\vee \otimes \mathcal{O}_C(-C^2)) \ni e \otimes H^0(\mathcal{O}_C(C^2)) \longrightarrow H^1(K_C^\vee). \tag{3.50}$$

Hence, the hypernet of the quadratic form (3.38) is in our case

$$\begin{gathered} H^0(\mathcal{O}_C(C^2)) \otimes e \otimes H^0(\operatorname{ad} E \otimes K_C) \longrightarrow H^1(\operatorname{ad} E) \\ e \in H^1(K_C^\vee \otimes \mathcal{O}_C(-C^2)). \end{gathered} \tag{3.51}$$

We consider the case, where S is a 3-surface and the vector bundle is the restriction $E|_C$ of an H-stable vector bundle E on S. In that case we have $T^\vee S = TS$, $\mathcal{O}_C(-C^2) = K_C^\vee$ and the cocycle $e \in H^1(K_C^{-2})$ (3.49) is the dual of the image of the Wahl-map for the curve C (see [5]).

Now, we can extend (3.49) by the diagram (0.19) to

$$\begin{tikzcd}
 & & 0 & 0 \arrow[d] \\
 & & H^1(\operatorname{ad} E) \arrow[u] & H^1(\operatorname{ad} E) \arrow[d] \\
H^0(K_C) \ \otimes e \in & H^1(K_C^{-2}) \otimes & H^0(\operatorname{ad} E|_C \otimes K_C) \arrow[u] \arrow[r] & H^1(\operatorname{ad} E|_C) \arrow[d] \\
H^0(\mathcal{O}_S(C)) \otimes & H^1(\mathcal{O}_S(-2C)) \otimes \arrow[u] & H^0(\operatorname{ad} E(C)) \arrow[u] \arrow[r] & H^2(\operatorname{ad} E(-C)) \arrow[d] \\
 & 0 \arrow[u, equal] & 0 \arrow[u] & 0
\end{tikzcd} \tag{3.52}$$

where we assume $H^1(\operatorname{ad} E(C)) = 0$, that is, we have the second case of (0.20). But $H^1(\mathcal{O}_S(2S)) = 0$. Using this, we can say

Proposition 3.1. *For every smooth curve $C' \in |C|$ and $E \in M_H(2, 0, k)$ every quadratic of the hypernet* (3.51) *for $E|_{C'}$ is vanishing on the space of the extendible Higgs field* (see the Definition 0.1).

This is the "infinitesimal part" of the proof of Conjecture 3.1, if we are in the case of K3-surfaces.

§ 4 The constants.

In this paragraph we consider the question of computing the constant (3.43) which is independent of the moduli of the curve C. Thus $\gamma_C^{d(k)}(C)$ is a function of the homology class of C and of the surface S. Indeed, if $p_g > 0$ (or $S = \mathbb{P}^2$) and $d(k) > 3p_g + 3$, it is a function of the smooth structure of S. An explanation of this situation will be given below.

Let S be considering as a smooth simply connected four-dimensional manifold with an intersection form

$$q_S \in S^2 H^2(S, \mathbb{Z}) \tag{4.1}$$

of rank b_2 (the second Betti number of S) and of index $I = b_2^+ - b_2^-$.

Recall that by the Hodge-index theorem

$$b_2^+ = 2p_g + 1, \tag{4.2}$$

where p_g is the geometric genus of S.

The intersection form q_S (4.1) is unimodular and defined uniquely by its rank, its index and its parity, that is the form is even iff $q_S(\sigma)$ is even for $\sigma \in H^2(S, \mathbb{Z})$ (by the Poincaré duality we can identify $H^2(S, \mathbb{Z})$ with $H_2(S, \mathbb{Z})$) and q_S is odd in the other case. The form q_S defines the homotopy type of S.

S. Donaldson (see [6]) recently defined on $H_2(S, \mathbb{Z})$ a series of homogenous integer-valued polynomials

$$\gamma_S^d \in S^d H^2(S, \mathbb{Z}) \tag{4.3}$$

of degree d indexed by the elements of an arithmetical progression

$$d(k) = 4k - 3p_g - 3 \tag{4.4}$$

starting from the smallest d_0 satisfying the inequality

$$d_0 > 3p_g + 3. \tag{4.5}$$

These polynomials are called *Donaldson polynomials*; they generalise the degree 2 polynomial q_S. But whereas q_S is a homotopy invariant of the 4-manifold S, the Donaldson polynomials γ_S^d are only invariants of the smooth structure if $p_g \geqslant 1$ or $S = \mathbb{P}^2$.

For the definition we consider an ideal situation (non real-life). Let H be a polarization of S, that is $H \in V^+(S)$. Suppose the moduli space $M = M_H(2,0,k)$ of H-stable 2-vector bundles of S with Chern-classes $c_1 = 0$, $c_2 = k$ is compact, smooth and of the right dimension:

$$\dim_{\mathbb{C}} M = d(k) = 4k - 3(p_g + 1) = \tfrac{1}{2}\dim{}_{\mathbb{R}} M. \tag{4.6}$$

Then on the lattice $H^2(M,\mathbb{Z})$ one has the homogeneous intersection form of degree d

$$q_M(\omega) = \int_M \underbrace{\omega \wedge \cdots \wedge \omega}_{d \text{ times}}. \tag{4.7}$$

Suppose a universal bundle $\mathfrak{E}$ exists on $S \times M$. Then the (2,2)-Künneth-component of the second Chern-class

$$c_2(\mathfrak{E})^{(2,2)} \in H^2(S,\mathbb{Z}) \otimes H^2(M,\mathbb{Z}) \tag{4.8}$$

can be interpreted as a homomorphism

$$c : H^2(S,\mathbb{Z}) \longrightarrow H^2(M,\mathbb{Z}), \tag{4.9}$$

which we call *homological correspondence*. It induces on $H_2(S,\mathbb{Z})$ the homogeneous form of degree d

$$\begin{aligned} \gamma^d &= c^*(q_M), \\ \gamma^d(\sigma) &= q_M(c(\sigma)). \end{aligned} \tag{4.10}$$

This is the Donaldson polynomial of degree $d(k)$, the complex dimension of M (see (4.6)). The class $c_2(\mathfrak{E})$ is algebraic, hence $c_2^{(2,2)}$ is a class of Hodge-type (2.2) and hence the homological correspondence (4.9) preserves the Hodge-decomposition.

But real life is more complicated:

(1) The existence of M is a very hard problem.

(2) M is not compact.

(3) The universal bundle does not exist and so on.

However, if following Donaldson, we solve these technical problems, then we obtain the collection of polynomials which for $p_g > 0$ or $S = \mathbb{P}^2$ are the invariants of the differentiable structure of S, not the homotopical one.

It is easy to see that the restriction $\gamma_S^d|_{V^+(S)}$ defines $\gamma_S^d|_{\mathrm{Pic}\, S}$.

How can $\gamma_S^d(H)$ for $H \in V^+(S)$ be computed?

The polynomial γ_S^d is homogenous, hence we can pass, if necessary, to a multiple of H. Then by definition for $C \in |NH|$ with N big

$$\gamma_S^{d(k)}(C) = \deg f_{C\subset S}(\overline{M}_H(2,0,k)) \tag{4.11}$$

(see (3.43)). What do we know about these constants?

First of all, they are positive and hence non vanishing. We can compute them by the following method. Consider a curve $C_i \in |C|$ and some point $p_i \in \mathbb{P}\mathcal{H}$ and the hyperplane $\check{p}_i$ in $\mathbb{P}\mathcal{H}^\vee$. Then we have the divisor

$$\mathcal{D}_i = f^{-1}_{C_i\subset S}(f_{C_i\subset S}(\overline{M}_H(2,0,k)\cap \check{p}_i)) \tag{4.12}$$

of $\overline{M}_H(2,0,k)$. Choose $d(k) = \dim \overline{M}_H(2,0,k)$ of such divisors $\{\mathcal{D}_i\}$, $i = 1,\dots,d(k)$. It is easy to see that if we take the collection of these curves $\{C_i\}$ and points $\{p_i\}$ in general position, then the intersection $\bigcap\limits_{i=1}^{d(k)} \mathcal{D}_i$ is the finite set of points of multiplicity 1 and

$$\bigcap_{i=1}^{d(k)} \mathcal{D}_i \subset M_H(2,0,k), \tag{4.13}$$

that is, the points of the intersection are not contained in the corona. Hence

$$\# \bigcap_{i=1}^{d(k)} \mathcal{D}_i = \gamma_S^{d(k)}(C). \tag{4.14}$$

Remark. The assertion about the general position is easy to see, indeed, but it is not trivial (see for example the formulas (4.23) – (4.26) and inequality (4.27) from [19]).

We can take the points $p_i \in \mathbb{P}\mathcal{H}$ in the following way. Consider a theta-characteristic $K_{C_i}^{\frac{1}{2}}$ on C_i and the divisor on $M_{C_i}(2,0)$:

$$\Delta_i = \{E \in M_{C_i}(2,0) | h^0(E\otimes K_{C_i}^{\frac{1}{2}}) > 0\}. \tag{4.15}$$

Then Δ_i defines the hyperplane $\check{p}_i$ such that

$$\Delta_i = M_{C_i}(2,0)\cap \check{p}_i. \tag{4.16}$$

Now, we take this construction as the definition of $\gamma_S^{d(k)}(C)$ (Donaldson in [6] does it similarly). We gain the possibility to consider curves of small genus, too. Working with elliptic and rational curves in the special case of K3-surface, it is possible to compute constants (4.11).

First of all for K3-surface we have that

$$d(k) = 4k - 6$$

is even and it is convenient to consider the number

$$n(k) = \tfrac{1}{2}d(k) = 2k - 3. \tag{4.17}$$

Then one has the

THEOREM OF FRIEDMAN–MORGAN. ([8]) *For a K3-surface S*

$$\gamma_S^{d(k)}(C) = (g(C) - 1)^{n(k)} \cdots \frac{(2n(k))!}{n(k)!}. \tag{4.18}$$

(Independently this constant has been determined by .Ó. Grady.)

The simplest surface of course is $\mathbb{P}^2$ and the simplest curve in $\mathbb{P}^2$ is a line $\mathbb{P}^1$. In that case

$$\gamma_{\mathbb{P}^2}^{4k-3}(C) = c_k(\deg C)^{4k-3}, \tag{4.19}$$

where c_k is an absolute constant, an invariant of the natural differentiable structure of $\mathbb{P}^2$ (over $\mathbb{C}$). To compute these constants it is convenient to use the definition (4.15) for a smallest curve that is a line on $\mathbb{P}^2$.

Denote

$$M(k) = M_H(2, 0, k) \tag{4.20}$$

(we can omit the subscript H, because the polarization of $\mathbb{P}^2$ is unique). On every line $l \subset \mathbb{P}^2$, there exists a unique theta-characteristic

$$K_l^{1/2} = \mathcal{O}_l(-1). \tag{4.21}$$

Hence the divisor (4.12) is

$$\mathcal{D}_l = \{E \in M(k) | h^0(E|_l \otimes \mathcal{O}_l(-1)) > 0\} \tag{4.22}$$

by (4.15).

For a generic line l in $\mathbb{P}^2$ the restriction

$$E|_l = \mathcal{O}_l \oplus \mathcal{O}_l \tag{4.23}$$

is trivial. If

$$E|_l = \mathcal{O}_l(d) \oplus \mathcal{O}_l(-d), \quad |d| > 0,$$

then l is called a *jumping line* (see [3]).

For every $E \in M(k)$ the set of jumping lines

$$C(E) = \{l \in \check{\mathbb{P}}^2 | h^0(E|_l(-1)) > 0\} \tag{4.24}$$

is a curve of degree k. Hence we have a morphism

$$M(k) \xrightarrow{j} |kh|, \tag{4.25}$$

where h is a line in the dual plane $\check{\mathbb{P}}^2$. This is a finite map and the preimage of a smooth curve C is

$$j^{-1}(C) = \{K_C^{\frac{1}{2}} \in \operatorname{Pic} C | h^0(K_C^{\frac{1}{2}}) = 0\}. \tag{4.26}$$

It is the set of non-vanishing theta-characteristics of C (see [3]). The closure $\overline{j(M(k))}$ provides a new compactification of $M(k)$ (it is easy to see that for a general quasitrivial sheaf F the curve of jumping lines is

$$C(F) = \bigcup \check{p}_i, \tag{4.27}$$

where $\overset{k}{\underset{i=1}{\oplus}} \mathcal{O}_{p_i}$ is the cotorsion sheaf of F (see (2.32)). From this it is easy to see that this compactification is distinguished from the dense packing one. Let

$$B_k = \overline{j(M(k))} \subset |kh| = \mathbb{P}^{\frac{(k+1)(k+2)}{2} - 1} \tag{4.28}$$

be the closure of the set of curves of jumping lines of stable 2-vector bundles with $c_2 = k$ on $\mathbb{P}^2$. Consider the map

$$M(k) \xrightarrow{j} B_k \subset |kh|.$$

It is not hard to prove that, for k big, $\deg j = 1$. Now, we can see that the constant c_k (4.19) is

$$c_k = \deg B_k \cdot \deg j. \tag{4.30}$$

EXAMPLES.

(1) $c_2 = 1$. In that case the map j (4.25) is an embedding.

(2) $c_3 = 3$ (see the end of [3]). In that case

$$B_3 = |3h|, \quad \deg j = 3. \tag{4.31}$$

We consider the first non-trivial case c_4 in detail below.

Now, consider Barth's interpretation of $M(k)$ (see [3]). Let H be a vector space of rk k and $\mathcal{P}_k^2$ be the "quantum projective plane" of triples of symmetric matrices

(4.32)
$$\mathcal{P}_k^2 = \frac{\{(A_1, A_2, A_3) \mid A_i \in S^2 H^\vee,\ \exists\, (\lambda_1, \lambda_2, \lambda_3) \in \mathbb{C}^3 : \det(\sum \lambda_i A_i) \neq 0\}}{\mathrm{GL}(H)}$$

where $\mathrm{GL}(H)$ acts on (A_1, A_2, A_3) by similarity transformation:

(4.33)
$$g(A_1, A_2, A_3) = (gA_1 g^\vee, gA_2 g^\vee, gA_3 g^\vee)\,, \quad g \in \mathrm{GL}(H)\,,\ g^\vee \in \mathrm{GL}(H^\vee)\,.$$

There exists a map of the k-quantum projective plane

$$\mathcal{P}^2_k \xrightarrow{j_b} |kh|, \tag{4.34}$$

where h is a line in the projective plane $\mathbb{P}^2_{(\lambda_1,\lambda_2,\lambda_3)}$ with projective coordinates $(\lambda_1, \lambda_2, \lambda_3)$ and the curve of degree k $\ j_b(A_1, A_2, A_3)$ is defined by equation

$$\det(\lambda_1 A_1 + \lambda_2 A_2 + \lambda_3 A_3) = 0. \tag{4.35}$$

This map is onto and for a generic smooth curve $C \in |kh|$

$$j_b^{-1}(C) = 2^{(g(C)-1)}\left(2^{g(C)} + 1\right), \tag{4.36}$$

the number of nonvanishing theta-characteristics, i.e., the number of even theta-characteristics $\{K_C^{\frac{1}{2}}\}$ in the generic case.

In his article [3] Barth defines an embedding

$$M(k) \overset{b}{\hookrightarrow} \mathcal{P}^2_k,$$

which can be extended to a commutative diagram

$$\begin{tikzcd} M(k) \arrow[rr, hook, "b"] \arrow[dr, "j"'] & & \mathcal{P}^2_k \arrow[dl, "j_b"] \\ & {|kh|} & \end{tikzcd} \tag{4.37}$$

and describes the image of b by equations for the triples (A_1, A_2, A_3) of symmetric matrices.

Let A_1^{adj} be the adjoint matrix of A_1. If $\det A_1 \neq 0$, then

$$A_1^{\mathrm{adj}} = A_1^{-1} \cdot \det A_1. \tag{4.38}$$

Barth defines a rational map of the k-quantum plane

$$\begin{gathered} \mathcal{P}^2_k \xrightarrow{\mathrm{kom}} \mathbb{P}\Lambda^2 H^\vee, \\ \mathrm{kom}(A_1, A_2, A_3) = A_2 A_1^{\mathrm{adj}} A_3 - A_3 A_1^{\mathrm{adj}} A_2. \end{gathered} \tag{4.39}$$

This map blows down the hyperplane

$$\mathcal{D}_1 = \{\det A_1 = 0\} \subset \mathcal{P}^2_k \tag{4.40}$$

to the Grassmannian

$$G(2, H) \subset \mathbb{P}\Lambda^2 H^\vee,$$

and $b(M(k))$ is a component of the preimage

$$\mathrm{kom}^{-1}(G(2, H)) - \mathcal{D}_1 \tag{4.41}$$

(see [3]). Using the diagram

(4.42)
$$\begin{tikzcd}
 & \mathcal{P}_k^2 \arrow[dl, "\mathrm{kom}"'] \arrow[dr, "j_b"] \arrow[d, phantom, "\bigcup"] & \\
\mathbb{P}\Lambda^2 H^\vee \arrow[d, phantom, "\bigcup"] & b(M(k)) \arrow[dl] \arrow[dr, "j"] & {|kh|} \arrow[d, phantom, "\bigcup"] \\
G(2,H) & & B_k
\end{tikzcd}$$

we can compute (following an idea of Tichomirow) the first non-trivial constant c_4 (4.24) – (4.30).

Example. The divisor of Lüroth quartics.

It is easy to see that

(4.43)
$$B_4 \subset |4h| = \mathbb{P}^{14}$$

is a hypersurface and Barth proved in [3] that B_4 is a closure of the set of Lüroth quartics. For the definition of it take

(1) a smooth conic Q on $\mathbb{P}^2$,

(2) 5 points on Q: $p_1, \ldots, p_5$,

(3) 5 tangent lines at the points: $l_1, \ldots, l_5$,

(4) all the points of intersections $\{l_i \cap l_j\}$.

(5) Move the 5 points on Q in a linear pencil, then the 10 points of intersections of tangent lines sweep out a curve of degree 4. This is by definition a *Lüroth-quartic*.

The set of Lüroth-quartics in $|4h| = \mathbb{P}^{14}$ is a hypersurface (the conics give 5 parameters, the pencils $\mathbb{P}^1 \subset \mathbb{P}^5$ of degree 5 give $\dim G(2,6) = 8$ parameters and we have $5 + 8 = 14 - 1$).

It is easy to see that a form f of degree 4 defines a Lüroth quartic iff there are 5 linear forms $L_1, \ldots, L_5 \in \check{\mathbb{P}}^2$ such that

(4.44)
$$f = \sum_{i=1}^{5} L_1 \ldots L_i^\vee \ldots L_5,$$

where the sign $^\vee$ over L means omitting the linear form.

Denote the hypersurface of Lüroth quartics by

(4.45)
$$\mathcal{D}_L \subset |4k|.$$

It is invariant under the action of $\mathbb{P}\mathrm{GL}(3,\mathbb{C})$. On the other hand, in $|4h|$ there is an other $\mathbb{P}\mathrm{GL}(3,\mathbb{C})$-invariant hypersurface

(4.46)
$$\mathcal{D}_C \subset |4h|$$

of Clebsch-quartics.

A form f of degree 4 defines a Clebsch quartic iff there are 5 linear forms $L_1, \dots, L_5 \in \check{\mathbb{P}}^2$ such that

$$f = \sum_{i=1}^{5} L_i^4. \tag{4.47}$$

In 1865 Clebsch proved (see [12])

CLEBSCH'S THEOREM. *Let* $V = \mathbb{C}^3$, $\mathbb{P}V = \mathbb{P}^2$ *and* $f \in S^4V^\vee$ *a homogenous form of degree* 4. *Consider* f *as a homomorphism*

$$S^2V \xrightarrow{f} S^2V^\vee. \tag{4.48}$$

Then $f \in \mathcal{D}_C \iff \det f = 0$.

From this it is clear that

$$\deg \mathcal{D}_C = 6. \tag{4.49}$$

Now, in 1868 Lüroth (see [12]) constructed a rational map

$$\phi : |4h| \longrightarrow |4h| \tag{4.50}$$

such that

$$\phi(\mathcal{D}_C) = \mathcal{D}_L \tag{4.51}$$

and ϕ is $(3,\mathbb{C})$-equivariant.

The definition of ϕ is a nice exercise from *Symbolic Calculus*:

1) Let $f \in S^4V^\vee$ be a form of degree 4 and (x_1, x_2, x_3) coordinates in V. Then we have

$$f = \sum_{\underline{i}} \binom{4}{\underline{i}} A_{\underline{i}} \underline{x}_{\underline{i}}, \tag{4.52}$$

where the symbols are the usual symbols of symbolic calculus:

$$\underline{i} = (i_1, i_2, i_3), \quad \binom{4}{\underline{i}} = \frac{4!}{i_1! i_2! i_3!}, \quad \underline{x}^{\underline{i}} = x_1^{i_1} x_2^{i_2} x_3^{i_3}.$$

2) Consider the formal system of equations

$$f = a_x^4 = b_x^4 = c_x^4 = d_x^4, \tag{4.53}$$

where

$$\begin{array}{rcl} a_x & = & a_1x_1 + a_2x_2 + a_3x_3 \\ b_x & = & b_1x_1 + b_2x_2 + b_3x_3 \\ c_x & = & c_1x_1 + c_2x_2 + c_3x_3 \\ d_x & = & d_1x_1 + d_2x_2 + d_3x_3, \end{array} \tag{4.54}$$

and consider the system of formal symbols:

$$\underline{a}^{\underline{i}} = a_1^{i_1} a_2^{i_2} a_3^{i_3}$$

with the unique condition:

$$\begin{gathered} \text{if} \quad i_1 + i_2 + i_3 = 4, \quad \text{then} \quad \underline{a}^{\underline{i}} = A_{\underline{i}} \ (4.52) \\ (\text{if} \quad i_1 + i_2 + i_3 \neq 4, \quad \text{then } \underline{a}^{\underline{i}} \quad \text{has no meaning}). \end{gathered} \tag{4.55}$$

3) Let the symbol

$$(abc) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \tag{4.56}$$

be the determinant. Similarly define $(abd), (acd), \dots$. At last we can define the equivariant rational map (4.45):

4) For a form f (4.52)

$$\phi(f) = (abc)(abd)(acd)(bcd) a_x b_x c_x d_x. \tag{4.57}$$

This means that if we decompose determinants, multiply ones and substitute $\underline{a}^{\underline{i}} = \underline{b}^{\underline{i}} = \underline{c}^{\underline{i}} = \underline{d}^{\underline{i}} = A_i$ (see (4.52) and (4.55)), we obtain the new form $\phi(f)$.

It is easy to see that the linear system with base conditions which give the map ϕ is

$$|4H - \sum B_i|, \tag{4.58}$$

where H is a the hyperplane in $|4h| = \mathbb{P}^{14}$.

By Lüroth theorem ([12] (and it easy to see from (4.57)) every Clebsch quartic is transformed to a Lüroth quartic. Hence

$$6 \leqslant \deg \mathcal{D}_L \leqslant 6 \cdot 4^{13}. \tag{4.59}$$

However, the number on the right hand side is too big. From Barth's diagram (4.42) we can prove

Proposition 4.1. $\deg \mathcal{D}_L = 54$.

Sketch of proof. (The idea of the proof is due to A.S. Tichomirow.) First of all, we must define on our k-quantum plane (4.32) a family of lines.

Consider a projective line $\mathbb{P}^1_{(t_0,t_1)}$ with homogenous coordinates (t_0, t_1) as a family of triples of symmetric matrices

$$(t_0 A_1 + t_1 A_1', t_0 A_2 + t_1 A_2', t_0 A_3 + t_1 A_3'). \tag{4.60}$$

If the triples (A_1, A_2, A_3) and (A_1', A_2', A_3') are in general position, then the image $\mathbb{P}^1_{(t_0,t_1)}$ in $\mathcal{P}^2_k$ is called a quantum line l on $\mathcal{P}^2_k$.

Remark. In contrast to a natural line on $\mathbb{P}^2$ a quantum line may be degenerated.

Lifting to the set of triples of matrices $(S^2H^\vee)^3$, one can prove that with respect to the projection j_b (4.42) we have an equation of homology classes

$$j_b^*(j_b)_*(l) = \deg j_b \cdot l \tag{4.61}$$

(this is not trivial, because a quantum plane is a singular variety and we have to be careful).

It is easy to see that for the two projections of the diagram (4.42)

$$\deg j_b(l) = k, \quad \deg \operatorname{kom}(l) = k+1 \tag{4.62}$$

(see (4.34) and (4.39)).

Now, in the case $k = 4$, we can prove that in (4.41) we have the equality

$$\operatorname{kom}^{-1}(G(2,4)) = \mathcal{D}_1 \cup b(M(4)), \tag{4.63}$$

that is, there are no false components. We will consider subvarieties as as homology classes below. Then for the quantum line l we have by the projection formula:

$$[\mathcal{D}_1 + b(M(4))] \cdot l = \operatorname{kom}^*(G(2,4)) \cdot l = G(2,4).\operatorname{kom}(l) = 2 \cdot 5 = 10 \tag{4.64}$$

(see (4.62)). Now, by the projection formula and (4.62)

$$\mathcal{D}_1 \cdot l = j_b^*(H) \cdot l = H \cdot j_b(l) = 4, \tag{4.65}$$

where H is the class of hyperplanes in $|4h|$. Hence

$$b(M(4)) \cdot l = 6. \tag{4.66}$$

By the projection formula, the constant c_4 is

$$\begin{aligned} c_4 = \deg B_4 = (j_b)_*(M(4)) \cdot \tfrac{1}{4} j_b(l) = M(4) \cdot j_b^*(\tfrac{1}{4}(j_b)_*(l)) \\ = b(M(4)) \cdot \tfrac{\deg j_b}{4} l = 9 \cdot b(M(4)) \cdot l = 54, \end{aligned}$$

and we are done.

The next step would be computing the degree of the set of Darboux quintics in $|5h|$ (see [3]) and so on.

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The classical geometry of vector bundles

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Introduction.

One of the expected products of this summer school is an answer to the following question:

What is ALGEBRAIC GEOMETRY?

(We write AG for short.) Actually this is a hard task, because everybody already has the fixed conviction that the objects of AG are algebraic varieties.

An irreducible algebraic variety X has a dimension $\dim X$, and this number is usually a rough indication of the level of completeness of the geometric theory describing it. Algebraic varieties of small dimension carry special names:

$\dim X = 0$:	set of points
$\dim X = 1$:	curves (Xavier Gomez-Mont's lectures)
$\dim X = 2$:	surfaces (Rick Miranda's lectures)
$\dim X = 3$:	3-folds (Miles Reid's lectures)
	etc.

To explain how my lectures fit into this list, I would like to remark that two algebraic varieties of different dimension can be *geometrically identical.* To see this, consider the following chain of examples:

$\dim = 0$:	a set of 6 distinct points on $\mathbb{P}^1$ up to $\mathbb{P}\mathrm{GL}(2,\mathbb{C})$ action;
$\dim = 1$:	a curve of genus 2;
$\dim = 2$:	a cubic surface in $\mathbb{P}^3$ with one ordinary double point;
$\dim = 3$:	a nonsingular intersection of two quadrics in $\mathbb{P}^5$.

The identifications between the objects in $\dim < 3$ are absolutely obvious: the canonical map of a curve (see Rick Miranda's lectures [8] in this volume) of genus 2 is a double cover of $\mathbb{P}^1$ ramified in 6 points; considering $\mathbb{P}^1$ as a conic in $\mathbb{CP}^2$, blowing up [8] 6 points on this conic and constructing the anticanonical map [8] of the resulting surface, we get a cubic in $\mathbb{P}^3$ with an ordinary double point.

The threefold in our list carries the imposing full name of *Fano threefold of index* 2 *and degree* 4; its half-anticanonical map [8] displays it as the base locus of a pencil of quadrics in $\mathbb{P}^5$. The six singular quadrics of this pencil take us back to a set of 6 distinct points on $\mathbb{P}^1$.

This example of a chain of identifications is of course very classical and simple. A more recent example is Mukai's construction [10] of an identification of a plane quartic with a Fano variety V_{22}.

Slogan. An algebraic geometer is skillful enough if he or she can recognize the geometric person under many guises of different dimensions.

My first aim is to give you some experience in this direction. But my task is a little more complicated, because there is some new person in our game:

ALGEBRAIC VECTOR BUNDLE

In some sense this geometric object does not have any dimension (or, if you prefer, is infinite dimensional). But in any case, we can not avoid it. Even in our simplest chain of identifications, the intersection of two quadrics in $\mathbb{P}^5$ is a moduli space of stable vector bundles on the corresponding curve of genus 2.

So my second aim is to construct a simple but a new chain of geometric identifications including a vector bundle as a geometric object.

This new chain is not quite as simple as the previous one, but it is perhaps the simplest illustration of a new geometric observations showing that CLASSICAL AG is a slice of much more general GEOMETRY. Namely, some time ago Gromov observed that many results of ENUMERATIVE AG are true in SYMPLECTIC GEOMETRY. But a recent observation due to Donaldson is much more unexpected: *many constants of* ENUMERATIVE AG *are invariants of the underlying smooth structure of algebraic surfaces.*

Thus my third aim is to explain these relations between AG and differential geometry.

§ 1 Clebsch and Darboux curves.

Let $\mathbb{CP}^2$ be the complex projective plane:

$$\mathbb{CP}^2 = \mathbb{P}T, \quad \text{where } T = \mathbb{C}^3, \quad \text{so that Pic } \mathbb{CP}^2 = \mathbb{Z} \cdot l,$$

where l is a line. Then $|d \cdot l| = \mathbb{P}S^d T^*$ is the complete linear system of curves of degree d in $\mathbb{CP}^2$. So a homogeneous polynomial $\phi_C \in S^d T^*$ of degree d is the equation of a curve

$$C = \{\phi_C = 0\} \subset \mathbb{CP}^2, \quad \text{that is,} \quad C \in |d \cdot l|.$$

It is a classical enumerative problem in invariant theory to compute the degree $\deg V$ of some $\mathbb{P}\mathrm{GL}(3,\mathbb{C})$-invariant subvariety $V \subset |d\cdot l|$.

Example. The discriminant hypersurface in $|d\cdot l|$:

$$V_{\mathrm{Sing}} = \left\{ C \in |d\cdot l| : \mathrm{Sing} C \neq \varnothing \right\}.$$

This is obviously a subvariety of $|d\cdot l|$ invariant under $\mathbb{P}\mathrm{GL}(3,\mathbb{C})$, and an easy calculation shows that

$$\deg V_{\mathrm{Sing}} = 3\cdot(d-1)^2.$$

Now, a curve in $\mathbb{CP}^2$ which splits completely as a union of lines

$$\Delta_r = \bigcup_{i=1}^{r} l_i \tag{2.1}$$

is called a *polygon* or an *r-gon*. In §4 we will define an r-gon to be *regular* if all its sides l_i and all its vertices $l_i \cap l_j$ are distinct. Let $P_r \subset |d\cdot l|$ be the subvariety of all r-gons.

Useful exercise. What is $\deg P_r$?

Definition 2.1. We say that a curve C *circumscribes* a regular r-gon Δ_r if for every pair (i,j) the vertex (= intersection of sides) $l_i\cap l_j \in C$.

Let

$$\mathrm{MP}_r^d = \{\Delta_r, C_d\} \subset P_r \times |d\cdot l| \tag{2.2}$$

be the closure of the incidence variety of pairs consisting of a regular r-gon Δ_r and a curve C_d of degree d circumscribing it. We have two projection maps:

$$\begin{tikzcd} & \mathrm{MP}_r^d \arrow[dl, "p_\Delta"'] \arrow[dr, "p_C"] & \\ P_r & & |d\cdot l| \end{tikzcd} \tag{2.3}$$

Thus the subvariety

$$p_C(\mathrm{MP}_r^d) \subset |d\cdot l| \tag{2.4}$$

of curves of degree d circumscribing some r-gon is invariant under $\mathbb{P}\mathrm{GL}(3,\mathbb{C})$.

Problem. What is $\deg p_C(\mathrm{MP}_r^d)$? More precisely, what is

$$s_r(d) = \deg p_C \cdot \deg p_C(\mathrm{MP}_r^d)? \tag{2.5}$$

In terms of the defining equations, it is easy to see that

$$(\Delta, C) \in \mathrm{MP}_r^{r-1} \iff \phi_C = \sum_{i=1}^{r} (\phi_\Delta/\phi_{l_i}) \tag{2.6}$$

where $\Delta = \bigcup\limits_{i=1}^{r} l_i$.

Historically, the problem (2.5) is closely related to the following problem:

Definition 2.2. We say that a polygon $\Delta = \bigcup_{i=1}^{r} l_i$ is *apolar* to a curve C if

$$\phi_C = \sum_{i=1}^{r} (\phi_{l_i})^d. \tag{2.7}$$

Let

$$\mathrm{MPA}_r^d = \{\Delta_r, C_d\} \subset P_r \times |d \cdot l| \tag{2.8}$$

be the space of apolar pairs of polygons and curves. We again have two projection maps

$$\begin{tikzcd} & \mathrm{MPA}_r^d \arrow[dl, "p_\Delta"'] \arrow[dr, "p_C"] & \\ P_r & & |d \cdot l| \end{tikzcd} \tag{2.9}$$

Thus the subvariety

$$p_C(\mathrm{MPA}_r^d) \subset |d \cdot l| \tag{2.10}$$

is also invariant under $\mathbb{P}\mathrm{GL}(3, \mathbb{C})$.

Problem. What is

$$c_r(d) = \deg p_C \cdot \deg p_C(\mathrm{MPA}_r^d)? \tag{2.11}$$

These problems were solved recently by Geir Ellingsrud and Stein Strømme. Using Bott's formula, they computed the constants $c_r(d)$ for $r < 9$ and $s_r(r-1)$ for $r = 6, 7, 8, 9, 10$. For example, they find

$$\begin{aligned} 5! \cdot c_5(d) = d^{10} - 100d^8 + 150d^7 + 3680d^6 - 10260d^5 - 52985d^4 + \\ + 224130d^3 + 127344d^2 - 1500480d + 1664640\ . \end{aligned} \tag{2.12}$$

The following particular cases of the general enumerative problem will be important for us:

Definition 3.

(1) A curve $C \in p_C(\mathrm{MP}_{d+1}^d)$ is called a *Darboux curve*.

(2) A curve $C \in p_C(\mathrm{MPA}_{d+1}^d$ is called a *Clebsch curve*.

For special reasons, Darboux curves of degree 4 are called *Lüroth quartics*. These names have a historical explanation. Namely it is easy to see that the virtual (expected) dimension

$$\mathrm{v.}\dim \mathrm{MP}_{d+1}^d = \mathrm{v.}\dim \mathrm{MPA}_{d+1}^d = 3d + 2. \tag{2.13}$$

Remark. This dimension is one more than the dimension of the subvariety of rational curves of degree d.

But in 1865, Clebsch observed the following:

Clebsch's Theorem.

- *The image* $p_C(\mathrm{MPA}_5^4)$ *is a hypersurface*[1] *in* $|4 \cdot l| = \mathbb{P}^{14}$.
- *If* C *is nonsingular then* $C \in p_C(\mathrm{MPA}_5^4) \iff p_C^{-1}(C) = \mathbb{P}^1$.
- $\deg p_C(\mathrm{MPA}_5^4) = 6$.

Exactly the same facts hold for MP_5^4, that is, for Lüroth quartics, as Lüroth observed in 1868. But the degree of the hypersurface of Lüroth quartics was only computed in 1918 by F. Morley [9]:

$$\deg p_C(\mathrm{MP}_5^4) = 54. \tag{2.14}$$

This constant was reproduced in modern investigation (Tyurin, Le Potier, Ellingsrud and Strømme) under absolutely new motivations related to PDEs.

Remarks.

- It follows from Clebsch's Theorem, that the polynomial (2.12) satisfies
$$c_5(4) = 0,$$
that is, 4 is a root of $s_5(d)$. Can you see this from the display of this polynomial (2.12)?
- The fibres of the projection p_C of the diagram (2.9) were used by S. Mukai to describe special Fano varieties: let
$$p_C : \mathrm{MP}_6^4 \longrightarrow |4 \cdot l|$$
be the right side of the diagram (2.3). Then

  (i) for general C, the inverse image $P_C^{-1}(C)$ is a Fano threefold;

  (ii) if $C = 2q$ is double nonsingular conic then $P_C^{-1}(2q)$ is a compactification of $\mathbb{C}^3$ [10].

  (iii) The exact formulas of Ellingsrud and Strømme also work when the degree of curve is not small with respect to the number of sides of polygons. More precisely, if $d \geq r-1$, S. Mukai proved that $p_C(\mathrm{MPA}_7^5) = |5 \cdot l|$, and that the map p_C is birational. But you can see that the constant $c_7(5)$ is negative.

[1]this is in spite of the fact that $\mathrm{v.dim}\, \mathrm{MPA}_5^4 = \dim |4 \cdot l| = 14$

§ 2 Vector bundles on an algebraic surface and their sections.

Let me recall briefly the main constructions of sheaf theory on algebraic surfaces. The starting point is the structure sheaf $\mathcal{O}_S = \mathcal{O}$ of an algebraic surface S. For a first approach, it is enough to consider a nonsingular surface. Thus the stalk $\mathcal{O}_P$ of $\mathcal{O}$ at a point $p \in S$ is a 2-dimensional regular local ring. Every coherent sheaf F has a stalk F_P at each point $P \in S$, which is a module of finite type over $\mathcal{O}_P$; moreover, in a neighborhood U of any point, there is a resolution

$$\mathcal{O}_U^p \longrightarrow \mathcal{O}_U^q \longrightarrow F|_U \longrightarrow 0.$$

Thus each sheaf F on S defines a filtration $S_2 \subset S_1 \subset S$ by the homological dimension of the stalk. If this filtration is trivial then F is called a *vector bundle* and we will note it as E.

For a sheaf F on S the canonical homomorphism

$$\text{(3.1)} \qquad \mathrm{can}: F \longrightarrow \mathrm{Hom}(\mathrm{Hom}(F, \mathcal{O}), \mathcal{O}) = F^{**}$$

can be completed to a 4-term exact sequence

$$\text{(3.2)} \qquad 0 \longrightarrow T(F) \longrightarrow F \xrightarrow{can} F^{**} \longrightarrow C(F) \longrightarrow 0$$

We say that F is a *torsion sheaf* if $F = T(F)$, a *torsion-free sheaf* if $T(F) = 0$ and a *reflexive sheaf* if $F = F^{**}$.

It is easy to see that on a surface, *a reflexive sheaf is a vector bundle.* Moreover, for a torsion-free sheaf F, we have $\dim \mathrm{Supp}\, C(F) = 0$; that is, in this case $C(F)$ is an *Artinian sheaf.*

A pair of sheaves F_1 and F_2 defines three vector spaces

$$\mathrm{Ext}^i(F_1, F_2), \quad i = 0, 1, 2$$

with the usual functorial properties.

In the short exact sequence of sheaves

$$(*) \qquad 0 \longrightarrow F_2 \longrightarrow F \longrightarrow F_1 \longrightarrow 0,$$

the sheaf F is called an *extension* of F_1 by F_2; such an extension is given by an element $e(F)$ in the vector space $\mathrm{Ext}^1(F_1, F_2)$, so the set of classes of such extensions has the structure of the vector space $\mathrm{Ext}^1(F_1, F_2)$. For the zero class $0 \in \mathrm{Ext}^1(F_1, F_2)$ we have $F = F_1 \oplus F_2$.

Exercises.

- Prove that on an algebraic curve C, every coherent sheaf F is a direct sum
$$F = T(F) \oplus F^{**}.$$

- Suppose that we have two extensions F and F' of F_1 by F_2 and F_1' by F_2, together with a homomorphism $\phi : F_1' \longrightarrow F_1$. Then the identity map $F_2 = F_2$ and the given map ϕ extend to a homomorphism $F' \longrightarrow F$ if and only if the homomorphism

$$\widetilde{\phi} : \operatorname{Ext}^1(F_1, F_2) \longrightarrow \operatorname{Ext}^1(F_1', F_2)$$

induced by ϕ satisfies

$$\widetilde{\phi}(e(F)) = e(F'). \tag{3.3}$$

Of course, we would prefer to work only with vector bundles, which is enough for working over algebraic curves. But over algebraic surfaces, it is absolutely necessary to use torsion-free sheaves.

Any rank 1 torsion-free sheaf J on an algebraic surface S admits an exact sequence of the form (3.2):

$$0 \longrightarrow J \longrightarrow \mathcal{O}_S(D) = J^{**} \longrightarrow C(J) \longrightarrow 0, \tag{3.4}$$

where D is some divisor on S, and we can untwist this sequence by tensoring with $\mathcal{O}_S(-D)$:

$$0 \longrightarrow J(-D) \longrightarrow \mathcal{O}_S \longrightarrow \mathcal{O}_\xi \longrightarrow 0. \tag{3.5}$$

The last sheaf is the structure sheaf of 0-dimensional subscheme (a *cluster*, or a "fine 0-cycle") ξ of S, and $J(-D) = \mathcal{I}_\xi \subset \mathcal{O}_S$ is the ideal sheaf of this subscheme. A cluster ξ defines a cycle of points

$$[\xi] = \sum \deg(\xi, p_i) \cdot p_i.$$

We say that ξ is *reduced* if we have

$$\deg(\xi, p_i) = 1 \quad \text{(or 0)} \quad \text{for every } i.$$

In this case $\xi = [\xi]$, and the cluster is a configuration of distinct points on S.

Thus a rank 1 torsion-free sheaf admits two invariants: $c_1(J) = c_1(J^{**})$ and $c_2(J) = \deg \xi = h^0(\mathcal{O}_\xi)$.

Now let $s : \mathcal{O}_S \longrightarrow E$ be a section of a vector bundle E of rank 2. We say that a section is *regular* if its zero set is a 0-dimensional subscheme: $(s)_0 = \xi$. In this case, by definition, $\deg \xi = c_2(E)$.

Remark. If the zero set of a section contains an effective curve C, we can untwist it by $-C$ to obtain a regular section $s : \mathcal{O}_S \longrightarrow E(-C)$. In this case

$$\deg \xi = c_2(E) - C \cdot (c_1(E) - C).$$

For a rank 2 vector bundle E, the dual map to a regular section s can be extended to the Koszul resolution (as in David Eisenbud's lectures)

$$0 \longrightarrow \Lambda^2 E^* \xrightarrow{\wedge S^*} E^* \xrightarrow{s^*} \mathcal{O}_S \xrightarrow{\text{can}} \mathcal{O}_\xi \longrightarrow 0 \tag{3.6}$$

of the zero set $(s)_0 = \xi$ of s.

The kernel of can is just the ideal sheaf of ξ from (3.5) and from the first part of the sequence (3.6) we get the exact sequence

$$0 \longrightarrow \Lambda^2 E^* \longrightarrow E^* \longrightarrow \mathcal{I}_\xi \longrightarrow 0.$$

Tensoring this sequence by the invertible sheaf $\Lambda^2 E = \det E$ we get finally the *short exact sequence of a regular section*:

$$0 \longrightarrow \mathcal{O}_S \xrightarrow{s} E \longrightarrow \mathcal{I}_\xi(c_1(E)) \longrightarrow 0. \tag{3.7}$$

As we know, an extension of this type is given by an element $e \in \mathrm{Ext}^1(\mathcal{I}_\xi(c_1(E))$. For the last space, by Serre duality we have

$$\mathrm{Ext}^1(\mathcal{I}_\xi(D), \mathcal{O}_S) = \mathrm{Ext}^1(\mathcal{O}_S, \mathcal{I}_\xi(D + K_S))^* = H^1(\mathcal{I}_\xi(D + K_S))^*, \tag{3.8}$$

where K_S is the canonical class of S.

Thus a pair (s, E) consisting of a vector bundle and a section is given by a cluster $(s)_0 = \xi$ and a hyperplane $p \subset H^1(\mathcal{I}_\xi(c_1(E) + K_S))$.

§ 3 The first interpretation – moduli spaces of stable pairs.

Now we will consider the space MP^d_{d+1} of pairs (2.2) only. A polygon $\Delta = \sum l_i$ is called *regular* if $i \neq j \implies l_i \neq l_j$, and $l_i \cap l_j = l_k \cap l_n \implies (i, j) = (k, n)$. That is, all the sides of Δ are distinct, and all the vertices of Δ (= intersections of sides) are different too.

A pair (Δ, C) is called *regular* if Δ is regular and C is nonsingular.

Let P^0_r be the open subset of regular polygons. Then we have the open subset

$$M_0 P^d_{d+1} = p_\Delta^{-1}(P^0_r) \cap p_C^{-1}\Big(p_C(\mathrm{MP}^d_{d+1}) \setminus V_{\mathrm{Sing}} \cap p_C(\mathrm{MP}^d_{d+1})\Big) \tag{4.1}$$

of regular pairs.

Every regular polygon $\Delta = \bigcup_{i=1}^{r} l_i$ defines a cycle of points

$$\Delta^* = l_1^* + \cdots + l_r^* \tag{4.2}$$

on the *dual plane* $\mathbb{P}^{2*}$. It is a fine cycle, and we want to consider it as a cluster (0-dimensional subscheme) of the dual plane.

On the other hand, Δ also determines the cycle of vertices

$$\mathrm{Ver}\,\Delta = \bigcup_{i,j} l_i \cap l_j \tag{4.3}$$

on the plane $\mathbb{CP}^2$ itself. The cluster Δ^* defines an ideal sheaf $\mathcal{I}_{\Delta^*}$, and the family of extensions

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^{2*}} \longrightarrow E \longrightarrow \mathcal{I}_{\Delta^*}(2) \longrightarrow 0 \tag{4.4}$$

parameterised by the space $\mathbb{P}H^1(\mathcal{I}_{\Delta^*}(-1))^*$ (see the end of the previous section).

From the exact sequence

$$0 \longrightarrow \mathcal{I}_{\Delta^*}(-1) \longrightarrow \mathcal{O}_{\mathbb{P}^{2*}}(-1) \longrightarrow \mathcal{O}_{\Delta^*}(-1) \longrightarrow 0$$

we get the isomorphism

$$H^1(\mathcal{I}_{\Delta^*}(-1) = H^0(\mathcal{O}_{\Delta^*}(-1)), \tag{4.4}$$

so the extension (4.4) is given by a hyperplane in $H^0(\mathcal{O}_{\Delta^*}(-1))$.

On the other hand, the space of curves of degree d circumscribing Δ is the following:

$$|d \cdot l - \mathrm{Ver}\,\Delta| = \mathbb{P}H^0(\mathcal{I}_{\mathrm{Ver}\,\Delta}(d)). \tag{4.5}$$

It's easy to see that ranks of the spaces (4.4) and (4.5) are equal. We would like to prove that

$$H^1(\mathcal{I}_{\Delta^*}(-1) = H^0(\mathcal{I}_{\mathrm{Ver}\,\Delta}(d))^*. \tag{4.6}$$

Let me emphasize again that on the left-hand side we have a sheaf on $\mathbb{P}^2$ but on the right-hand side we have a sheaf on the *dual* plane $\mathbb{P}^{2*}$.

Remark. Actually the proof of this equality is a very good exercise for David Eisenbud's lectures.

Here is the heart of our lectures: for a geometric object (Δ, C) on the plane $\mathbb{P}^2$ we get a new interpretation as a pair (s, E) (see the end of the §3) on the dual plane $\mathbb{P}^{2*}$.

Let $\mathbb{P}^2 = \mathbb{P}T$, where $T = \mathbb{C}^3$, and $\mathbb{P}^{2*} = \mathbb{P}T^*$. Let

$$H^1(\mathcal{I}_{\Delta^*}(k)) = V_k. \tag{4.7}$$

Then every line l on $\mathbb{P}^{2*}$ defines a homomorphism

$$H^1(\mathcal{I}_{\Delta^*}(-1)) = V_{-1} \longrightarrow H^1(\mathcal{I}_{\Delta^*}) = V_0 \tag{4.8}$$

given by multiplication by ϕ_l.

When l sweeps out $\mathbb{P}T = (\mathbb{P}^{2*})^*$, we get a homomorphism

$$T \otimes V_{-1} \longrightarrow V_0, \tag{4.9}$$

which we can consider as a homomorphism of vector bundles on $\mathbb{P}^2$:

$$V_{-1} \otimes \mathcal{O}_{\mathbb{P}^2}(-1) \xrightarrow{\phi} V_0 \otimes \mathcal{O}_{\mathbb{P}^2}. \tag{4.10}$$

The homomorphism ϕ is nothing other than a $(d+1)\times d$ matrix of linear forms on $\mathbb{P}^2$, which we can extend to the exact sequence
(4.11)
$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^2}(-d-1) \longrightarrow V_{-1}\otimes\mathcal{O}_{\mathbb{P}^2}(-1) \xrightarrow{\phi} V_0\otimes\mathcal{O}_{\mathbb{P}^2} \longrightarrow \mathrm{coker} \longrightarrow 0.$$

It is easy to see that $\operatorname{Supp}\mathrm{coker} = \operatorname{Ver}\Delta$.

Now applying the functor $\operatorname{Hom}(*, \mathcal{O}_{\mathbb{P}^2}(-1))$ to this exact sequence, we get *Eagon–Northcott resolution*

$$V_0^*\otimes\mathcal{O}_{\mathbb{P}^2}(-1) \longrightarrow V_{-1}^*\otimes\mathcal{O}_{\mathbb{P}^2} \longrightarrow \mathcal{I}_{\operatorname{Ver}\Delta}(d) \longrightarrow 0 \tag{4.12}$$

of the ideal sheaf $\mathcal{I}_{\operatorname{Ver}\Delta}(d)$.

Remark. Our Eagon-Northcott resolution is a slight generalization of the Koszul complex (see Eisenbud's lectures).

Now the cohomology long exact sequence of (4.11) provides the required equality (4.6) and an embedding

$$M_0P^d_{d+1} \hookrightarrow \mathcal{MP}(2,2,d+1) \tag{4.13}$$

to the moduli space $\mathcal{MP}(2,2,d+1)$ of *stable pairs* (s,E) where E is a vector bundle of rank 2 with $c_1 = 2$, $c_2 = d+1$. Here the zero set of s is a simple cluster in the dual plane, that is, a $(d+1)$-gon in $\mathbb{P}^2$.

Now the left-hand side of (4.13) admits a projection map p_C to $|d\cdot l|$, and the right-hand side admits the projection on the second component – the vector bundle.

To compare these projections and to compute the fibres of p_C, we have to consider a new geometric object, the *noncommutative plane*.

§ 4 Noncommutative planes.

For any pair $(\Delta, C) \in M_0P^d_{d+1}$, the nonsingular curve C contains the effective divisor

$$\operatorname{Ver}\Delta = \bigcup_{i,j}(l_i\cap l_j) \tag{5.1}$$

(4.3) of degree $\frac{1}{2}d(d+1)$. Let $\mathcal{O}_C(h) = \mathcal{O}_{\mathbb{P}^2}(1)|_C$.

Lemma 5.1 *The divisor class*

$$\operatorname{Ver}\Delta - 2h = \theta \tag{5.2}$$

is a regular theta characteristic of C*. That is,* $2\theta = K_C$ *is the canonical class of* C*, and* $h^0(\mathcal{O}_C(\theta)) = 0$*, in other words, this theta characteristic is ineffective.*

Proof. Consider the polygon as a curve of degree $d+1$. Then the support of the intersection
$$\mathrm{Supp}\,(\Delta \cdot C) = \mathrm{Ver}\,(\Delta),$$
because for every line l_i the intersection
$$C \cap l_i = l_i \cap \big(\bigcup_{j \neq i} l_j\big).$$
(Both of sides have degree d and by definition the curve C contains the set in the right-hand side of this equality). Now by definition
$$\mathrm{Ver}\,\Delta = \mathrm{Sing}\,\Delta$$
and every singular point of Δ is quadratic. Hence as divisor classes on C, we have
$$2\mathrm{Ver}\,\Delta = C \cdot \Delta = (d+1)h,$$
and
$$2\mathrm{Ver}\,\Delta - 4h = (d-3)h = K_C$$
by the adjunction formula.

Now if $\mathrm{Ver}\,\Delta - 2h = \eta$ is effective then
$$\eta = (d-1)h - \mathrm{Ver}\,\Delta$$
and there exists a curve C' of degree $(d-1)$ which contains $\mathrm{Ver}\,\Delta$. But then C' and Δ have a common component, because
$$C' \cdot \Delta \geq 2 \deg \mathrm{Ver}\,\Delta = d(d+1) > \deg C' \cdot \deg \Delta = (d-1) \cdot (d+1).$$
Thus,
$$C' = C_0 + \bigcup_{i=1}^{n} l_i,$$
where C_0 does not contain lines. Repeating this arguments for C_0 and Δ_{d+1-n}, we get a contradiction. QED.

Now the pair (C, θ) defines a net of quadrics. Namely, if θ is an ineffective theta characteristic on C then the complete linear system $|\theta + h|$ is base point free and $h^0(\mathcal{O}_C(\theta + h)) = d$. Consider $\mathcal{O}_C(\theta + h)$ as a $\mathcal{O}_{\mathbb{P}^2}$-sheaf, and the canonical surjective map
$$H^0(\mathcal{O}_C(\theta + h)) \otimes \mathcal{O}_{\mathbb{P}^2} \longrightarrow \mathcal{O}_C(\theta + h) \longrightarrow 0.$$
We have the exact sequence
$$0 \longrightarrow \ker \xrightarrow{\alpha} H^0(\mathcal{O}_C(\theta + h)) \otimes \mathcal{O}_{\mathbb{P}^2} \longrightarrow \mathcal{O}_C(\theta + h) \longrightarrow 0,$$
and it is easy to see that
$$\ker = H^0(\mathcal{O}_C(\theta + h))^* \otimes \mathcal{O}_{\mathbb{P}^2}(-1)$$

and we have the net of correlations

$$\alpha : H \otimes \mathcal{O}_{\mathbb{P}^2}(-1) \longrightarrow H^* \otimes \mathcal{O}_{\mathbb{P}^2} \tag{5.3}$$

where $H = H^0(\mathcal{O}_C(\theta + h))^*$.

Under any identification $H = H^*$ and a choice of the homogeneous coordinates $(\lambda_0, \lambda_1, \lambda_2)$ of $\mathbb{P}^2$, we can consider the homomorphism (5.3) as a linear combination of a triple of symmetric $d \times d$ matrices

$$\lambda_0 \cdot A_0 + \lambda_1 \cdot A_1 + \lambda_2 \cdot A_2, \tag{5.4}$$

where the equation of curve C is

$$\phi_C = \det(\lambda_0 \cdot A_0 + \lambda_1 \cdot A_1 + \lambda_2 \cdot A_2), \tag{5.5}$$

and we can consider the line bundle $\mathcal{O}_C(\theta + h)$ as the family of cokernels of the net of correlations (5.3).

Now the group $\mathrm{GL}(d, \mathbb{C})$ acts on the set of triples in the usual way:

$$g(A_0, A_1, A_2) = (gA_0g^*, gA_1g^*, gA_2g^*) \tag{5.6}$$

Let $\{(A_0, A_1, A_2)\}^{\mathrm{ss}}$ be the set of semistable points with respect to this action. Then the variety

$$\{(A_0, A_1, A_2)\}^{\mathrm{ss}} \,/\, \mathbb{P}\mathrm{GL}(d, \mathbb{C}) = \mathcal{P}^2_d \tag{5.7}$$

is called the *noncommutative plane.*

C.T.C. Wall proved that

$$(A_0, A_1, A_2) \in \{(A_0, A_1, A_2)\}^{\mathrm{ss}} \implies \phi_C = \det(\sum_{i=0}^{2} \lambda_i \cdot A_i) \neq 0. \tag{5.8}$$

Thus we have a regular map

$$p_C : \mathcal{P}^2_d \longrightarrow |d \cdot l| \tag{5.9}$$

sending a triple to ϕ_C (5.5). Thus

$$\deg p_C = 2^{g-1} \cdot (2^g + 1), \quad \text{where } g = \tfrac{1}{2}(d-1)(d-2)$$

is the number of even theta characteristics of a nonsingular plane curve of degree d.

Now assume that a triple (A_0, A_1, A_2) satisfies $\det A_0 \neq 0$, and consider the skew symmetric matrix

$$[A_0, A_1 \wedge A_2] = A_1 \cdot A_0^{-1} \cdot A_2 - A_2 \cdot A_0^{-1} \cdot A_1. \tag{5.10}$$

It easy to see that the rank of this matrix $\mathrm{rank}\alpha$ is an invariant of a class of a net of quadrics. Thus we have a filtration

$$\mathcal{P}^2_d(0) \subset \mathcal{P}^2_d(2) \subset \cdots \subset \mathcal{P}^2_d, \tag{5.11}$$

where

$$\mathcal{P}_d^2(2r) = \{(A_0, A_1, A_2) \mid \operatorname{rank}[A_0, A_1 \wedge A_2] \le 2r\}.$$

Now we have the pseudoclassical

PROBLEM. What is

$$\deg p_C(\mathcal{P}_d^2(2r)) \cdot \deg p_C? \tag{5.12}$$

The relationship between our geometric objects is given by the following

PROPOSITION 5.1.

1) $p_C(\mathrm{MP}_{d+1}^d) \subset p_C(\mathcal{P}_d^2(2))$;

2) $d \le 5 \implies p_C(\mathrm{MP}_{d+1}^d) = p_C(\mathcal{P}_d^2(2))$.

To prove these statements, we need a final interpretation of the geometric objects.

Consider the flag diagram

$$\begin{tikzcd} & F = \{p \in l\} \arrow[dl, "p"'] \arrow[dr, "q"] & \\ \mathbb{P}^2 & & \mathbb{P}^{2*} \end{tikzcd} \tag{5.13}$$

Then we can apply the functor $q_* \circ p^*$ for any net of quadrics α (5.3). We get the exact sequence of sheaves on the dual plane $\mathbb{P}^{2*}$:

$$0 \longrightarrow H \otimes \mathcal{O}_{\mathbb{P}^{2*}}(-1) \xrightarrow{q_*\circ p^*(-1)(\alpha(1))} H^* \otimes \Omega(1) \longrightarrow \operatorname{coker}\alpha \longrightarrow 0 \tag{5.14}$$

and as second invariant of a net we have the number

$$\operatorname{rank}\operatorname{Hom}(\operatorname{coker}\alpha, \mathcal{O}_{\mathbb{P}^{2*}}).$$

But actually it is not a new invariant, because of the following result:

BARTH'S THEOREM (see [1]).

$$\operatorname{rank}\operatorname{Hom}(\operatorname{coker}\alpha, \mathcal{O}_{\mathbb{P}^{2*}}) = d - \operatorname{rank}\alpha$$

where $2r$ is the rank of the net of quadrics (5.10)–(5.11).

Thus for every net α of rank 2 on $\mathbb{P}^2$ we have the complex on its dual plane

(5.15)

$$0 \longrightarrow H \otimes \mathcal{O}_{\mathbb{P}^2*}(-1) \xrightarrow{q_*\circ p^*(-1)(\alpha(1))} H^* \otimes \Omega(1) \xrightarrow{\text{can}} \mathbb{C}^{d-2} \otimes \mathcal{O}_{\mathbb{P}^2*} \longrightarrow 0$$

which is called a *monad*, and the middle cohomology

$$\ker \text{can} / \operatorname{im} q_* \circ p^*(-1)(\alpha(1)) = E \tag{5.16}$$

is a semistable torsion-free sheaf of rank 2 with the Chern classes $c_1 = 0$, $c_2 = 2$.

So we have the map

$$\mathcal{P}^2_d(2) \longrightarrow \overline{M(2,0,d)}$$

to the Gieseker closure of the moduli space of stable vector bundles on $\mathbb{P}^{2*}$. The construction of the inverse map is as follows: a point $p \in \mathbb{P}^2$ gives a line $l_p \in \mathbb{P}^{2*}$ in the dual plane. A line $l_p \in \mathbb{P}^{2*}$ is called a *jumping line* for E if

$$E|_{l_p} \neq \mathcal{O}_{l_p} \oplus \mathcal{O}_{l_p}, \quad \text{which happens iff} \quad h^0(E(-1)|_{l_p}) \neq 0. \tag{5.17}$$

Thus in the dual plane $\mathbb{P}^{2*}$ we have the curve $C(E)$ of jumping lines of E.

Now it is easy to see that $H^1(E(-2)) = H = \mathbb{C}^d$ and the Serre dual space $H^1(E(-1)) = H^*$. Now multiplication by the equation of any line ϕ_{l_p} defines the correlation

$$H^1(E(-2)) = H \longrightarrow H^1(E(-1)) = H^* \tag{5.18}$$

as an element of the net of correlations (5.4) $C(E) = C(5.5)$.

Now consider a regular pair $(\Delta, C) \in M_0 P^d_{d+1}$ (see (5.1)), where $\Delta = \bigcup l_i$. Then we have the chain of identifications:

$$(\Delta, C) = (\mathbb{C}^* \cdot s, E) \tag{5.19}$$

where s is a regular section of E;

$$E = (C, \theta) = (\alpha) \tag{5.19$'$}$$

So a pair (Δ, C) is geometrically equivalent to the exact sequence on $\mathbb{P}^{2*}$:

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^{2*}}(-1) \longrightarrow E \longrightarrow \mathcal{I}_{\Delta^*}(1) \longrightarrow 0 \tag{5.20}$$

where $\mathcal{I}_\Delta$ is the ideal sheaf of 0-dimensional cycle Δ^* on $\mathbb{P}^{2*}$.

Corollary. *The space of circumscribed* $(d+1)$*-gons to a Darboux curve* C *is a rational irreducible variety.*

Indeed, it is birationally equivalent to $\mathbb{P}H^0(E)$!

These geometric identifications were done for “regular” geometric objects. It is reasonable to construct some “natural” nonsingular compactification of the space of regular objects sending the computation of constants of type (5.12), (2.11), (2.5) to the regular procedure of computations of Chern classes of standard vector bundles on “moduli space” our geometric figures.

§ 5 Compactifications.

A regular $(d+1)$-gon (2.1) is of course a curve of degree $d+1$ on $\mathbb{P}^2$, and so the space P_{d+1} of all polygons is a compact irreducible subvariety in the complete linear system $|(d+1)\cdot l|$. Geometrically,

$$P_{d+1} = S^{d+1}\mathbb{P}^{2*} \tag{6.1}$$

is the $(d+1)$-st symmetric power of $\mathbb{P}^{2*}$, a rather singular algebraic variety. Fortunately for algebraic surfaces, there is a canonical desingularisation of it. Let us recall that a regular polygon Δ defines a zero dimensional subscheme Δ^* (4.2). Thus as a compactification of the space of regular polygons on $\mathbb{P}^2$, we can consider the moduli space of zero dimensional subscheme of $\mathbb{P}^{2*}$ of degree d:

$$\mathrm{Hilb}^{d+1} = M(1,0,d+1) \tag{6.2}$$

which is called the *Hilbert scheme* on $\mathbb{P}^{2*}$. The beautiful and very important theory of Hilbert schemes says that for a nonsingular algebraic surface, this scheme is again nonsingular (see [5]). The space of extensions of type (5.20) is given by the projectivisation of the vector space $H^1(\mathcal{I}_\xi(-1))$ (see (3.8)), because of $c_1(E(1)) = 2h$, and because the canonical class of the plane is given by $K_{\mathbb{P}^{2*}} = -3h$. Thus, it is natural to represent the space of all nontrivial extensions (5.20) as a projectivisation of a vector bundle on Hilb^{d+1}. Of course, this variety is nonsingular.

From now on, all of our geometric objects are defined on the dual projective plane $\mathbb{P}^{2*}$, and we omit the star.

First of all, on the Hilbert scheme we have the special divisor class H defined by clusters intersecting a fixed line. On the other hand, the Hilbert scheme defines the universal subscheme $Z \subset \mathbb{P}^2 \times \mathrm{Hilb}$, and the two projection maps to the direct components define the diagram

$$\begin{tikzcd} & Z_{d+1} \arrow[dl, "p_S"'] \arrow[dr, "p_H"] & \\ \mathbb{P}^2 & & \mathrm{Hilb}^{d+1} \end{tikzcd} \tag{6.3}$$

For any divisor class $\mathcal{O}_{\mathbb{P}^2}(k)$, consider the vector bundle

$$\mathcal{E}_k = R^0 p_H(p_S^*\mathcal{O}_{\mathbb{P}^2}(k)). \tag{6.4}$$

These sheaves are locally free, because the canonical homomorphism is surjective.

In particular, in our case $k=-1$, the fibre of this vector bundle over $\xi \in \mathrm{Hilb}$ is $H^0(\mathcal{O}_\xi(-1))$. Now the cohomology long exact sequence of

$$0 \longrightarrow \mathcal{I}_\xi(-1) \longrightarrow \mathcal{O}(-1) \longrightarrow \mathcal{O}_\xi \longrightarrow 0 \tag{6.5}$$

gives the isomorphism

$$H^0(\mathcal{O}_\xi(-1)) = H^1(\mathcal{I}_\xi(-1)) \tag{6.6}$$

Thus the space of all extensions of type (5.20) is the projectivisation

$$\mathbb{P}\mathcal{E}^*_{-1}. \tag{6.7}$$

The next thing we have to understand is that our constant $s_{d+1}(d)$ (2.5) is the top Segre class of the standard vector bundle on Hilb:

$$s_{d+1}(d) = s_{\mathrm{top}}(\mathcal{E}_{-1}(H)) \tag{6.8}$$

To see this, we have to return to the isomorphism (4.6) and remark that for $s \in H^0(\mathcal{I}_{\mathrm{Ver}\,\Delta}(d)$ the condition "a curve $C = (s)_0$ passes through a point $l^* \in \mathbb{P}^{2*}$" defines a section of $\mathcal{E}_{-1}(H)$! Thus by the definition of the Segre class, $3d+2$ general points determine a general $(3d+2)$-dimensional subspace W of $H^0(\mathcal{E}_{-1}(H))$. Now the canonical homomorphism

$$W \otimes \mathcal{O}_{\mathrm{Hilb}} \xrightarrow{\mathrm{can}} \mathcal{E}_{-1}(H)$$

is general enough and

$$\deg \operatorname{coker} \mathrm{can} =: s_{\mathrm{top}}(\mathcal{E}_{-1}(H)) = s_{d+1}(d) \tag{6.9}$$

is the number of Darboux curves through $3d+2$ general points.

Using this beautiful interpretation, G. Ellingsrud and S. Strømme computed (2.12) (using Bott's formula for the $\mathbb{C}^*$-action on Hilb, see [4]):

$$\begin{aligned} s_5(4) &= 54 \\ s_6(5) &= 2540 \\ s_7(6) &= 583020 \\ s_8(7) &= 99951390 \\ s_9(8) &= 16059395240 \\ s_{10}(9) &= 2598958192572. \end{aligned}$$

This list can be extended if your computer is good enough and you have S. A. Strømme as a collaborator. But to understand the nature of these numbers (it's the new shape of mathematical questions, is not it?), you have to use new identifications, proposed below, and the collection of beautiful new results provided by differential topologists such as Fintushel and Stern, Kotschick and Lisca and many others. We will discuss this in the next section.

Thus this story is not finished yet. The nonsingular compactification (6.7) of the moduli space of pairs (Δ, C) (on $\mathbb{P}^{2*}$!) is called the moduli space of stable pairs $(\mathcal{C}^* \cdot s, E)$ (see [15], Lecture 6):

$$\mathbb{P}\mathcal{E}^*_{-1} = \mathcal{MP}(2, 2, d+1). \tag{6.10}$$

Let us consider the general diagram of our identifications:

$$
\begin{tikzcd}
 & \mathbb{P}\mathcal{E}^*_{-1} \arrow[r, equal] \arrow[dl, "p_\Delta"'] \arrow[d, "p_C"] & \mathcal{MP}(2,2,d+1) \arrow[d, "p_E"] \\
\mathrm{Hilb}^{d+1} & {|d \cdot l|} & M(2,2,d+1) \arrow[l, "p_j"']
\end{tikzcd} \tag{6.11}
$$

where $M(2,2,d+1)$ is the moduli space of semistable bundles, the map p_E sends a pair (E,s) to the vector bundle E, and p_j sends the vector bundle E to its curve of jumping lines (5.17). Now twisting sheaves by $\mathcal{O}_{\mathbb{P}^2}(1)$ gives the isomorphism

$$M(2,0,d) = M(2,2,d+1)$$

and we can use the chain of identifications (5.19).

Remark. The extension of the map p_j to the compactification of moduli spaces is a nontrivial task.

The map p_E is only a rational map, as treated in detail in [12]: we described there what has to be blown up and what gets blown down. But in any case, we can describe the image of this rational map: the final moduli space contains the Brill–Noether locus

$$M_1 = \{E \in M(2,2,d+1) \mid h^0(E) \geq 1\}. \tag{6.12}$$

Thus, we have

$$M_1 = p_E(\mathcal{MP}(2,2,d+1)) \tag{6.13}$$

This is just what we need. Now, by the R–R theorem,

$$d \leq 5 \implies M_1 = M(2,2,d+1). \tag{6.14}$$

Moreover, if $d = 4$, that is, for the Lüroth quartics we have the following construction: let Q be a nonsingular plane conic and $|\eta|$ be a general linear pencil of divisors of degree 5 on Q. For an element of this pencil $p_1 + \cdots + p_5$, consider the pentagon $\Delta = \bigcup l_i$, where l_i is the tangent line to Q at p_i. When elements sweep out this pencil, the cycles $\operatorname{Ver} l(\Delta)$ sweep out a quartic curve C.

So we have the divisor classes

$$\mathcal{O}_{\mathbb{P}\mathcal{E}^*_{-1}}(1) = p_C^*(\mathcal{O}_{|d\cdot l|}(1)) = \mathcal{O}_{\mathcal{MP}(2,2,d+1)}(D), \tag{6.15}$$

$$\mu(l) = p_j^*(\mathcal{O}_{|d\cdot l|}(1)) \tag{6.16}$$

on $\mathcal{MP}(2,2,d+1)$ and M_1, which are related by the birational map p_E. This situation is a beautiful exercise in practical birational geometry. As you know from Miles Reid's lectures, a birational map may well alter the degree of a divisor. But in our case (this beautiful observation is due to Dmitry Orlov) the existence of the regular maps p_C and p_j relating our divisors as in (6.15) and (6.16) gives the equality

$$s_{\mathrm{top}}(\mathcal{E}_{-1}(H)) = c_{d+1}(d) = D^{3d+2} = (\mu(l))^{3d+2}. \tag{6.17}$$

The last link of this beautiful chain of identifications of geometric objects is following: the points of each of the moduli spaces $\mathcal{MP}(2,2,d+1)$, $M(2,2,d+1)$ and M_1 describe geometric objects on the same algebraic surface, the plane $\mathbb{P}^2$. But the construction of the divisor class $\mu(l)$ sends us to geometric objects (jumping curves) on the dual plane $\mathbb{P}^{2*}$. From the *algebraic geometric* point of view this is reasonable, but we promised to extend these constructions to objects of *differential geometry.* As a differential topological object, the projective plane does not define the dual plane. Avoiding this obstacle, we would like to describe the constants $s_{d+1}(d)$ in terms of $\mathbb{P}^2$. That is, we want to define μ-class $\mu(l)$ in terms of objects on $\mathbb{P}^2$ only.

Now on the direct product $\mathbb{P}^2 \times M(2,2,d+1)$, the universal sheaf $\mathcal{F}$ exists locally only (for technical details see [14]), but the Pontrjagin class

$$p_1(\mathcal{F}) = 4c_2(\mathcal{F}) - c_1^2(\mathcal{F}) \in \mathrm{Pic}\ \mathbb{P}^2 \otimes \mathrm{Pic}\ M(2,2,d+1) \tag{6.18}$$

is defined correctly in any case. The intersection number on Pic $\mathbb{P}^2$ gives the isomorphism $(\mathrm{Pic}\ \mathbb{P}^2)^* = \mathrm{Pic}\ \mathbb{P}^2$ so we can consider the Pontrjagin class (6.18) as the homomorphism

$$\mu = 1/4 p_1(\mathcal{F}) : \mathrm{Pic}\ \mathbb{P}^2 \longrightarrow \mathrm{Pic}\ M(2,2,d+1). \tag{6.19}$$

Now it's easy to see that the divisor class $\mu(l)|_{M_1}$ is just (6.16).

Now to get our classical enumerative algebraic geometry constants (2.11) in the more general set-up, we have to use the equality

$$s_{d+1}(d) = (\mu(l))^{3d+2}, \tag{6.20}$$

and to extend the definitions of $M_1 \subset M(2,2,d+1)$ and $\mu(l)$ in differential geometric terms. Of course now $M_1 \subset M(2,2,d+1)$ will be compact spaces and $\mu(l) \in H^2(M(2,2,d+1), \mathbb{Z})$ is a 2-cohomology class only but this is quite enough to define the constant (6.20)!

§ 6 Differential geometry.

Algebraic geometry can be considered as a part of differential geometry, namely as Kähler geometry. Then we have to use new notions like connections, differential forms and so on; you can learn this approach from the standard monograph [6]. We also strongly recommend the monograph [2] as a unique source of new style to use these ideas. But it is very important to understand that classical algebraic geometry is a foundation of almost all the local constructions of Riemannian geometry. We will discuss the special case of the projective plane $\mathbb{P}^2$, but you can determine the generality quantor yourself.

Let M be the underlying 4-manifold of the complex projective plane $\mathbb{P}^2$. Any Riemannian metric g on M defines a decomposition of the complexified tangent bundle $T_{\mathbb{C}}M$ as a tensor product

$$T_{\mathbb{C}}M = (W^-)^* \otimes W^+$$

of two rank 2 Hermitian vector bundles $W^\pm$ with

$$c_1(W^\pm) = -3h,$$

where h is the generator of $H^2(M, \mathbb{Z})$.

Write $*$ for the Hodge star operator on $\Omega^2(M)$ determined by the metric g. Moreover, for any $U(2)$-bundle E on M of topological type ($c_1 = 2$, $c_2 = d+1$) and any Hermitian connection $a \in \mathcal{A}_h$ on E, putting any Hermitian connection ∇_0 on $\Lambda^2 W^\pm$ gives a coupled Dirac operator

(7.1) $$D_a^{g,\nabla_0} : \Gamma^\infty(E \otimes W^+) \longrightarrow \Gamma^\infty(E \otimes W^-).$$

Now the orbit space of irreducible connections modulo the gauge group

$$\mathcal{B}(2,2,d+1) = \mathcal{A}_h^*(2,2,d+1)/\mathcal{G}$$

contains the subspace

(7.2) $$\mathcal{M}^g(2,2,d+1) = \big\{\, (a) \in \mathcal{M}^g(2,2,d+1) \;\big|\; *F_a = -F_a \big\} \subset \mathcal{B}(E)$$

of antiselfdual connections with respect to the Riemannian metric g (here F_a is the curvature form of a connection a).

Now we can consider the subspace of jumping connections:

(7.3)
$$\mathcal{M}_1^g(d) = \big\{(a) \in \mathcal{M}^g(2,2,d+1) \;\big|\; \operatorname{rank}\ker D_a^{g,\nabla_0} \geq 1\big\} \subseteq \mathcal{M}^g(2,2,d+1).$$

The virtual codimension of $\mathcal{M}_1^g(d)$ (that is, the expected codimension determined by the Atiyah–Singer index theorem) is given by

(7.4) $$\text{v.codim}\mathcal{M}_1^g(d) = 2 - 2\chi = 2(5-d),$$

where χ is the index of the coupled Dirac operator (7.1), which depends only on the Chern classes of E.

So you can see that for $d \leq 5$

$$\mathcal{M}_1^g(d) = \mathcal{M}^g(2,2,d+1) \tag{7.5}$$

Please compare this fact with (6.14)!

For a generic metric g, the moduli spaces $\mathcal{M}^g(2,2,d+1)$ and $\mathcal{M}_1^g(d)$ (7.3) are smooth manifolds of the expected dimension with regular ends (see [2] and [11], Chap. 2, § 3). Moreover, $\mathcal{M}^g(2,2,d+1)$ admits a natural orientation (see [2]) inducing an orientation on $\mathcal{M}_1^g(d)$, because its normal bundle has a natural complex structure. This orientation is described in detail in [11], Chap. 1, § 5.

Moreover, there exists the so-called Uhlenbeck compactification of our moduli spaces

$$\overline{\mathcal{M}^g(2,2,d+1)} \supseteq \overline{\mathcal{M}_1^g(d)}. \tag{7.6}$$

Now for any element of our filtration the first Pontrjagin class of the universal connection on the direct product $M \times \mathcal{M}^g(2,2,d+1)$ (by the slant product) defines cohomological correspondences

$$H_i(M,\mathbb{Z}) \xrightarrow{\mu_d} H^{4-i}(\overline{\mathcal{M}^g(2,2,d+1)},\mathbb{Z}),$$
$$H_i(M,\mathbb{Z}) \xrightarrow{\mu_d^1} H^{4-i}(\overline{\mathcal{M}_1^g(d)},\mathbb{Z}),$$

and two collections of numbers

$$D_g(d) = (\mu_d(h))^{4d-3}, \tag{7.7}$$

the so-called *Donaldson numbers* (Donaldson polynomials) of $\mathbb{P}^2$, and

$$s\gamma_g^d = (\mu_d^1(h))^{3d+2}. \tag{7.8}$$

Now, suppose as a special case that our metric g is the Fubini–Study metric g_{F-S}. In this case, by the Donaldson–Uhlenbeck identification theorem, we have

$$\mathcal{M}^{g_{F-S}}(2,2,d+1) = M(2,2,d+1),$$

where the right-hand side is the moduli space of holomorphic stable bundles on $\mathbb{P}^2$ (6.11).

Making this identification $(a) = E$, we have identifications

$$\ker D_a^{g_{F-S}} = H^0(E) \oplus H^2(E) \qquad \text{and} \qquad \operatorname{coker} D_a^{g_{F-S}} = H^1(E), \tag{7.9}$$

where $H^i(E)$ denote coherent cohomology groups (see [2]).

But by Serre duality $H^2(E) = H^0(E(-2))^* = 0$, by the stability of E. Thus, the subspace $\mathcal{M}^{g_{F-S}}$ is

$$\mathcal{M}_1^{g_{F-S}}(d) = \big\{E \in M(2,2,d+1) \;\big|\; h^0(E) \geq 1\big\}, \tag{7.10}$$

that is (see (6.12)),

$$\mathcal{M}_1^{g_{F-S}}(d) = M_1$$

and our constants (7. 8)

$$s\gamma^d_{F-S} = s_{d+1}(d) \tag{7.11}$$

are constants (2.5) and (6.8).

These integers do not depend on the metric g, because the space of all Riemannian metrics is contractible. (In fact, to be rigorous, we have to use much more sophisticated bordism arguments, similar to those in [2], where the same statement was proved for the Donaldson's numbers (7. 7)).

Let us remark that the initial terms of both collections (7.7) and (7.8) are coincidence by (7.5) and (6.14)

$$D_g(4) = s\gamma^4_g = 54;$$
$$D_g(5) = s\gamma^5_g = 2540;$$

and the collection of Donaldson's constants was extended to infinity by Ellingsrud and Göttsche (see [3]):

$$\begin{aligned} D_g(6) &= 233208;\\ D_g(7) &= 35825553;\\ D_g(8) &= 8365418914;\\ D_g(9) &= 2780195996868;\\ D_g(10) &= 1253558847090600 0; \quad \text{etc.} \end{aligned}$$

Let me remark that we have got the following striking fact:

Theorem. *The constants* (7.11) = (2.5) = (6.8) *are invariants of the underlying differentiable structure of* $\mathbb{P}^2$.

It is well known that the complex structure on M is unique. Recently it was proved that the symplectic structure on M is unique. The next question is the following differentiable version of the Poincaré conjecture for $\mathbb{P}^2$:

Conjecture (DPC for $\mathbb{P}^2$). *The complex projective plane* $\mathbb{P}^2$ *has a unique differentiable structure.*

In particular this statement would imply the following fact

Corollary. *The constants* (7.11) = (2.5) = (6.8) *are invariants of the topological structure of* $\mathbb{P}^2$.

There is overwhelming direct evidence for this statement, and hence a partial confirmation of Conjecture DPC for $\mathbb{P}^2$. Namely, there are two possible methods to construct our constants using the topological structure of the plane only. The first approach is related to the following fundamental problem:

Hilbert scheme problem. *Can we give a purely topological construction of the Hilbert scheme* Hilb^d *and of the standard vector bundles* (6.3)*?*

Of course this problem is interesting in full generality for all 4-manifolds. If we could realize the scheme (6.3) topologically then, as proposed by G. Ellingsrud, we could use induction over d to prove

Proposition 7.1. *The constants* (6.8) *are topological invariants.*

The second approach to prove the statement of Corollary is related to proving of the same fact for Donaldson's constants (7.7) proposed by Kotschick and Lisca using the Kotschick–Morgan Conjecture from [7].

The idea of this program is following: let us blow up one or two points on $\mathbb{P}^2$ and for this new surface $\widetilde{\mathbb{P}^2}$ let us consider the same numbers as (7.7). But now these numbers depend on the Riemannian metric in essential way. However it can be shown that the dependence of these numbers on the metric can be controlled explicitly! Going from $\mathbb{P}^2$ to $\widetilde{\mathbb{P}^2}$, we have to consider the collection $\{\alpha\}_d \subset H^2(\widetilde{\mathbb{P}^2}, \mathbb{Z})$ of classes such that

$$\alpha = 0 \mod 2 \quad \text{and} \quad -4d \leq \alpha^2 < 0. \tag{7.12}$$

The intersection of the positive cone in $H^2(\widetilde{\mathbb{P}^2}, \mathbb{R})$ with the hyperplane $\alpha^\perp$ is called a *d-wall* (or the d-wall defined by α). Let Δ_d be the set of open chambers into which the positive cone is divided by all d-walls.

Proposition 7.2.

1) *The constant $D_g(d)$ of (7.7) for $\widetilde{\mathbb{P}^2}$ depends on the chamber $C \in \Delta_d$, which contains the g-self dual harmonic 2-form.*

2) *If C and C' are chambers then*

$$D_C(d) - D_{C'}(d) = \sum_\alpha \delta_d(\alpha), \tag{7.13}$$

where the sum is taken over all d-walls α such that

$$\alpha \cdot C' < 0 < \alpha \cdot C,$$

that is over all walls dividing C and C'.

Now the following result has been proved:

Proposition 7.3. *The constant $D_g(d)$ of (7.7) for $\widetilde{\mathbb{P}^2}$ are determined by the difference terms δ_d only.*

The following fact is "almost" proved:

Kotschick–Morgan Conjecture (see [7]). *The difference terms δ_d are homotopy invariants.*

So finishing the proof of this conjecture implies the topological definition of the Donaldson constants (7.7).

Remark. Mixing the Hilbert scheme method and the difference terms method is a very fruitful technique. G. Ellingsrud and L. Göttsche [3] can use it to compute any Donaldson number exactly, but the real nature of these expressive numbers remains an open question.

Finally, to prove the topological nature of the constants $s\gamma^d_{F-S} = s_{d+1}(d)$ (7.11), we have to mimic these constructions for the moduli spaces of jumping instantons (7.3) or, in full generality, for the spin polynomials (see [13]) in place of the Donaldson polynomials.

I would like to finish by drawing the following conclusion:

MORAL. *Different interpretations of classical algebraic geometric figures provide very fruitful approaches to understanding the nature of results of ENUMERATIVE AG.*

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The Weil–Petersson metric on the moduli space of stable vector bundles and sheaves on an algebraic surface

A twistor description of the Weil–Petersson metric on the moduli space of stable vector bundles on a K3-surface with hyper-Kähler structure is given, and this metric is extended to the compactification of the moduli space by torsion-free sheaves.

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Dedicated to the memory of my sister

Introduction.

The moduli space of stable vector bundles on a compact Kähler surface (S, ω) has a canonical Kähler metric, which it is natural to call *the Weil–Petersson metric*. Recall that this metric was originally constructed by Weil and Petersson on the Teichmüller space of distinguished Riemann surfaces of genus g; in that context it was studied by Ahlfors [1], Royden [14], and Wolpert [20]. The basic result of this research was to describe the geometrical properties of the curvature. In particular, Wolpert proved that the metric has negative sectional curvature.

The actual construction of the Weil–Petersson metric is based on the existence of a constant curvature metric on a Riemann surface; hence, it became possible to carry this construction over to multidimensional Kähler manifold only after the Calabi–Yau problem was solved, i.e., after Kähler–Einstein metrics were constructed on compact Kähler manifolds with negative or zero first Chern class (see [21]).

Koiso [10] verified the Kähler condition for the Weil – Petersson metric in the multidimensional case, and Siu [15] developed the method of canonical lifting of tangent vectors in order to study the curvature properties of this metric. This led Siu to a precise (and relatively simple) description of the complete curvature tensor and a proof that the holomorphic bisectional curvature is negative in certain important but special cases.

Finally, Schumacher [17] and Nannicini [13] studied the curvature properties of the Weil–Petersson metric for manifolds with zero Chern class, and they proved (using different methods) that the holomorphic bisectional curvature is negative in the case of symplectic manifolds. In particular, in the case of polarized K3-surfaces, Schumacher proved that the Weil–Petersson metric coincides with the Bergman metric of the type III symmetric domain that is induced on the moduli space by the period mapping [15].

The Weil–Petersson metric is defined on the moduli space of stable vector bundles on a compact Riemann surface using the identification of the moduli space of stable vector bundles with the moduli space of Hermitian–Einstein connections on a fixed $\mathbb{C}^\infty$-bundle.

Zograf and Takhtadzhyan made an exhaustive investigation of the properties of this metric, connecting the properties of the curvature tensor with classical problems in the theory of Riemann surfaces [23].

In particular, the symplectic form of the Weil–Petersson Kähler metric coincides with the $\bar{\partial}$-derivative of the canonical section (given by the Hermitian–Einstein connection) of the affine bundle on the connection over the moduli space of vector bundles. The same form is proprtional to the curvature form of the Quillen metric on the determinant bundle of the family of $\bar{\partial}$-operators.

By explicitly computing the curvature tensor and interpreting its components, one obtains positivity of the scalar curvature and nonnegativity of holomorphic bisectional curvatures.

In the multidimensional case, the solution of the Kobayashi–Hitchin problem concerning the existence of a Hermitian–Einstein connection in a holomorphic stable vector bundle leads to the construction of the Weil–Petersson metric on the moduli space of vector bundles (see [19] for the greatest generality).

Buchdahl [2] proved the Kobayashi–Hitchin conjecture for complex-analytic surfaces with Hermitian metric g, whose symplectic form is $\bar{\partial}\partial$-closed.

In the case when a Kähler surface (S, ω) is the base of vector bundles, the notion of a Hermitian–Einstein connection coincides with the notion of an anti-self-dual connection (ASD) connection [8], and so it is equivalent to the notion of a holomorphic stable bundle. In this case Ito [8] computed the curvature form for the Weil–Petersson metric on the moduli space of vector bundles; in certain cases this made it possible to connect the sign of the curvature of the surface with the moduli space [9]. In particular, if the metric g on S is hyper-Kählerian, then so is the Weil–Petersson metric on the moduli space of vector bundles (see [8] and [9]).

On the other hand, if S is an algebraic surface and g is the Hodge metric giving the polarisation H, then, using modular operations ([24], Chapter II), one can construct a series of components of the moduli space of H-stable vector bundles of standard birational type; in particular, they are birationally equivalent to the surface S itself. In this case one would like to compare the original metric on S and the Weil–Petersson metric on S, as metrics on the moduli space of vector bundles. The obstacle to doing this is the fact that, in general, the components of the moduli space of stable bundles are not compact. They can be compactified (see Gieseker [4]) using semistable torsion-free sheaves. Thus, one does not have a direct differential geometric construction of the Weil–Petersson metric on the points of the boundary (see [8]), and we are left with two problems:

1) to extend the Weil–Petersson metric to the compactification of a component of the moduli space of stable vector bundles;

2) to study the geometry of the curvature tensor of the extended metric at boundary points of the moduli space.

In this paper we solve both of these problems in the case when S has a nondegenerate symplectic structure, i.e., in the case of 3-surfaces and abelian surfaces. We give a description of all of the constructions only in the case of

3-surfaces, since the constructions are exactly the same for abelian surfaces. We shall apply the twistor construction of hyper-Kähler metrics (see [5], [7], and [16]).

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I would like to thank A. Todorov for interesting and stimulating discussions of the problem.

§ 1 Hyper-Kähler metrics.

A triple of infinitesimal structures on the tangent bundle TX of a smooth manifold X having the form

$$
\begin{array}{llll}
& TX \xrightarrow{g} T^*X & g^* = g & \text{(Riemannian structure)} \\
(1.1) & TX \xrightarrow{I} TX & I^2 = -1 & \text{(complex structure),} \\
& TX \xrightarrow{\omega} T^*X & \omega^* = -\omega & \text{(symplectic structure)}
\end{array}
$$

together give a single almost complex Hermitian structure

$$
h : TX \longrightarrow T^*X, \quad \bar{h}^* = h, \tag{1.2}
$$

if

$$
IgI^* = g, \quad I\omega I^* = \omega = Ig. \tag{1.3}
$$

If the structure is I-integrable and the form ω is closed, then we say that it is a *Kähler structure.*

Remark. Thurston [18] constructed an example of an almost complex Hermitian structure with closed form ω, which is not Kähler.

If $\dim X = 4$, then there is an important generalisation of an almost complex structure (1.1) — an infinitesimally quaternionic structure, which is given by three almost complex structures:

$$
\begin{array}{lll}
I : TX \longrightarrow TX, & I^2 = J^2 = K^2 = -1, \\
J : TX \longrightarrow TX, & IJ = -JI = K, \\
K : TX \longrightarrow TX, & KI = -IK = J.
\end{array} \tag{1.4}
$$

If X has a metric g (1.1), which is invariant under I, J, and K (see (1.3)), then we can define the three skew-symmetric forms:

$$
\omega_I = Ig, \quad \omega_J = Jg, \quad \omega_K = Kg. \tag{1.5}
$$

In contrast to Thurston's example in [18], we have the following

Lemma (Hitchin [6], Lemma 6:8). *If the forms ω_I, ω_J and ω_K in* (1.5) *are closed, then any almost complex structure*

$$\{aI + bJ + cK | a, b, c \in \boldsymbol{R}, \quad (aI + bJ + cK)^2 = -\mathrm{id}\} \tag{1.6}$$

is integrable.

In this case the four-tuple (g, I, J, K), or $(g, \omega_I, \omega_J, \omega_K)$, is called a *hyper-Kähler structure on* X, and $I, J, K, \omega_I, \omega_J, \omega_K$ are called the *components* of the hyper-Kähler structure on X.

The complex structures I, J, K are not in any way distinguishable from the other points of the two-dimensional sphere $S^2 = q = \mathbb{P}^1$ which parametrizes the complex structures (1.6) on X. It is easy to see that the form

$$\omega_I + i\omega_K \tag{1.7}$$

is a nowhere degenerate holomorphic 2-form for the complex structure I on X. We shall call the family (1.6) the *conic* of complex structures of the hyper-Kähler structure.

If $\dim X = 4$, then a Riemannian metric g on X can be a component of only one hyper-Kähler structure; and a Kähler structure is hyper-Kähler if and only if it is self-dual and Ricci-flat [16]. In this case the holomorphic forms (1.7) for the family (1.6) of complex structures generate a family of one-dimensional subspaces in $H^2(X, \mathbb{C})$, which sweep out a conic in the projectivisation $\mathbb{P}H^2(X, \mathbb{C})$ as the complex structure varies. This explains the terminology "conic of complex structures (1.6)".

In addition to having a simple integrability criterion (that three forms be closed; see Hitchin's lemma), hyper-Kähler structures enjoy one other remarkable property: they have a characterisation in terms of the complex geometry of the space of twistors (see [5] and 16]).

Theorem (HKLR [5], Theorem 3.3). *Let Z be a smooth complex manifold of dimension $2n + 1$ for which the following conditions hold*:

1) *There exists a holomorphic surjection*

$$Z \xrightarrow{\pi} \mathbb{P}^1 = q \tag{1.8}$$

(*i.e.*, $d\pi : TZ_z \longrightarrow Tq_{\pi(z)}$ *is an epimorphism for every* $z \in Z$).

2) *The relative tangent bundle*

$$TZ_{/\pi} = \ker d\pi \tag{1.9}$$

has an isomorphism

$$TZ_{/\pi} \xrightarrow{\tilde{\omega}} (TZ_{/\pi})^* \otimes \pi^* \mathcal{O}_q(2) \tag{1.10}$$

such that

$$\tilde{\omega}^* \otimes \mathcal{O}_q(2) = -\tilde{\omega}. \tag{1.11}$$

3) *There exists a holomorphic section*

$$s : q \longrightarrow Z \tag{1.12}$$

such that the normal bundle of the curve

$$l = s(q) \tag{1.13}$$

has the form

$$N_{l \subset Z} = \mathbb{C}^{2^n} \otimes \mathcal{O}_q(1); \tag{1.14}$$

4) *There exists a real structure*

$$\sigma : Z \longrightarrow Z, \tag{1.15}$$

acting on the fibre of the bundle (1.8) *which is compatible with* (1.10), *preserves the section* s *in* (1.12), *and induces the antipodal map on* $q = \mathbb{P}^1$:

$$\sigma : z \longrightarrow 1/z. \tag{1.16}$$

Then a hyper-Kähler structure is induced on the manifold of σ*-real sections. Conversely, every hyper-Kähler structure determines a smooth complex manifold* Z *with all of the features* (1.8) – (1.15), *where the base manifold* X *can be identified with the manifold of* σ*-real sections and the hyper-Kähler structure coincides with the induced hyper-Kähler structure.*

We may always suppose that there is a σ-real section of the form (1.12) passing through every point of the fibre of the surjection (1.8), and this section is unique, because the real sections do not intersect.

The manifold Z is called the *twistor space* of the hyper-Kähler structure (g, q) on X. The real sections of the bundle (1.8) are disjoint, and they determine a $\mathbb{C}^\infty$-bundle

$$Z \xrightarrow{p} X, \tag{1.17}$$

whose fibre

$$p^{-1}(x) = l \tag{1.18}$$

is a smooth rational curve in Z with normal bundle of the form (1.14).

The family of fibres of the surjection (1.8) coincides with the family of complex structures (1.6) that is parametrized by the conic $q = \mathbb{P}^1 = S^2$.

Let G be the space of sections of (1.8) of the form (1.12) – (1.14), and let l be a σ-real section, which we can identify with a point $x \in X$ so that $p^{-1}(x) = l$. Then the space of lines which intersect l determines a symmetric correlation on the subspace of σ-real sections of the space $H^0(N_{l \subset Z}) = TG_l$, which we can identify with the complexification of the tangent space TX_x. This symmetric correlation also determines the quadratic form g_x of the metric g on TX_x.

Now let (X, g) be a K3-surface with Ricci-flat Kähler structure. Then the twistor space $Z(S)$ is a three-dimensional compact manifold with two surjections

(1.19)
$$\begin{tikzcd} & Z(S) \arrow[dl, "p"'] \arrow[dr, "\pi"] & \\ S & & q \end{tikzcd} ,$$

where π is a holomorphic map, p is a $\mathbb{C}^\infty$-map, and the pair $p \times \pi$ gives a $\mathbb{C}^\infty$-identification of $Z(S)$ with $S \times q$.

The fibres $\pi^{-1}(z)$ are Kähler K3-surface. For any such surface

$$S_z = \pi^{-1}(z), \tag{1.20}$$

the surface $S_{\sigma(z)}$ has a complex structure conjugate to S_z. We shall later have need of this type of variation of complex structure.

Definition 1.1.

1) *A simple variation of a complex analytic surface* S_0 is a smooth three-dimensional manifold Z and a smooth manifold T with distinguished point t_0 connected by maps

(1.21)
$$\begin{tikzcd} & Z \arrow[dl, "p"'] \arrow[dr, "\pi"] \arrow[d, phantom, "\cup"] & \\ S_0 & S_0 \arrow[l] \arrow[r, "\pi"] & t_0 \in T \end{tikzcd}$$

such that

a) π is a smooth holomorphic map of $\pi^{-1}(t_0) = S_0$, and

b) p is an everywhere nondegenerate $\mathbb{C}^\infty$-map which gives a $\mathbb{C}^\infty$-isomorphism

$$Z \xrightarrow{p \times \pi} S_0 \times T. \tag{1.22}$$

2) *A simple variation of a holomorphic vector bundle E on S_0 which covers the simple variation* (1.21) is a vector bundle $\mathcal{E}$ on Z such that

a) $\mathcal{E}|_{\pi^{-1}(t_0)} = E$, and

b) there exists a $\mathbb{C}^\infty$-isomorphism of bundles $f : p^* E \longrightarrow \mathcal{E}$

The twistor space (1.19) is an example of a global simple variation.

§ 2 Stratification of the moduli space.

We consider a torsion-free sheaf F on a compact smooth algebraic surface or a Kähler surface S. Such a sheaf determines the standard exact triple

$$0 \longrightarrow F \xrightarrow{\text{can}} F^{**} \xrightarrow{\varphi} C(F) \longrightarrow 0, \tag{2.1}$$

where can is the canonical homomorphism from the sheaf to the double dual sheaf

$$F^{**} = \operatorname{Hom}((F, \mathcal{O}_S), \mathcal{O}_S) \ .$$

F^{**} is a locally free sheaf, i.e., it is the sheaf of germs of sections of a holomorphic bundle E on S, and $C(F)$ is a sheaf supported in the finite set of points

$$\operatorname{supp} C(F) = \sum_{i=1}^{n} x_i(F), \quad x_i(F) \in S, \tag{2.2}$$

where F is not locally free. The bundle $E = F^{**}$ is called the *reflexive span* of the sheaf F.

The symple cycle (2.2) and the reflexive space $E = F^{**}$ are the first invariants of the sheaf F.

The Artinian sheaf $C(F)$ in (2.1) splits into a direct sum of sheaves supported at $x_i(F)$:

$$C(F) = \bigoplus_{i=1}^{n} C(F)_i, \quad \operatorname{supp} C(F)_i = x_i(F), \tag{2.3}$$

and the epimorphism φ in (2.1) also splits:

$$\varphi = (\varphi_1, \cdots, \varphi_n), \tag{2.4}$$

where $\varphi_i : F^{**} = E \longrightarrow C_i(F)$ is a local epimorphism which is nonzero only over the point $x_i(F)$. Each such epimorphism can be completed to an exact triple:

$$0 \longrightarrow F_i \longrightarrow F^{**} = E \xrightarrow{\varphi_i} C(F)_i \longrightarrow 0, \tag{2.5}$$

and it is easy to see that

$$F = \bigcap_{i=1}^{n} F_i \subset E. \tag{2.6}$$

The components of the direct sum (2.3) are called the *local components* of the Artinian sheaf, and the subsheaves F_i are called the *local spans* of F.

The geometrical object we define next is a generalisation of the notion of a zero-dimensional scheme, or a fine cycle ([25], § 2), or a cluster (in Miles Reid's terminology).

Definition 2.1. An *equipped local cluster of rank r supported at the point $x \in C$* is a triple (A_x, V, φ), where A_x is an Artinian sheaf supported at x, $V = \mathbb{C}^r$ is a vector space, and φ is an epimorphism of sheaves

$$V \otimes \mathcal{O}_S \xrightarrow{\varphi} A_x \longrightarrow 0; \tag{2.7}$$

this epimorphism is called the cluster's *equipment.* A *local cluster Ξ_x of rank r* (supported at $x \in S$) is a class of equipped clusters (2.7) considered modulo linear automorphisms of the vector space V and automorphisms of the sheaf A_x. A *cluster of rank r* is a formal sum

$$\Xi = \sum_{i=1}^{n} \Xi_i, \tag{2.8}$$

where Ξ_i is a local cluster supported at x_i, (here $\{x_i\}$ is a set of distinct points on S).

Example 1. It is easy to see that a rank 1 cluster is the same as a family of sheaves

$$\{\mathcal{O}_\xi \otimes L\}, \quad L \in \operatorname{Pic} S, \tag{2.9}$$

where $\mathcal{O}_\xi$ is the structure sheaf of a zero-dimensional subscheme $\xi \subset S$. Such a cluster determines a subsheaf $J_\xi \otimes L$ in each invertible sheaf L, where J_ξ is the sheaf of ideals of the subscheme $\xi \in S$.

Example 2. Any torsion-free sheaf F of rank r determines a rank r cluster $\Xi(F)$.

Namely, for any point $x_i(F)$ in (2.2) there is a neighborhood U_i such that $F^{**} = E|_{U_i} = \mathbb{C}^r \times \mathcal{O}_{U_i}$, and the epimorphism in the exact triple (2.1) determines the local clusters

$$\Xi(F)_i = \Xi(F_i) \tag{2.10}$$

and the rank r cluster (2.8).

The equipment of a local cluster (2.7) factors as a composition of homomorphisms

$$\begin{tikzcd} \mathbb{C}^r \otimes \mathcal{O}_S \arrow[rr, "\varphi"] \arrow[dr, "\bar\varphi \otimes \mathcal{O}_S"'] & & A_x \\ & H^0(A_x) \otimes \mathcal{O}_S \arrow[ur, "\mathrm{can}"'] & \end{tikzcd}, \tag{2.11}$$

where

$$\bar{\varphi} : \mathbb{C}^r \longrightarrow H^0(A_x) \tag{2.12}$$

is a certain homomorphism which may have a kernel or a cokernel.

The rank of the kernel

$$\operatorname{rk}\ker\bar{\varphi} = k \tag{2.13}$$

will be called the *local defect* of the cluster, and the rank

$$d = \operatorname{rk} H^0(A_x) \tag{2.14}$$

will be called the *local degree* of the cluster.

Proposition 2.1. *Let $K^r_d(k, A_x)$ be the manifold of local clusters of rank r, degree d local defect k and Artinian sheaf A_x* (see (2.7)). *Then either $K^r_d(k, A_x) = \varnothing$, or else*

$$K^r_d(k, A_x) = \operatorname{Gr}(r-k, H^0(A_x))_0/\operatorname{Aut} A_x \tag{2.15}$$

is a Zariski open subset of the Grassmannian of $(r-k)$-dimensional subspaces in $H^0(A_x)$ of rank d, considered modulo the action of the group $\operatorname{Aut} A_x$.

Proof. In fact, if no $(r-k)$-dimensional subspace of $H^0(A_x)$ spans A_x, then $K^r_d(k, A_x) = \varnothing$. Otherwise, there is a Zariski open subset of the Grassmannian consisting of the subspaces of sections which span A_x. We then have (2.15).

Now if Ξ is a rank r cluster with local decomposition (2.8), then the vector

$$\bar{d} = (d_1, d_2, \cdots, d_n), \tag{2.16}$$

where d_i is the local degree of the cluster Ξ_i (see (2.14)), is called the *vector degree* of the cluster, and the vector

$$\bar{k} = (k_i, k_2, \cdots, k_n), \tag{2.17}$$

where k_i is the local defect of Ξ_i (see (2.13)), is called the *vector defect* of the cluster.

Let $K^r_{\bar{d}}(\bar{k})$ be the moduli space of clusters of rank r, degree $\bar{d}$, and defect $\bar{k}$, and let

$$K^r_{\bar{d}}(\bar{k}, \sum x_i) \subset K^r_{\bar{d}}(\bar{k}) \tag{2.18}$$

be the subspace of clusters supported in $\sum x_i$.

Proposition 2.2. *The space $K^r_{\bar{d}}(\bar{k}, \sum x_i)$ does not change under a simple variation of the base surface S* (see Definition 1.1).

Proof. What the proposition says is that the space (2.18) for any of the surfaces S_t in a simple variation, where $t \in T$ (see (1.21)), is canonically biholomorphic to the same space for S_0. Note that

$$K^r_{\bar{d}}(\bar{k}, \sum x_i) = K^r_{d_1}(\bar{k}_1, x_1) \times \cdots \times K^r_{d_n}(\bar{k}_n, x_n), \tag{2.19}$$

and so it suffices to prove the proposition for a single point $x \in S_0$. But a simple variation determines an isomorphism of fibres of the structure sheaves

$$(\mathcal{O}_{S_0})_x = (\mathcal{O}_{S_t})_x, \tag{2.20}$$

and hence also an isomorphism of the spaces, of the algebras $\{A_x\}$, which are finitely generated over these rings, and of all of the features of the cluster (see Proposition 2.1). This gives us the desired canonical biholomorphic equivalence.

In the same way as rank 1 cluster — a zero-dimensional subscheme ξ (see Example 1) — determines a subsheaf $J_\xi \otimes L$ in each rank 1 bundle L, a rank r cluster Ξ determines a family of subsheaves in each vector bundle E:

$$\mathcal{F}(E, \Xi) = \{F \subset E | F^{**} = E, \Xi(F) = \Xi\}. \tag{2.21}$$

In particular, in the case of an invertible sheaf L and a rank 1 cluster $\xi \subset S$, the family $\mathcal{F}(L, \xi)$ in (2.21) consists of the single sheaf $J_\xi \otimes L$.

Let $B(E, \Xi)$ be the base of the family of sheaves in (2.21). To describe this base it is useful to decompose Ξ into local clusters (2.8) and represent a sheaf F in (2.21), $[F] \in B(E, \Xi)$, as the intersection of its local spans.

The geometrical interpretation of the features of a cluster, along with Proposition 2.1, implies the following fact.

Proposition 2.3.

1) *If* $[F] \in B(E, \Xi)$, *then its local span* F_i (see (2.6)) *satisfies*

$$[F_i] \in B(E, \Xi_i), \tag{2.22}$$

where the Ξ_i *are the local components* (2.8) *of the cluster* Ξ.

2)

$$B(E, \Xi_x) = \mathrm{Gr}(k, E_x) \tag{2.23}$$

is the grassmannian of k*-dimensional subspaces* (*the kernels of the homomorphism* $\bar{\varphi}$*; see* (2.12)) .

3)

$$B(E, \Xi) = B(E, \Xi_1) \times \cdots \times B(E, \Xi_n) \tag{2.24}$$

and the components of $[F] \in B(E, \Xi)$ *have the form*

$$[F] = ([F_1], \cdots, [F_n]). \tag{2.25}$$

Given a bundle E of rank r, we can define the family of subsheaves

$$\mathcal{F}(E, \bar{d}, \bar{k}) = \{F \subset E | F^{**} = E, \Xi(F) \in K^r_{\bar{d}}(\bar{k})\}. \tag{2.26}$$

where $K^r_{\bar{d}}(\bar{k})$ is the space of clusters of vector degree $\bar{d}$ and vector defect $\bar{k}$ (see (2.16) and (2.17)) with base $B(E, \bar{d}, \bar{k})$, and also the subfamily

$$\mathcal{F}(E, \bar{d}, \bar{k}, \sum x_i) = \{F \subset E | \Xi(F) \in K^r_{\bar{d}}(\bar{k}, \sum x_i)\}. \tag{2.27}$$

(see (2.18)) with base $B(E, \bar{d}, \bar{k}, \sum x_i)$. The bases of these families are stratified as follows:

$$
\begin{array}{ccc}
B(E,\bar d,\bar k) & \xrightarrow{\quad\pi\quad} & K^r_{\bar d}(\bar k) \\
\bigcup & & \bigcup \\
B(E,\bar d,\bar k,\sum x_i) & \xrightarrow{\pi} & K^r_{\bar d}(\bar k,\sum x_i)
\end{array}
\qquad \pi^{-1}(\Xi) = B(E,\Xi), \tag{2.28}
$$

over the spaces in (2.24).

Proposition 2.4. *The space $B(E, \bar{d}, \bar{k}, \sum x_i)$ does not change under a simple variation of the base surface S and the bundle E.*

Proof. Besides the argument used to prove Proposition 2.2, we need only add the observation that the spaces (2.23) are canonically biholomorphic to one another under a simple variation of the bundle E (see Definition 1.1, 2)).

We are now ready to describe the compact smooth components of the moduli spaces of sheaves on S. Let $H = [\omega]$ be the class of the Kähler form which is used to define the notion of stability [19].

Consider a torsion-free stable sheaf F and the exact triple (2.1). Then

$$c_1(F) = c_1(F^{**}), \quad c_2(F) = c_2(F^{**}) + \operatorname{rk} H^0(c(F)). \tag{2.29}$$

The first of these equalities implies the important fact that:

$$F \text{ is stable} \Leftrightarrow F^{**} \text{ is stable.} \tag{2.30}$$

In fact, the definition of H-stability (in the sense of Mumford and Takemoto) involves only the ranks and first Chern classes, and passing to the reflexive spans is functorial: $F_1 \subset F \Rightarrow F_1^{**} \subset F^{**}$.

The homotheties of F determine an embedding $\mathcal{O}_S \longrightarrow \operatorname{End} F = \operatorname{Hom}(F, F)$ which, along with the trace homomorphism $t_r : \operatorname{End} F \longrightarrow \mathcal{O}_S$, gives the decomposition

$$\operatorname{End} F = \mathcal{O}_S \otimes \operatorname{ad} F. \tag{2.31}$$

A flat sheaf $\mathcal{F}$ on the product $S \otimes T$, where $t_0 \in T$ is the germ of the analytic space for which $\mathcal{F}|_{S\times t_0} = F$, is called a *local variation* of F. The stable sheaf F is simple, i.e.,

$$h^0(\operatorname{ad} F) \overset{\text{SD}}{=\!=\!=} h^2(\operatorname{ad} F) = 0, \tag{2.32}$$

where " $\overset{\text{SD}}{=\!=\!=}$ " means "equal by Serre duality". Hence, for F we have

Theorem (Mukai [11]).

1) *The sheaf F has a universal variation with base $[F] \in \operatorname{Spl} F$.*

2) *The germ of the analytic space $[F] \in \operatorname{Spl} F$ is reduced and smooth in $[F]$, and the tangent space to $\operatorname{Spl} F$ at $[F]$ satisfies the relation*

$$T\operatorname{Spl} F_{[F]} = \operatorname{Ext}^1(F, F).$$

By (2.30), the reflexive span F^{**} is a stable bundle, and we have another result:

THEOREM (Artamkin [22], Lemma 6.2). *If F has rank greater than 1, then the generic sheaf of the universal local family* $\operatorname{Spl} F$ *is locally free, i.e., it is the sheaf of germs of sections of the bundle.*

The bases of the universal local families of stable sheaves glue together to form the global components of the moduli space $M_H(r, c_1, c_2)$ of stable sheaves of rank r with Chern classes c_1 and c_2.

Here we shall consider the so-called "fine" components of $M_H(r, c_1, c_2)$.

Definition 2.2. A component M of the moduli space $M_H(r, c_1, c_2)$ is said to be *fine*, if

1) M is a complete smooth manifold, and
2) there exists a family $\mathcal{F}$ of sheaves on $S \times M$ which is flat over M and has the property that the Kodaira–Spencer map

$$TM_{[F]} = \operatorname{Ext}^1(F, F) \tag{2.33}$$

is an isomorphism for every $[E] \in M$.

Combining the criteria of Mukai [11] and Maruyama [12] gives

Proposition 2.5. *If*

$$g.c.d.\left(r, \frac{c_1^2}{2} - c_2\right) = 1, \tag{2.34}$$

then all of the components of $M_H(r, c_1, c_2)$ are fine.

Returning to cluster spaces, we can now construct the family of sheaves

$$\mathcal{F}(r, c_1, c_2, \bar{d}, \bar{k}) = \left\{F \;\middle|\; F^{**} \in M_H(r, c_1, c_2 - \textstyle\sum d_i)\,,\ \Xi(F) \in K^r_{\bar{d}}(\bar{k})\right\} \tag{2.35}$$

with base $B(r, c_1, c_2, \bar{d}, \bar{k})$, from which we have two maps

$$\begin{tikzcd}[column sep=small] & B(r, c_1, c_2, \bar{d}, \bar{k}) \arrow[dl, "R"'] \arrow[dr, "f"] & \\ M_H\left(r, c_1, c_2 - \sum d_i\right) & & K^r_{\bar{d}}(\bar{k}) \end{tikzcd} \qquad \begin{array}{l} R([F]) = [F^{**}] \\ f([F]) = \Xi(F) \end{array} \tag{2.36}$$

such that the direct product

$$B\left(r, c_1, c_2, \bar{d}, \bar{k}\right) \xrightarrow{(R,f)} M_H\left(r, c_1, c_2 - \textstyle\sum d_i\right) \times K^r_{\bar{d}}(\bar{k}) \tag{2.37}$$

has as a fibre the space

$$(R, f)^{-1}(E, \Xi) = B(E, \Xi). \tag{2.38}$$

We proceed to describe the stratification of the moduli space $M_H(r, c_1, c_2)$ of torsion-free stable sheaves of rank r with Chern classes c_1 and c_2.

In the first place, $M_H(r, c_1, c_2)$ contains the closed submanifold

$$C = \{[F] \in M_H(r, c_1, c_2) | F \neq F^{**}\} \tag{2.39}$$

of non-locally-free sheaves and the dense quasiprojective submanifold

$$M^0_H(r, c_1, c_2) = M_H(r, c_1, c_2) - C \tag{2.40}$$

of moduli of stable bundles.

The integer vectors $\bar{d}$ in (2.16) and $\bar{k}$ in (2.17) determine a stratification of the manifold C in (2.39):

$$C_{\bar{d}}(\bar{k}) = \{[F] \in C | \Xi(F) \in K^r_{\bar{d}}(\bar{k})\}. \tag{2.41}$$

Each stratum has an embedding

$$C_{\bar{d}}(\bar{k}) \xrightarrow{g} B(r, c_1, c_2, \bar{d}, \bar{k}) \tag{2.42}$$

into the manifold (2.36).

Proposition 2.6. *Each component of $C_{\bar{d}}(\bar{k})$ embeds into a Zariski dense subset of a component of the manifold $B(r, c_1, c_2, \bar{d}, \bar{k})$*, (2.36).

Proof. If the component B of the manifold (2.36) contains the point $[F]$ corresponding to the stable sheaf F, then there exists a Zariski dense subset B_H in B, whose points correspond to the stable sheaves. For any sheaf $[F] \in B_H$, the universal variation (2.33) contains $[F]$ as a smooth point and determines a component M of the manifold $M_H(r, c_1, c_2)$. Then the intersection $M \bigcap C_{\bar{d}}(\bar{k})$ is a component of the manifold (2.41) which maps isomorphically onto B_0.

Remark. Here we have made essential use of the properties of K3-surfaces.

§ 3 Twistor space of a component of the moduli space of bundles.

Now let M be a fine component of the moduli space of $H(= [\omega])$-stable $(r, 0, c_2)$-sheaves, as in Proposition 2.5, and let

$$M = M^0 \bigcup C, \tag{3.1}$$

where M^0 is the component (2.40) of the moduli space of stable bundles and C is the component (2.39) of the moduli space of non-locally-free sheaves.

Suppose we have an algebraic K3-surface $S = S^0$ with complex structure I, a Ricci-flat metric g that is Hermitian compatible with I (see § 1), and a symplectic form ω. We consider the features of Yang–Mills theory on S (see [25], § 3).

Let E be a $\mathbb{C}^\infty$-complex vector bundle of rank r on S with Chern classes $c_1 = 0$ and $c_2 > 0$, and Hermitian structure h. Let $\mathcal{A}^{\mathbb{C}}$ be the affine space of $\mathrm{SL}(r,\mathbf{C})$-connections on E, let $\mathcal{G}^{\mathbb{C}} = \operatorname{Aut} E$ the group of $\mathrm{SL}(r,\mathbf{C})$-gauge transformations, and let

$$F : \mathcal{A}^{\mathbb{C}} \longrightarrow H^0(\text{ad } E \otimes \Omega^2_{\mathbb{C}}) \tag{3.2}$$

be the curvature map with its Hodge decomposition

$$F = F^{2,0} \oplus F^{1,1} \oplus F^{0,2} \tag{3.3}$$

for the complex structure I on S.

Let $\mathcal{A}''_0$ be the space of integrable semiconnections on E (i.e., d''-operators with $d'' \cdot d'' = 0$), and let

$$\mathcal{A}^{\mathbb{C}} = \mathcal{A}' \otimes \mathcal{A}'' \tag{3.4}$$

be the d'-, d''-decomposition of the space of connections, where$\mathcal{A}_h \subset \mathcal{A}^{\mathbb{C}} = \mathcal{A}' \times \mathcal{A}''$ is the subset of $SU(r)$-connections.

Let $\mathcal{G} = \operatorname{Aut} E_h$ be the subgroup of $SU(r)$-gauge transformations,

$$\mathcal{A}_h^{1,1} = \{a \in \mathcal{A}_h | F^{2,0}(a) = F^{0,2}(a) = 0\} \tag{3.5}$$

the subset of $SU(r)$-connections with curvature of type (1,1), and

$$\begin{tikzcd}
 & \mathcal{A}^{\mathbb{C}} = \mathcal{A}' \otimes \mathcal{A}'' \arrow[dl, "\pi'"'] \arrow[dr, "\pi''"] \arrow[d, phantom, "\cup"] & \\
\mathcal{A}' & \mathcal{A}_h \arrow[l, "x'"'] \arrow[r, "x''"] & \mathcal{A}''
\end{tikzcd} \tag{3.6}$$

the projections onto the direct factors, giving the identifications

$$\mathcal{A}_h = \mathcal{A}'', \quad \overline{\mathcal{A}_h} = \overline{\mathcal{A}''} \times \mathcal{A}', \tag{3.7}$$

and determining a complex structure on $\mathcal{A}_h$.

This complex structure on the tangent space to a connection $a \in \mathcal{A}_h$ has the form

$$(T\mathcal{A}_h)_a = H^0(\text{ad } E \otimes \Omega^{0,1}). \tag{3.8}$$

The quadratic form

$$G(\omega) = 2i \int_S \operatorname{tr}(\omega \wedge \omega^*) \tag{3.9}$$

on the space (3.8) determines a *canonical metric* on $\mathcal{A}_h$. This metric gauge invariant.

Let $\mathcal{A}_h^0$ be the subset of irreducible connections, and let

$$\mathcal{X} = \mathcal{A}_h/\mathcal{G} \tag{3.10}$$

be the orbit space of the gauge group. Then the gauge invariant formula (3.9) gives a metric on the space $\mathcal{X}$ in (3.10).

The metric g on S determines an operator $*$ on the 2-forms on S, and it splits the curvature tensor (3.2) into two components

$$F(a) = F_+(a) + F_-(a), \quad *F_+(a) = F_+(a), \quad *F_-(a) = -F_-(a). \tag{3.11}$$

A connection a is said to be *anti-self-dual* (ASD) if

$$F_+(a) = 0. \tag{3.12}$$

This equation is invariant relative to the gauge group, and

$$\mathcal{M}^g_{ASD} = \{a \bmod \mathcal{G} | F_+(a) = 0\} \subset \mathcal{X} \tag{3.13}$$

is the space of orbits of irreducible ASD-connections in the space $\mathcal{X}$ in (3.10).

Definition 3.1. The restriction to $\mathcal{M}^g_{ASD}$ of the metric G in (3.9) is called the *Weil–Petersson metric* on $\mathcal{M}^g_{ASD}$.

It is easy to see that this is a Kähler metric (the same is true for G on $\mathcal{X}$) in the complex structure determined by the isomorphism (3.8).

We shall call the class of the Kähler form $[\omega] = H$ a *polarisation* of S even if it is not an integral class. In that case the definition of a stable holomorphic bundle E does not change: for any subsheaf $F \subset E$, $\operatorname{rk} F < \operatorname{rk} E$, we have

$$\frac{c_1(F) \cdot H}{\operatorname{rk} F} < \frac{c_2(F) \cdot H}{\operatorname{rk} E} \tag{3.14}$$

(even if the numbers in the numerator are not integers).

Using the local $*$-decomposition of the bundle

$$\Omega^2 = \Lambda_+ \oplus \Lambda_-, \quad *|_{\Lambda_+} = \mathrm{id}, \quad *|_{\Lambda_-} = -\mathrm{id} \tag{3.15}$$

in the case of a Kähler metric it is easy to see that

$$\Lambda_+ = R\omega_g \oplus \Omega^{2,0}. \tag{3.16}$$

Hence, the curvature tensor of an ASD-connection a satisfies the relation

$$F^{2,0}(a) = F^{0,2}(a) = 0. \tag{3.17}$$

i.e., $a \in \mathcal{A}^{1,1}_h$, and the d''-connected component of a determines a holomorphic structure on E.

According to a result of Uhlenbeck and Yau, this construction gives an isomorphism

$$\mathcal{M}^g_{\mathrm{ASD}}(r, 0, c_2) = M^0_H(r, 0, c_2) \tag{3.18}$$

between the space of ASD-connections on a bundle of rank r with $c_1 = 0$ and $c_2 > 0$ on the one hand, and the moduli space of stable bundles on the other.

We can now proceed to the construction of the twistor space (1.8) for the Weil–Petersson metric G on the moduli space (see (3.9)). We return to the fine component M of the moduli space of sheaves (see (3.1))and the dense subset M^0 of vector bundles. We first construct the twistor space $Z(M^0)$ in (1.8) for the subset M^0. Let $\mathcal{M}$ be the component of the moduli of ASD-connections, so that (3.18) gives an isomorphism

$$\mathcal{M} = M^0. \tag{3.19}$$

We consider the simple variation of complex structures (1.19):

$$\begin{array}{ccc} Z(S) & \xrightarrow{\pi} & q \\ \cup & & \cup \\ S = \pi^{-1}(z)_z & \longrightarrow & z, \end{array} \tag{3.20}$$

that is given by the hyper-Kähler structure on S_0. Suppose that the complex structure on S_z is given by means of the automorphism I_z in (1.1), and ω_z is the Kähler form of the Ricci-flat metric g on S_0. Let M_z^0 be the component of the moduli space of stable bundles which are holomorphic on S_z, i.e., holomorphic relative to I_z. Then the family of complex manifolds

$$M_z^0, \quad z \in q, \tag{3.21}$$

sweeps out a complex manifold $Z(\mathcal{M})$ with two projections

$$\begin{tikzcd} & Z(\mathcal{M}) \arrow[dl, "p"'] \arrow[dr, "\pi"] & \pi^{-1}(z) = M_z^0, \\ \mathcal{M} & & q = \mathbb{P}^1, \end{tikzcd} \tag{3.22}$$

where the fibre of the holomorphic projection π is the moduli space M_z^0 of holomorphic stable bundles on the surface S_z in (3.20), and the $\mathbb{C}^\infty$-projection p gives an isomorphism (see (3.19) and (3.18)):

$$p : \pi^{-1}(z) = M_z^0 \longrightarrow \mathcal{M}. \tag{3.23}$$

Thus, the sphere of complex structures M_z, $z \in q = S^2$, on $\mathcal{M}$ gives a simple variation of each fibre of the projection π.

We must now construct the features 2) (1.10), 3) (1.14), and 4) (1.15) of the geometric interpretation of hyper-Kähler structure in the HKLR theorem in § 1.

In the first place, on the union $Z(\mathcal{M})$ of the moduli spaces M_z^0 there exists a real anti-involution σ which covers the anti-involution (1.16), since we obviously have

$$M_{\bar{z}}^0 = \overline{M}_z^0, \tag{3.24}$$

so that condition 4) HKLR theorem is fulfilled.

Next, the isomorphism (1.10) is given on the fibres by the Mukai symplectic form [11] on the moduli space of simple vector bundles on a K3-surface. More precisely, let

$$[E] \in Z(\mathcal{M}), \quad \pi([E]) = Z, \quad [E] \in M_z^0. \tag{3.25}$$

Then E is a holomorphic bundle on S_z. The fibre of the relative tangent bundle to $Z(\mathcal{M})$ at the point $[E]$ is

$$(TZ(\mathcal{M})_{/\pi[E]}) = (TM_z^0)_{[E]} = H^1(S_z, \text{ad } E)$$

(see (3.26)).

Since the canonicall class on the surface S_z is trivial, it follows by Serre duality that there exists a skew-symmetric isomorphism

$$H^1(S_z, \text{ad } E) \longrightarrow H^1(S_z, \text{ad } E)^*, \tag{3.27}$$

which gives the Mukai symplectic structure on M_z^0 (see [11]) and determines the isomorphism (1.10) up to an invertible sheaf lifted from q. Comparing the determinants of the relative tangent bundle and its dual (and taking the skew symmetry into account), we obtain the isomorphism (1.10). This gives part 2) of the HKLR theorem. It remains for us to construct a real section (1.12) with normal sheaf (1.14).

To do this, we consider a self-dual connection $a \in \mathcal{M}$ on the bundle E_h (see the beginning of the section). Then a complex structure $z \in q$ determines a holomorphic bundle structure E_z on (E_h, a), and the family $\{E_z\}$ of holomorphic sections on the family $\{S_z\}$ of Kahler surfaces determines a holomorphic bundle $\widetilde{E}$ on $Z(S)$ such that

$$\widetilde{E}|_{S_z} = E_z, \quad z \in q. \tag{3.28}$$

By analogy with the theory of instantons we shall call $\widetilde{E}$ a *physical anti-instanton* on the twistor space $Z(S)$ in (1.29). It is easy to see that $\widetilde{E}$ on $Z(S)$ determines a simple variation of any bundle E_z on S_z (see Definition 1.1).

On the other hand, the family (3.28) determines a section of the twistor space $Z(\mathcal{M})$ in (3.22) over q:

$$\begin{gathered} q \xrightarrow{s} Z(\mathcal{M}) \\ s(z) = [E_z] \in M^0 z\ , \quad l = s(q)\ , \quad \sigma(l) = l. \end{gathered} \tag{3.29}$$

Now we need only verify that the normal sheaf $N_{l \in Z(\mathcal{M})}$ of a curve l in the twistor space $Z(\mathcal{M})$ has the form (1.14). Comparing the exact triples on l

$$0 \longrightarrow Tl \longrightarrow TZ(\mathcal{M})|_l \longrightarrow N_{l\subset Z(\mathcal{M})} \longrightarrow 0$$
$$0 \longrightarrow TZ(\mathcal{M})_{/\pi}|_l \longrightarrow TZ(\mathcal{M})|_l \xrightarrow{d\,\pi|_l} \pi^* Tl|_l \longrightarrow 0$$

we see that

$$N_{l\subset Z(\mathcal{M})} = TZ(\mathcal{M})_{/\pi}|_l \tag{3.30}$$

and nondegenerate skew-symmetric pairing (3.27) implies the equality

$$TZ(\mathcal{M})_{/\pi}|_l = (TZ(\mathcal{M})_{/\pi})^* \otimes K^*_{Z(S)/q}, \tag{3.31}$$

where $K_{Z(S)/q}$ is the relative tangent bundle.

On the three-dimensional manifold $Z(S)$ we have

$$K_{Z(S)/q} = \pi^*\mathcal{O}_q(-4). \tag{3.32}$$

In fact, in (1.19) the complex "line" $l = p^{-1}(x) \subset Z(S)$ has normal sheaf

$$N_{l\in Z(S)} = \mathcal{O}_l(1) \oplus \mathcal{O}_l(1) \tag{3.33}$$

and hence

$$K_{Z(S)|_l} = \mathcal{O}_l(-4). \tag{3.34}$$

From this we obtain (3.32).

The real line $l \subset Z(\mathcal{M})$ is determines by the real anti-instanton $\widetilde{E}$ in (3.28). Hence,

$$TZ(\mathcal{M})_{/\pi}|_l = R^1\pi \text{ ad } \widetilde{E} \tag{3.35}$$

and, by relative Serre duality,

$$R^1\pi \text{ ad } \widetilde{E} = (R^1\pi \text{ ad } \widetilde{E}^* \otimes K_{Z(S)/\pi})^*, \tag{3.36}$$

from which we obtain (3.31).

The construction of the anti-instanton $\widetilde{E}$ starting with any fibre E_z, $z \in q$, determines a section

$$H^1(S_z, \text{ad } E_z) \longrightarrow H^1(\text{ad } E). \tag{3.37}$$

Because

$$H^0(S_z, \text{ad } E_z) = H^2(S_z, \text{ad } E_z) = 0, \tag{3.38}$$

a standard spectral sequence with second term $H^i(R\pi \text{ ad } \widetilde{E})$ gives the equality

$$H^0(R^1\pi \text{ ad } \widetilde{E}) = H^1(\text{ad } \widetilde{E}) \tag{3.39}$$

and (3.37) is a real section of the canonical map

$$H^0(R^1\pi \text{ ad } \widetilde{E}) \otimes \mathcal{O}_q \xrightarrow{\text{can}} R^1\pi \text{ ad } \widetilde{E}. \tag{3.40}$$

This implies that can is an epimorphism over z, and hence over any point. Thus,

$$R^1\pi \text{ ad } \widetilde{E} = \bigoplus_{i=1}^{N} \mathcal{O}_q(d_i), \quad d \geqslant 0, \tag{3.41}$$

and, by (3.31) and (3.32), we have

$$\bigoplus_{i=1}^{N} \mathcal{O}_q(d_i) = \oplus\, \mathcal{O}_q(2-d_i), \tag{3.42}$$

from which it follows that d can only be 0, 1, or 2.

If a component exists with $d=0$, it determines a σ-real point in $\mathbb{P}(H^1(\text{ad } \widetilde{E}))$; however σ determines quaternionic real structure in $\mathbb{P}(H^1(\text{ad}\, \widetilde{E}))$ having no real points.

Consequently,

$$N_{l\subset Z(\mathcal{M})} = N \otimes \mathcal{O}_l(1) \tag{3.43}$$

and we have constructed all of the geometrical features of a hyper-Kähler structure on $M_z^0 = \mathcal{M}$ which are enumerated in the HKLR theorem in § 1. We have thus proved

Theorem 1. *A hyper-Kähler structure on a compact smooth Kähler surface S induces a hyper-Kähler structure on any fine component M^0* (see (3.1)) *of the stable vector bundles. Here the hyper-Kähler metric on $M^0 = \mathcal{M}$ is a Weil–Petersson metric* (see Definition 3.1).

In the next section we compactify the twistor space $Z(\mathcal{M})$ by means of real lines, obtaining a smooth compact complex manifold $Z(\mathcal{M})$, and we construct a hyper-Kähler structure on a fine component of the moduli space M of torsion-free stable sheaves.

§ 4 The twistor space of a fine component of the moduli space of stable sheaves.

We return to the union (3.1) and the equality (3.18) at the beginning of the last section:

$$\begin{array}{c} M = C \cup M^0 \\ \qquad\quad \| \\ \qquad\quad \mathcal{M}. \end{array} \tag{4.1}$$

We would like to adjoin to the twistor space $Z(\mathcal{M})$ the family of "lines"

$$\{l_{[F]}\}, \quad [F] \in C, \tag{4.2}$$

which can be depicted by the diagram

$$
(4.3) \qquad p^{-1}([F]) = l, \quad
\begin{tikzcd}
 & Z(C) \arrow[dl, "p"'] \arrow[dr, "\pi"] & \\
C & & q
\end{tikzcd}
\quad \pi^{-1}(z) = C_z,
$$

so that diagram (3.22) is completed to a diagram

$$
(4.4) \qquad
\begin{tikzcd}
Z(\mathcal{M}) \cup Z(C) \arrow[d, "p"'] & = Z(M) & \\
M^0 \cup C & & q
\end{tikzcd}
\quad \pi^{-1}(z) = M_z,
$$

where M_z, $z \in q$, is a fine component of the moduli space of torsion-free stable sheaves on the Kähler surface S_z, and $Z(M)$ is a smooth compact complex manifold.

Remark. We shall identify the original algebraic surface S with the fibre of the morphism π of the twistor bundle (1.19) over a point $z_0 \in q$, and we shall omit the indices: $M = M_z$, $C = C_z$, $M^0 = M_z^0$, and so on.

To construct $Z(M)$ it suffices to complete the family of physical anti-instantons of the form (3.28) using torsion-free sheaves $\widetilde{F}$ on the twistor space $Z(S)$, each of which gives a family of torsion-free sheaves:

$$
(4.5) \qquad \widetilde{F}|_{S_z} = F_z, \quad [F_z] \in C_z, \quad z \in q,
$$

on the family of surfaces S_z, $z \in q$, (1.19).

The sheaf $\widetilde{F}$ on $Z(S)$ determines a σ-real line if and only if

$$
(4.6) \qquad \mathrm{Ext}^1_{\mathcal{O}_q}(\widetilde{F}, \widetilde{F}) = \mathbb{C}^N \otimes \mathcal{O}_q(1)
$$

(see [11] and (1.1.23) in [24]), $[\widetilde{F}|_s] \in C \subset M$, and for the standard exact triple on $Z(S)$

$$
(4.7) \qquad 0 \longrightarrow \widetilde{F} \longrightarrow F^{**} \longrightarrow C(\widetilde{F}) \longrightarrow 0,
$$

where F^{**} is the reflexive span of $\widetilde{F}$, we have the following

Proposition 4.1. *If $\widetilde{F}$ determines a σ-real line which compactifies $Z(\mathcal{M})$, then $F^{**} = \widetilde{E}_\infty$ is a physical anti-instanton on $Z(S)$, and* $\mathrm{Supp}C(\widetilde{F}) = \bigcup_{i=1}^{n} p^{-1}(x_i(F))$, *where $\{x_i(F)\}$ are the points* (2.3).

Proof. On the surface $S = S_0$ the sheaf $F = F_{z_0}$ is the limit of a sequence of bundles E_i, $i \longrightarrow \infty$, $[E_i] \in M^0$ (see (4.1)). Let $\{a_i\}$, $i \longrightarrow \infty$, be a sequence of ASD connections in $\mathcal{M} = M^0$ which converge to the point $[F] \in C$.

By Uhlenbeck's weak compactness theorem, one can choose a subsequence $\{a_j\}$, $j \longrightarrow \infty$, of the sequence $\{a_i\}$ which, after a suitable gauge transformation, converges to an ASD connection

$$
(4.8) \qquad a_\infty \in \mathcal{M}^g_{\mathrm{ASD}}(c_2(F) - h^0(C(F)))
$$

(see (2.29)) uniformly (with bounded curvature norms) over $S - \{x_1(F), ..., x_n(F)\}$, where $\{x_i(F)\}$ is the set of points on S where F is not locally free (see (2.3)).

The ASD-connection (4.9) determines a physical anti-instanton $\widetilde{E}_\infty$ on $Z(S)$ whose restriction to each fibre S_z coincides with F_z^{**}, i.e., $\widetilde{E}_\infty = \widetilde{F}^{**}$ and

$$\operatorname{Supp} C(\widetilde{F}) = \bigcup_{i=1}^{n} p^{-1}(x_i(F)). \tag{4.9}$$

Remark. In the framework of the theory of sheaves on the three-dimensional manifold $Z(S)$ one can show that for a torsion-free sheaf $\widetilde{F}$ the single condition (4.6) is sufficient for the reflexive span to be locally free. To construct a sheaf $\widetilde{F}$ which gives a "real line", it now suffices to describe the local components

$$c(\widetilde{F})_i, \quad \operatorname{Supp} C(\widetilde{F})_i = p^{-1}(x_i(F)), \tag{4.10}$$

in terms of the local components $C(\widetilde{F})_i$ (see (2.3)) and the local epimorphisms

$$\varphi_i : \widetilde{E}_\infty \longrightarrow C(\widetilde{F})_i, \tag{4.11}$$

which give the families of equipped clusters of rank r (see (2.7)).

We divide the procedure for constructing $\widetilde{F}$ into three steps:

1) lowering the rank of the cluster;

2) lowering the local degree; and

3) making the construction in the simplest case, to which we are led by steps 1) and 2).

We first describe the procedure for a single fibre S_0 of the bundle $Z(S) \longrightarrow q$.

1. Lowering the rank of a cluster. Let Ξ_x be a local cluster of the sheaf F on $S = S_0$ with zero defect, i.e., the local equipment φ, (see (2.11)), is an embedding.

We choose a flag $V_1 \subset V_2 \subset \cdots \subset V_r$ in the space V of the cluster's equipment. The image of these spaces determine a filtration $A_1 \subset A_2 \subset \cdots \subset A_x$ of the Artinian sheaf A_x in (2.7) and a sheaf filtration $F_1 \subset F_2 \subset \cdots \subset F_{r-1} \subset F$, where

$$F_i = \ker(\varphi_i : F^{**} \longrightarrow A_x/A_i). \tag{4.12}$$

Then

$$(V_i/V_{i+1}) \otimes \mathcal{O}_s \longrightarrow A_i/A_{i-1} = \mathcal{O}_{s_i} \tag{4.13}$$

is a rank 1 cluster, i.e., a zero-dimensional subscheme $\xi \subset S$ (see Example 1 in § 2), and

$$F_i/F_{i-1} = J_{\xi_i} \tag{4.14}$$

is the sheaf of ideals of the subscheme ξ_i.

Thus, locally in a neighborhood of $x \in S$ the sheaf F can be represented as a chain of extensions of sheaves of ideals.

2. Lowering the local degree. A local cluster Ξ_x of rank r and defect k determines a local cluster Ξ_x^0 of rank $(r-k)$ and defect 0:

$$(V/\ker\bar{\varphi}) \otimes \mathcal{O}_s \longrightarrow A_x \longrightarrow 0 \tag{4.15}$$

(see (2.11)), to which we can apply the rank lowering procedure, thereby reducing everything to the case of rank 1 cluster. In that case the structure sheaf $\mathcal{O}_\xi$ is filtered by subsheaves of nilpotents

$$N_\xi^d \subset N_\xi^{d-1} \subset \cdots \subset \mathcal{O}_\xi, \tag{4.16}$$

where $\mathcal{O}_\xi/N_\xi^i$ is the ith order reduction and

$$J_\xi \subset J_{\xi_1} \subset \cdots \subset J_{\xi_{d-1}} \subset \mathcal{O}_s \tag{4.17}$$

is the corresponding filtration of the sheaves of ideals of the subcycles of a fixed order.

By a *locally simple sheaf* we mean a sheaf which has a local cluster of degree 1.

Given a sheaf F, we consider the exact triple (2.1) and apply the functor $\mathrm{Ext}(F,\ )$ to it:

$$\begin{aligned} 0 &\longrightarrow \mathrm{Ext}^0(F,F) \longrightarrow \mathrm{Ext}^0(F,F^{**}) \longrightarrow \\ &\longrightarrow \mathrm{Ext}^0(F,C(F)) \longrightarrow \mathrm{Ext}^1(F,F) \longrightarrow \mathrm{Ext}^1(F,F^{**}) \longrightarrow \\ &\longrightarrow \mathrm{Ext}^1(F,C(F)) \xrightarrow{\delta} \mathrm{Ext}^2(F,F) \longrightarrow 0\,. \end{aligned} \tag{4.18}$$

We shall later see that in this case we may suppose that δ is an isomorphism. But it is obvious that the first monomorphism in this sequence is also an isomorphism. Hence, we have the exact triple :

$$0 \longrightarrow \mathrm{Ext}^0(F,C(F)) \longrightarrow \mathrm{Ext}^1(F,F) \longrightarrow \mathrm{Ext}^1(F,F^{**}) \longrightarrow 0. \tag{4.19}$$

We compute the last term in this triple by applying the functor $\mathrm{Ext}(\,*\,,F^{**})$ to (2.1):

$$\begin{aligned} 0 &\longrightarrow \mathrm{Ext}^1(F^{**},F^{**}) \longrightarrow \mathrm{Ext}^1(F,F^{**}) \longrightarrow \\ &\longrightarrow \mathrm{Ext}^2(C(F),F^{**}) \longrightarrow \mathrm{Ext}^2(F^{**},F^{**}) \longrightarrow 0. \end{aligned} \tag{4.20}$$

By Serre duality, we may change the end of this sequence so as to obtain the triple

(4.21)
$$0 \longrightarrow \mathrm{Ext}^1(F^{**}, F^{**}) \longrightarrow \mathrm{Ext}^1(F, F^{**}) \longrightarrow \mathrm{Ext}^0(F, C(F))^* \longrightarrow 0.$$

Thus, the space $\mathrm{Ext}^1(F, F)$ in (4.19) is an extension of $\mathrm{Ext}^1(F^{**}, F^{**})$ by means of spaces of the form $\mathrm{Ext}^0(F, C(F))$ and their duals, and these split into a sum

$$\mathrm{Ext}^0(F, C(F)) = \bigoplus_{i=1}^{n} \mathrm{Ext}^0(F, C(F)_i) \tag{4.22}$$

of local components. Furthemore, each local component splits into a sequence of extensions of local components corresponding to simple sheaves.

We now pass from the fibre S to the entire three-dimensional manifold $Z(S)$, and make all of our calculations over the base q.

Suppose that all of the local components of F have the form

$$C(F)_i = \mathcal{O}_{x_i}; \tag{4.23}$$

we consider the local epimorphisms $\varphi_i : \widetilde{E}_\infty \longrightarrow \mathcal{O}_{x_i} \longrightarrow 0$. Then in the neighborhood U_i of each line $l_i = p^{-1}(x_i)$ the sheaf $\widetilde{F} = \ker \bigoplus_{i=1}^{n} \varphi_i$ has the form

$$\widetilde{F}|_{U_i} = (\ker \bar{\varphi}_i) \otimes \mathcal{O}_{U_i} \otimes J_l, \tag{4.24}$$

where J_l is the sheaf of ideals of the real line $l = l_i$.

We now apply the functor $\mathrm{Ext}_{\mathcal{O}_q}(F, \ )$ to the triple

$$0 \longrightarrow \widetilde{F} \longrightarrow \widetilde{E}_\infty \longrightarrow \bigoplus_{i=1}^{n} \mathcal{O}_{l_i} \longrightarrow 0 \tag{4.25}$$

and obtain the exact triple of bundles on q

(4.26)
$$0 \longrightarrow \bigoplus_{i=1}^{n} \mathrm{Ext}^0_{\mathcal{O}_q}(\widetilde{F}, \mathcal{O}_{l_i}) \longrightarrow \mathrm{Ext}^1_{\mathcal{O}_q}(\widetilde{F}, \widetilde{F}) \longrightarrow \mathrm{Ext}^1_{\mathcal{O}_q}(\widetilde{F}, \widetilde{E}_\infty) \longrightarrow 0,$$

whose fibre coincides with the exact triple (4.19).

To compute the last term of this triple, we apply the functor $\mathrm{Ext}_{\mathcal{O}_q}(\ , \widetilde{E}_\infty)$ to (4.25) and obtain

$$\begin{aligned} 0 \longrightarrow \mathrm{Ext}^1_{\mathcal{O}_q}(\widetilde{E}_\infty, \widetilde{E}_\infty) \longrightarrow \mathrm{Ext}^1_{\mathcal{O}_q}(\widetilde{F}, \widetilde{E}_\infty) \longrightarrow \\ \longrightarrow \bigoplus_{i=1}^{n} \mathrm{Ext}^0\mathcal{O}_q(\widetilde{F}, \mathcal{O}_{l_i})^* \otimes K^*_{Z(S)/q} \longrightarrow 0 \end{aligned} \tag{4.27}$$

where $K_{Z(S)/q} = \mathcal{O}_q(-2)$ is the relative canonical class $Z(S) \xrightarrow{\pi} q$ (see (3.48)). (To interpret the last term we use relative Serre duality over q.)

Since $\widetilde{E}_\infty$ is a physical anti-instanton, it follows by Lemma 3.1 that

$$\mathrm{Ext}^1_{\mathcal{O}_q}(\widetilde{E}_\infty, \widetilde{E}_\infty) = R^1\pi \ \mathrm{ad}\ \widetilde{E}_\infty = N_\infty \otimes \mathcal{O}_q(1). \tag{4.28}$$

It remains only for us to compute the local terms in (4.26) and (4.27) for each i. Each such term occurs as a direct summand in the first and last component. Hence, it suffices to compute the local terms in a neighborhood of a single real line l.

Lemma 4.1

$$\operatorname{Ext}^0_{\mathcal{O}_q}(\widetilde{F}, \mathcal{O}_l) = \ker \bar{\varphi} \otimes \mathcal{O}_q \oplus N_{l \subset Z(S)}. \tag{4.29}$$

Proof. Since $\tilde{F}$ locally has the form (4.24), it is obvious that the first component in (4.29) occurs. It remains to compute $\operatorname{Ext}^0_{\mathcal{O}_q}(J_l, \mathcal{O}_l)$. To do this, we apply the functor $\operatorname{Ext}^0_{\mathcal{O}_q}(J_l, \ )$ to the standard triple

$$0 \longrightarrow J_l \longrightarrow \mathcal{O}_{Z(S)} \longrightarrow \mathcal{O}_l \longrightarrow 0, \tag{4.30}$$

obtaining

$$\begin{array}{c} 0 \longrightarrow \operatorname{Ext}^0_{\mathcal{O}_q}(J_l, \mathcal{O}_l) \longrightarrow \operatorname{Ext}^0_{\mathcal{O}_q}(J_l, J_l) \longrightarrow 0 \\ \qquad\qquad\qquad\qquad\qquad \| \\ \qquad\qquad\qquad\qquad\qquad N_{l \subset Z(S)}, \end{array} \tag{4.31}$$

which gives us (4.29).

Corollary 1.

$$\operatorname{Ext}^0_{\mathcal{O}_q}(\widetilde{F}, \mathcal{O}_l)^* \otimes K^*_{Z(S)/q} = \ker \bar{\varphi} \otimes N_{l \subset Z(S)}. \tag{4.32}$$

Corollary 2.

$$\operatorname{Ext}^1_{\mathcal{O}_q}(\widetilde{F}, \widetilde{F}) = \bigoplus_{i=1}^{n} \ker \bar{\varphi}_i \otimes \mathcal{E} \oplus \mathbb{C}^N \otimes \mathcal{O}_q(1), \tag{4.33}$$

where each component of $\mathcal{E}$ is determined by the line l_i and the vector $v \in \ker \bar{\varphi}_i$ and can be represented as an extension

$$0 \longrightarrow \mathcal{O}_q \longrightarrow \mathcal{E} \longrightarrow \mathcal{O}_q(2) \longrightarrow 0. \tag{4.34}$$

In fact, all of the components of the extensions (4.26) and (4.27) have only trivial extensions of one another, except for the components of the form

$$0 \longrightarrow \operatorname{Ext}^0_{\mathcal{O}_q}(\widetilde{F}, \mathcal{O}_l) \longrightarrow \mathcal{E} \longrightarrow \operatorname{Ext}^0_{\mathcal{O}_q}(\widetilde{F}, \mathcal{O}_q)^* \otimes K^*_{Z(S)/q} \longrightarrow 0, \tag{4.35}$$

where the first term lies in the first term of (4.26) and the second term lies in the last term of (4.27).

Lemma 4.2 *The extension* (4.34) *is nontrivial.*

Proof. We return to the fibre-by-fibre interpretation of the bundle (4.33), its first components and Serre duality

$$\operatorname{Ext}^1(F, F) \longrightarrow \operatorname{Ext}^1(F, F)^*. \tag{4.36}$$

Then it is not hard to see that the fibre of the elementary bundle (4.34) over point $z \in q$ can be identified with the tangent space to the surface S_z at the point $x = l \cap S_z$. Hence,

$$\mathcal{E} = TZ(S)/q|_l, \quad l = p^{-1}(x). \tag{4.37}$$

On the other hand, the restriction of the relative tangent bundle satisfies:

$$TZ(S)/q|_l = N_{l \subset Z(S)} = \mathcal{O}_l(1) \oplus \mathcal{O}_l(1), \tag{4.38}$$

i.e., the extension (4.34) is nontrivial.

COROLLARY.

$$\mathrm{Ext}^1_{\mathcal{O}_q}(\widetilde{F}, \widetilde{F}) = \mathbb{C}^N \otimes \mathcal{O}_l(1). \tag{4.39}$$

In the simplest case we have thereby constructed the real line $\widetilde{F}$ which compactifies $Z(\mathcal{M})$.

In the general case, when $[F] \in C_{\bar{d}}(\bar{k})$, (2.41), we apply the procedure for lowering the local degree (see (4.15) – (4.17)). At each step of this procedure only components of the form $\mathcal{O}_q(1)$ can appear in $\mathrm{Ext}^1_{\mathcal{O}_q}(\widetilde{F}, \widetilde{F})$. On the other hand, procedure for lowering the rank of a cluster leads to a representation of the components of $\mathrm{Ext}^1_{\mathcal{O}_q}(\widetilde{F}, \widetilde{F})$ as an extension of components of the same form $\mathcal{O}_q(1)$.

Thus, in the general case we also have the decomposition(4.39). We thereby obtain the compactification (4.4) of the twistor space $Z(\mathcal{M})$, and so have proved our basic result.

THEOREM 2. *The hyper-Kähler structure on a component of the moduli space M^0 of stable vector bundles that is induced by the hyper-Kähler structure on S extends to a hyper-Kähler structure on the fine component M of the moduli space of torsion-free stable sheaves on S.*

§ 5 Concluding remarks.

In the case when the surface S has positive or negative canonical class K_s, there does not exist a geometrical (twistor) characterisation of the special metrics on the multidimensional components M of the moduli space. However, here we shall describe an indirect way to obtain such a characterisation. In the case $K_s < 0$ the method is based on the

CALABI EFFECT. A *Kähler structure (X, ω) induces a hyper-Kähler structure on the cotangent bundle T^*X.*

Calabi first discovered this effect by studying the example of projective space [3].

In the same way as symplectic reduction modulo the action of $U(1)$ on $\mathbb{C}^n$ by homotheties with the standard Hermitian structure on $\mathbb{C}^n$ gives the Fubini–Study Kähler structure on $\mathbb{P}^{n-1}$, the same action of $U(1)$ on $\mathbb{C}^n \times (\mathbb{C}^n)^*$ leads to the Calabi hyper-Kähler structure on $T^*\mathbb{P}^{n-1}$ (see [6]).

Hitchin applied hypersymplectic reduction to observe the Calabi effect in the case when $X = N$ is the moduli space of stable bundles with fixed determinant on a curve, and ω_{P-W} is the Kähler form of the Weil–Petersson metric on N [6].

The Hitchin structure on T^*N is obtained as the hypersymplectic reduction of the standard flat hyper-Kähler structure on the cotangent bundle of the affine space of semiconnections modulo the action of the infinite-dimensional gauge group [6].

This construction has a direct generalisation to the case of a moduli space of sheaves on a surface.

Let M be a fine component of the moduli space of K_s^*-stable sheaves on S, and let $M^0 \subset M$ be a component of the moduli space of bundles. Let ω be the Kähler form of the Calabi–Yau metric on S, and let g_{W-P} be the Weil–Petersson metric on M^0. Then one has

THEOREM 3. *The Weil–Petersson metric g_{W-P} on M^0 induces a hyper-Kähler structure on T^*M^0, and this structure extends to a hyper-Kähler structure on T^*M. The image of the embedding of M into T^*M as the zero section is a totally geodesic submanifold of the hyper-Kähler manifold.*

In the case $K_s > 0$, instead of the cotangent bundle $T^*M \xrightarrow{\pi} M$, one must consider the bundle $HM \xrightarrow{\pi} M$, where HM is the moduli space of Higgs bundles (i.e., pairs $(E, \varphi : E \longrightarrow E \otimes T^*S)$). This leads to analogous constructions and results.

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On the superpositions of mathematical instantons

Progr. in Math. 1983. V. 36. P. 433–450.

To I.R. Shafarevich

Introduction.

A *mathematical c-instanton* is, by definition, a vector bundle F on a projective space $\mathbb{P}^3 = \mathbb{P}(T)$, $T = \mathbb{C}^4$, with the following properties:

$$
\begin{aligned}
&1)\quad \operatorname{rk} F = 2,\\
&2)\quad c_1(F) = 0, \quad c_2(F) = c,\\
&3)\quad h^0(F) = 0,\\
&4)\quad h^1(F(-2)) = 0,
\end{aligned}
\tag{0.1}
$$

(we refer to the papers [3], [4], [1] for the background and for a discussion of related topics). For any c-instanton F, $\dim H^1(\mathbb{P}^3, F(-1)) = c$.

Let $H = \mathbb{C}^n$ be a vector space over $\mathbb{C}$ and $H^* = \operatorname{Hom}_{\mathbb{C}}(H, \mathbb{C})$.

Definition 1. A pair (F, i), where F is a mathematical c-instanton and

$$H^1(F(-1)) \xrightarrow{i} H^*$$

is a monomorphism, is called an H-marked c-instanton. A pair (F, i) is called an *exactly marked c-instanton, if i is an isomorphism.*

Let M_c denote the moduli space of mathematical c-instantons, and let $M_c(H)$ denote the moduli space of H-marked c-instantons.

If we attach to every pair (F, i) its image $i(H^1(F(-1))) \subset H^*$, we have a fibering

$$\varphi_c : M_c(H) \longrightarrow G(c, n), \tag{0.2}$$

with the Grassmann variety of c-subspaces of H^* as a base and the moduli space $M_c(u)$ as the fiber over a point $u \in H^*$.

The group $\mathrm{GL}(n, \mathbb{C}) = \operatorname{Aut} H$ acts on $M_n(H)$ and defines a principal $\mathrm{GL}(n, \mathbb{C})$-bundle:

$$\pi : M_n(H) \longrightarrow M_n \tag{0.3}$$

over the moduli space of n-instantons. Consequently, these moduli spaces are birationally equivalent to direct products:

$$M_n(H) \overset{\text{bir}}{\sim} M_n \times \mathrm{GL}(n, \mathbb{C})$$
$$M_c(H) \overset{\text{bir}}{\sim} M_c \times \mathrm{GL}(c, n) \times \mathrm{GL}(c, n) \times \mathrm{GL}(c, \mathbb{C})$$

and the moduli space $M_c(H)$ has as many components as M_c has. Moreover, the unirationality of M_c is equivalent to the unirationality of $M_c(H)$.

There is a certain way to embed all these spaces $M_c(H)$ into the same vector space $\Lambda^2 T^* \otimes S^2 H^*$ (§1), and any H-marked n-instanton is a superposition of marked 1-instantons (or half-instantons, see Definitions 2 and 3). In this paper we describe the properties of such superpositions.

§1 $M_n(H)$ as Determinantal Locus (Determinantal Variety).

Let $(F, i) \in M_n(H)$ be any exactly marked n-instanton. A vector bundle F is a cohomology bundle of a complex of bundles over $\mathbb{P}^3 = \mathbb{P}(T)$, where $T = \mathbb{C}^4$):

$$(1.1)\quad 0 \longrightarrow H \otimes \mathcal{O}_{\mathbb{P}(T)}(-1) \xrightarrow{a} H^* \otimes \Omega\mathbb{P}(T)(1) \xrightarrow{c} \mathbb{C}^{2n-2} \otimes \mathcal{O}_{\mathbb{P}(T)} \longrightarrow 0,$$

where $\Omega\mathbb{P}(T) = (T\mathbb{P}(T))^*$ is the cotangent bundle of $\mathbb{P}(T)$.

For the dual complex

$$(1.2)\quad 0 \longrightarrow \mathbb{C}^{2n-2} \otimes \mathcal{O}_{\mathbb{P}(T)} \xrightarrow{c^*} H \otimes T\mathbb{P}(T)(-1) \xrightarrow{a^*} H^* \otimes \mathcal{O}_{\mathbb{P}(T)}(1) \longrightarrow 0,$$

the initial part of the corresponding cohomology sequence

$$(1.3)\qquad 0 \longrightarrow \mathbb{C}^{2n-2} \xrightarrow{\Gamma(c^*)} H \otimes T \xrightarrow{\Gamma(a^*)} H^* \otimes T^*$$

is exact, and defines a homomorphism

$$\alpha(F, i) : H \otimes T \xrightarrow{\Gamma(a^*)} H^* \otimes T^*$$

of rank $2n + 2$. This homomorphism can be considered as an element of the space $T^* \otimes T^* \otimes H^* \otimes H^*$. Hence we may identify such homomorphisms with the corresponding elements of the tensor space.

On the space $T^* \otimes T^* \otimes H^* \otimes H^*$ there are three involutions $*$, $*_H$ and $*_T$:

$$*(t^1 \otimes t^2 \otimes h^1 \otimes h^2) = t^2 \otimes t^1 \otimes h^2 \otimes h^1$$
$$*_H(t^1 \otimes t^2 \otimes h^1 \otimes h^2) = t^1 \otimes t^2 \otimes h^2 \otimes h^1$$
$$*_T(t^1 \otimes t^2 \otimes h^1 \otimes h^2) = t^2 \otimes t^1 \otimes h^1 \otimes h^2.$$

These operators commute pairwise, and

$$* = *_H \cdot *_T \tag{1.5}$$

The tensor $\alpha = \alpha(F, i), (F, i) \in M_n(H)$ has additional symmetries:

$$\alpha^* = -\alpha. \tag{1.6}$$

(this follows from Serre duality and the existence of a skew-symmetric form

$$F \xrightarrow{\gamma} F^*,$$

which induces an isomorphism $H^1(F(-1)) = H^1(F^*(-1)))$,

$$\alpha^{*_T} = -\alpha \tag{1.7}$$

(this follows from the representation of F as the cohomology bundles of the complex (1.1)). So we have $\alpha(M_n(H)) \subset S^2H^* \otimes \Lambda^2 T^*$, and $\alpha(F, i)$ is a hypernet of quadrics in H (see [7], § 1, where some different interpretations of this notion are given).

Conversely, a tensor $\alpha \in S^2H^* \otimes \Lambda^2 T^*$ is of the form $\alpha(F, i)$ if and only if

(α_0) $\mathrm{rk}\alpha = 2n + 2$

(α_1) the homomorphism $(t_1 \wedge t_2) \otimes H \xrightarrow{\alpha} H^*$ is an isomorphism for some $t_1, t_2 \in T$

(α_2) $\bigcap\limits_{t \in T} \ker((t_0 \wedge t) \otimes H \xrightarrow{\alpha} H^*) = 0$ for every vector $t_0 \in T$

An arbitrary homomorphism $\alpha : H \otimes T \longrightarrow H^* \otimes T^*$, tensored with $\mathcal{O}_{\mathbb{P}(T)}$, can be included in commutative diagram

$$\begin{tikzcd}
H \otimes T\mathbb{P}(T)(-1) \arrow[r, "a^*"] & H^* \otimes \mathcal{O}_{\mathbb{P}(T)}(1) \\
H \otimes T \otimes \mathcal{O}_{\mathbb{P}(T)} \arrow[u] \arrow[r, "\alpha"] & H^* \otimes T^* \otimes \mathcal{O}_{\mathbb{P}(T)} \arrow[u] \\
H \otimes \mathcal{O}_{\mathbb{P}(T)}(-1) \arrow[u] \arrow[uur, "\alpha'"'] & H^* \otimes \Omega\mathbb{P}(1). \arrow[u]
\end{tikzcd} \tag{1.8}$$

This diagram can be completed by a homomorphism a^* if and only if the homomorphism $\alpha' : H \otimes \mathcal{O}_{\mathbb{P}(T)}(-1) \longrightarrow H^* \otimes \mathcal{O}_{\mathbb{P}(T)}(1)$ is zero. If we consider this homomorphism as a tensor $\alpha' \in S^2T^* \otimes H^* \otimes H^*$, then we have

$$2\alpha' = \alpha + \alpha^{*_T}. \tag{1.9}$$

Therefore, α is $*_T$-skew-symmetric if and only if there exists a homomorphism a^* in (5).

We have, then, a complex

$$V \otimes \mathcal{O}_{\mathbb{P}(T)} \xrightarrow{c^*} H \otimes T\mathbb{P}(T)(-1) \xrightarrow{a^*} H^* \otimes \mathcal{O}_{\mathbb{P}(T)}(1), \tag{1.10}$$

where $V = H^0(\ker a^*)$, and c^* is the natural homomorphism of sections.

If F is the cohomology sheaf of the dual complex, then the property (α_2) is equivalent to F being a bundle, and the property (α_1) is equivalent to the existence of an exact H-marking of F, together with $\operatorname{rk} F = \operatorname{rk}\alpha - 2n$.

Consequently, any tensor $\alpha \in T^* \otimes T^* \otimes H^* \otimes H^*$ with $\alpha^{*_T} = -\alpha$ and the property

$$(\alpha_2') : \left\{ \bigcap_{t \in T} \ker(\alpha : (t_0 \wedge t) \otimes H \longrightarrow H^*) \text{ doesn't depend on } t_0 \right\},$$

defines H-marked bundle F, which is the cohomology sheaf of the complex (1.1). Hence

Proposition 1.1. $\alpha : M_n(H) \longrightarrow S^2H^* \otimes \Lambda^2 T^*$ *is an embedding.*

The variety $\alpha(M_n(H))$ is a linear cone in $\Lambda^2(H \otimes T)^*$, and we can pass to its projectivization $\mathbb{P}\alpha(M_n(H)) \subset \mathbb{P}\Lambda^2(H \otimes T)^*$.

Proposition 1.2. $\mathbb{P}\alpha(M_n(H))$ *contains a Zariski open subset of the complete intersection*

$$M_{2n+2} = \Omega^{2n+2} \cap \mathbb{P}(S^2H^* \otimes \Lambda^2 T^*) \tag{1.11}$$

in the projective space $\mathbb{P}(\Lambda^2(H \otimes T)^*)$*, where*

$$\Omega^{2n+2} = \{\alpha \in \mathbb{P}(\Lambda^2(H \otimes T)^*) \quad | \quad \operatorname{rk}\alpha \leqslant 2n+2\}.$$

To prove this fact, it is sufficient to show that the conditions (α_1) and (α_2) are open and that $5n(n-1)$ of the linear equations $\alpha - \alpha^{*_T} = 0$ are independent on Ω^{2n+2}. This means that

$$\dim \mathbb{P}\alpha(M_N(H)) = \dim \Omega^{2n+2} - \dim\ker(1 - *_H). \tag{1.12}$$

But we have from (0.3) that

$$\dim \mathbb{P}\alpha(M_n(H)) = \dim M_n + \dim \mathrm{PGL}(n, \mathbb{C}) = n^2 + 8n - 4.$$

On the other hand, $\dim \Omega^{2n+2} = 6n^2 + 3n - 4$ ([6], 10.4.3), and we obtain the equation (1.12).

So M_{2n+2} is a classical skew-symmetric determinantal variety of type[1]

$$(W|4n, 4n|_{2n+2}, [11n^2 + n - 1]) .$$

The ideal sheaf of this variety has the resolvent of Lascoux–Józefiak–Pragacz (see [5] and the reference in that paper). However the reducibility of this variety

[1] see [6, ch. X]

presents an obstacle to obtaining information about it from the general theory determinantal varieties.

The projective variety $M_{2n+2} \subset \mathbb{P}(S^2H^* \otimes \Lambda^2T^*)$ (1.11) is a union of its components

$$M_{2n+2} = \overline{M_n(H)} \cup M^0 \cup M^1 \cup M^2, \tag{1.13}$$

where $\overline{M_n(H)}$ is the closure of $\alpha(M_n(H))$, and where M^i is the union of those components on which the condition (αi) is invalid.

The irreducibility of M_{2n+2} was proved for $n \leqslant 4$ by W. Barth ([1]). In the paper [7] the existence of a subvariety of dimension $n^2 - 1 + \frac{n^2+3n+6}{2}$ of M^0 is proved. Therefore M^0 is non-empty for $n \geqslant 12$. I. Artamkin has observed that M^1 is non-empty for $n \geqslant 9$, and so is M^2 for $n \geqslant 12$.

To check this it is sufficient to consider any tensor $\omega \in S^2H^* \otimes \Lambda^2T^*$ of rank 2 and all the superpositions $\alpha + \omega$, where α belongs to the component M^0 of dimension $n^2 - 1 + \frac{n^2+3n+6}{2}$. We obtain a subvariety in M^2, whose general point satisfies the conditions (α_0) and (α_1) and whose dimension is greater than the dimension $n^2 + 8n - 4$ of $\overline{M_n(H)}$.

This shows that we have obtained a component of M^2.

§ 2 The Superpositions.

The filtration

$$\mathbb{P}\Lambda^2(H \otimes T)^* = \Omega^{4n} \supset \cdots \supset \Omega^{2n+2} \supset \Omega^{2n} \supset \cdots \supset \Omega^2 = G(2,4)\ , \tag{2.1}$$

of the projective space $\mathbb{P}\Lambda^2(H \otimes T)^*$, where $\Omega^{2k} = \{\alpha \in \mathbb{P}\Lambda^2(H \otimes T)^* | \mathrm{rk}\alpha \leqslant 2k\}$, induces the filtration

$$\mathbb{P}S^2H^* \otimes \Lambda^2T^* = M_{4n} \supset \cdots \supset M_{2n+2} \supset M_{2n} \supset \cdots \supset M_2\ , \tag{2.2}$$

where $M_{2k} = \Omega^{2k} \cap \mathbb{P}S^2H^* \otimes \Lambda^2T^*$. Any $\alpha \in \Omega^{2k}$ is a superposition of k matrices of rank 2:

$$\alpha = \sum_{i=1}^{k} \omega_i, \quad \omega_i \in \Omega^2 \tag{2.3}$$

(the dimension of the variety of such decompositions is $2k(k-1)$). Geometrically, Ω^{2k} is the union of k-chords of Ω^2. (A k-chord is a linear envelope of k points from Ω^2.)(See [6], 10.4.5).

Definition 2. A decomposition

$$\alpha(F, i) = \sum_{k=1}^{N} \alpha(F_k, i_k) \tag{2.4}$$

is called *a representation of the marked instanton as superposition of marked instantons.*

Definition 3. A hypernet $\omega \in \Omega^2$ is called an *H-marked half-instanton.*

Any marked 1-instanton is a superposition of two marked half-instantons.

Proposition 2.1. 1) *The space of marked half-instantons is a direct product: $M_2 = G \times \mathbb{P}(H^*)$, where $G = G(2, T)$ is the grassmannian of lines in $\mathbb{P}(T)$.*

2) *The embedding $G \times \mathbb{P}(H^*) = M_2 \subset \mathbb{P}S^2H^* \otimes \Lambda^2T^*$ is defined by all sections of the sheaf $p_1^*\mathcal{O}_G(1) \otimes p_2^*\mathcal{O}_{\mathbb{P}(H^*)}(2)$, where p_1 and p_2 are the projections of the direct product on its factors.*

3) *The moduli space of H-marked 1-instantons is a direct product:*

$$M_1(H) = (\mathbb{P}\Lambda^2T^* - G) \times \mathbb{P}(H^*) \subset \mathbb{P}\Lambda^2T^* \otimes S^2H^*,$$

where the projection of $M_1(H)$ on $\mathbb{P}(H^*)$ is the map (0.2).

4) *The closure $\overline{M_1(H)} \subset \mathbb{P}\Lambda^2T^* \otimes \mathbb{P}(H^*)$ lies between two elements of the filtration* (2.2):

$$M_2 \subset \overline{M_1(H)} \subset M_4.$$

5) $\mathcal{O}_{\mathbb{P}S^2H^* \otimes \Lambda^2T^*}(1)|_{\overline{M_1(H)}} = p_1^*\mathcal{O}_{\mathbb{P}\Lambda^2T^*}(1) \otimes p_2^*\mathcal{O}_{\mathbb{P}(H^*)}(2)$.

To prove these facts, it is sufficient to point out that if $\alpha \in \mathrm{Hom}(\Lambda^2T, S^2H^*)$, then $\mathrm{rk}\,\alpha \geqslant 2\mathrm{rk}\ \alpha(t_0 \wedge t_1)$ for every t_1, t_2 from T. Consequently, if $\mathrm{rk}\,\alpha = 2$, then $\mathrm{rk}\,\alpha(t_0 \wedge t_1) = 1$, and the projectivization $\mathbb{P}(\alpha(t_0 \wedge t_1))$ does not depend on $t_0, t_1 \in T$. Hence $\alpha = \kappa \otimes h^2, \quad \kappa \in \Lambda^2T^*, \quad h \in H^*$. But $\mathrm{rk}\,\kappa \otimes h^2 = \mathrm{rk}\,\kappa$, and consequently, $\mathrm{rk}\,\kappa = 2$. From this the assertion 1) follows. The other assertions are proved along the same lines.

Corollary. *Any H-marked instanton F is a superposition of marked half-instantons and is a superposition of marked 1-instantons.*

Indeed, $\mathbb{P}\Lambda^2T^* \otimes S^2H^*$ is the linear envelope of M_2 and $M_1(H)$.

Definition 4. If $\kappa \in \Lambda^2T^*, \quad \kappa \wedge \kappa = 0, \quad h^i \in H^*$, a hypernet $\gamma = \kappa \otimes h^1 \cdot h^2$ is called a *marked quasi-instanton.*

Any marked quasi-instanton $\gamma = \kappa \otimes h^1 \cdot h^2$ is a superposition of two half--instantons

$$2\kappa \otimes h^1 \cdot h^2 = \kappa \otimes h_+^2 - \kappa \otimes h_-^2, \quad h_\pm = h^1 \pm h^2. \tag{2.5}$$

Definition 5. Any mathematical instanton F defines three numbers

$$(h(F),\, d(F),\, q(F))$$

as follows:

$h(F)$ is the minimal number of terms in a decomposition of the exactly marked instanton (F, i) into half-instantons. The numbers $d(F)$ and $q(F)$ have the corresponding meaning for decomposition into 1-instantons and quasi-instantons.

Remark. These numbers do not depend on the choice exact markings.

The functions $h(F), d(F)$ and $q(F)$ are semi-continuous on M_n.

Evidently $2q(F) \geqslant h(F) \leqslant 2d(F)$, $h(F) \geqslant n+1$ and $d(F) \geqslant n+1$.

The behavior of these functions is rather complicated.

Proposition 2.2. $h(F) = c_2(F)+1$ if and only if F is a t'Hooft-instanton, that is, $H^0(F(-1)) \neq 0$.

Indeed, if (F,i) is an exactly marked n-instanton and

$$\alpha(F,i) = \sum_{i=1}^{n+1} \kappa_i \otimes h_i^2, \quad \kappa_i \in \Lambda^2 T^*, \quad \kappa_i \wedge \kappa_i = 0, \quad h_i \in H^*, \tag{2.6}$$

then every κ_i defines a projective line $L_i \subset \mathbb{P}(T)$. For any projective plane $\mathbb{P}^2 \supset L_i$ $F|_{\mathbb{P}^2}$ is not a stable bundle, that is, there exists $s \in H^0(F|_{\mathbb{P}^2})$, $s \neq 0$, and $(s)_0 = \bigcup_{i \neq j}(L_j \cap \mathbb{P}^2)$. Hence $F(1)$ has a section vanishing on each L_i, $i = 1, \ldots, n+1$.

Conversely, if $F(1)$ has a section, vanishing on these lines L_i, then, fixing one L_1, we can define by the other lines $L_2, \ldots, L_{n+1}$ a β-commutative hypernet α_0 (see [7]), such that $\alpha(F,i) - \alpha_0 = \kappa_1 \otimes h_1^2$, where $\mathbb{P}\kappa_1 = L_1$ and $\alpha_0 = \sum\limits_{i=2}^{n} \kappa_i \otimes h_i^2$.

Corollary. *If $n \geqslant 3$, then* $\max\limits_{F \in M_n} h(F) \geqslant n+2$.

A trivial calculation shows that for $n \geqslant 13$,

$$\max_{F \in M_n} d(F) \geqslant n+2, \quad \max_{F \in M_n} q(F) \geqslant n+3,$$

and so on.

Any tensor α of the space $T^* \otimes T^* \otimes H^* \otimes H^*$ has the decomposition:

$$\begin{array}{rcl} 4\alpha &=& \alpha_+^+ + \alpha_-^- + \alpha_-^+ + \alpha_+^-, \\ \alpha_+^+ &=& \alpha + \alpha^{*_T} + \alpha^{*_H} + \alpha^* \in S^2T^* \otimes S^2H^*, \\ \alpha_-^- &=& \alpha - \alpha^{*_T} - \alpha^{*_H} + \alpha^* \in \Lambda^2T^* \otimes \Lambda^2H^*, \\ \alpha_-^+ &=& \alpha + \alpha^{*_T} - \alpha^{*_H} - \alpha^* \in S^2T^* \otimes \Lambda^2H^*, \\ \alpha_+^- &=& \alpha - \alpha^{*_T} + \alpha^{*_H} - \alpha^* \in \Lambda^2T^* \otimes S^2H^*, \end{array} \tag{2.7}$$

where $*_T, *_H$ and $*$ are the operators (1.5).

Notice that for any $\alpha \in T^* \otimes T^* \otimes H^* \otimes H^*$, rank α is the rank of the corresponding homomorphism $T \otimes H \longrightarrow T^* \otimes H^*$.

Any tensor ξ of rank 1 is defined by two homomorphisms

$$\begin{gathered} \varphi_i : T \longrightarrow H^*, \quad i = 1,2, \quad \text{and} \\ \xi_{\varphi_1,\varphi_2} = \varphi_1 \otimes \varphi_2 : T \otimes T \longrightarrow H^* \otimes H^*. \end{gathered} \tag{2.8}$$

If $\{t_i\}, i = 0, \ldots, 3$, is a basis of T and $\{t^i\}$ is the dual basis of T^*, then such a tensor has the following components (2.7):

$$
\begin{aligned}
\xi_+^+ &= \sum_{i,j} t^i \cdot t^j \otimes [\varphi_1(t_i)\cdot\varphi_2(t_j) + \varphi_1(t_j)\cdot\varphi_2(t_i)],\\
\xi_-^- &= \sum_{i<j} t^i \wedge t^j \otimes [\varphi_1(t_i)\wedge\varphi_2(t_j) - \varphi_1(t_j)\wedge\varphi_2(t_i)],\\
\xi_-^+ &= \sum_{i,j} t^i \cdot t^j \otimes [\varphi_1(t_i)\wedge\varphi_2(t_j) + \varphi_1(t_j)\wedge\varphi_2(t_i)],\\
\xi_+^- &= \sum_{i<j} t^i \wedge t^j \otimes [\varphi_1(t_i)\cdot\varphi_2(t_j) - \varphi_1(t_j)\cdot\varphi_2(t_i)].
\end{aligned} \tag{2.9}
$$

The last component ξ_+^- is a superposition of 12 quasi-instantons, or of 24 half-instantons.

Any tensor $\alpha \in T^* \otimes T^* \otimes H^* \otimes H^*$ of rank $2n+2$ is a superposition

$$\alpha = \sum_{k=1}^{2n+2} \xi_{\varphi_1^\kappa,\varphi_2^\kappa}, \quad \operatorname{rk}\xi_{\varphi_1^\kappa,\varphi_2^\kappa} = 1$$

of tensors of rank 1, and its $(\pm)$-component equals

$$\alpha_+^- = \sum_{i<j} t^i \wedge t^j \otimes \left(\sum_{k=1}^{2n+2} \varphi_1^\kappa(t_i)\cdot\varphi_2^\kappa(t_j) - \varphi_1^\kappa(t_j)\cdot\varphi_2^\kappa(t_i)\right). \tag{2.10}$$

If $\alpha \in \Lambda^2 T^* \otimes S^2 H^*$, then $\alpha = \alpha_+^-$ and (2.10) is a decomposition of it into $24(n+1)$ quasi-instantons, or $48(n+1)$ half-instantons.

Thus we obtain trivial estimates of the numbers $h(F)$ and $q(F)$:

$$q(F) \leqslant 24(n+1), \quad h(F) \leqslant 48(n+1).$$

But we have a more exact geometrical result:

The components ξ_-^- and ξ_+^- (2.9) of the general tensor ξ of rank 1 are $*_T$-skew-symmetric and have the property (α_2') of § 1. According to construction (1.8), they define H-marked bundles over $\mathbb{P}(T)$.

Let $\operatorname{ad} T\mathbb{P}(T)$ be the adjoint bundle to the tangent bundle $T\mathbb{P}(T)$ of the projective space $\mathbb{P}(T)$, i.e., the subbundle of $\operatorname{End} T\mathbb{P}(T)$ of endomorphisms with zero trace. We have an isomorphism

$$H^1(\operatorname{ad} T\mathbb{P}(T)(-1)) = T, \tag{2.11}$$

and any monomorphism $\varphi : T \longrightarrow H^*$ defines the H-marking of this bundle. By the well-known exact sequence for the tangent bundle of $\mathbb{P}^n$, we have

Proposition 2.3. *If $\varphi : T \longrightarrow H^*$ is a monomorphism and $\xi_{\varphi,\varphi}$ is the corresponding tensor of rank 1, then*

$$(\xi_{\varphi,\varphi})_-^- = \alpha(\operatorname{ad} T\mathbb{P}(T), \varphi). \tag{2.12}$$

Remark. The $*_H$-skew-symmetry of $\alpha(\operatorname{ad} T\mathbb{P}(T), \varphi)$ is a consequence of the fact that $\operatorname{ad} T\mathbb{P}(T)$ has the orthogonal structure defined by the Killing-form of $\operatorname{ad} T\mathbb{P}(T)$.

From now on we restrict our consideration to $*$-skew-symmetric tensors of $T^* \otimes T^* \otimes H^* \otimes H^*$, that is, to the subspace

$$S^2T^* \otimes \Lambda^2H^* \oplus \Lambda^2T^* \otimes S^2H^*.$$

Any $*$-skew-symmetric tensor ω of rank 2 is defined by a pencil of homomorphisms $T \longrightarrow H^*$, or by a homomorphism $\psi : T \otimes W_0 \longrightarrow H^*$, where $W_0 = \mathbb{C}^2$ is a fixed 2-vector space. Such a tensor has a decomposition

$$\begin{array}{rcl} 2\omega &=& \omega^+ + \omega^-, \\ \omega^+ &=& \omega + \omega^{*_T} = \omega - \omega^{*_H} \in S^2T^* \otimes \Lambda^2H^*, \\ \omega^- &=& \omega - \omega^{*_T} = \omega + \omega^{*_H} \in \Lambda^2T^* \otimes S^2H^*. \end{array} \tag{2.13}$$

If ψ is a monomorphism, then ω_ψ^- has the following geometric interpretation.

For the bundle $\operatorname{ad} T\mathbb{P}(T) \otimes W_0$, we have $H^1(\operatorname{ad} T\mathbb{P}(T) \otimes W_0(-1)) = T \otimes W_0$, and a homomorphism $\psi : T \otimes W_0 \longrightarrow H^*$ defines an H-marking of this bundle.

Proposition 2.4. *If $\psi : T \otimes W_0 \longrightarrow H^*$ is a monomorphism and ω_φ is the corresponding tensor of rank 2, then*

$$\omega_\varphi^- = \alpha(\operatorname{ad} T\mathbb{P}(T) \otimes W_0, \psi). \tag{2.14}$$

Remark. The $*_H$-symmetry of ω_φ^- is a consequence of the fact that $\operatorname{ad} T\mathbb{P}(T) \otimes W_0$ has a symplectic structure, defined by the tensor product of the Killing-form and $\Lambda^2 W_0$.

Any tensor $\alpha, \alpha^* = -\alpha$, of rank $2n+2$ is a superposition

$$\alpha = \sum_{i=1}^{n+1} \omega_i, \quad \omega_i^* = -\omega_i, \quad \operatorname{rk} \omega_i = 2\ .$$

From this we have

Proposition 2.5. *Any exactly H-marked n-instanton (F, i) is a superposition of $n+1$ marked bundles $\operatorname{ad} T\mathbb{P}(T) \otimes W_0$.*

A general homomorphism $\psi : T \otimes W_0 \longrightarrow H^*$ can be specialized to a homomorphism of rank 1:

$$\psi : T \otimes W_0 \xrightarrow{s} \mathbb{C} \longrightarrow H^*.$$

In that case, $T \xrightarrow{s} W_0^*$ defines the composition

$$T \xrightarrow{s} W_0^* \xrightarrow{\kappa_0} W_0 \xrightarrow{s^*} T^*, \tag{2.16}$$

where κ_0 is a standard skew-symmetric correlation. We get that

$$\kappa_\psi = s \cdot \kappa_0 \cdot s^* : T \longrightarrow T^*, \quad \kappa_\psi \in \Lambda^2 T^*, \quad \kappa_\psi \wedge \kappa_\psi = 0.$$

If $\mathbb{C} \cdot h_\psi = \operatorname{im} \psi \subset H^*$, the components (2.3) of the tensor ω_ψ are

$$\omega_\psi^+ = 0, \quad \omega_\psi^- = \kappa \otimes h_\psi^2.$$

The superposition $\alpha = \sum_{i=1}^{n+1} \omega_{\psi_i}$ for such ψ_i is an H-marked t'Hooft-instanton.

§ 3 The Special Superposition.

The group $\mathrm{GL}(n,\mathbb{C}) = \mathrm{Aut}\, H^*$ acts on the final element Ω^2 of the filtration (2.1). Consider certain orbits of this action:

I. $\omega \in M_2 \subset \Omega^2, \omega = \kappa \otimes h^2$ is a marked half-instanton, and $\dim M_2 = n+3$.

II. $t^i \in T^*$, $h^i \in H^*$, $i = 1, 2$,

$$\begin{array}{rcl} \omega & = & t^1 \otimes t^2 \otimes h^1 \otimes h^2 - t^2 \otimes t^1 \otimes h^2 \otimes h^1 \in \Omega^2, \\ \omega^+ & = & t^1 \cdot t^2 \otimes h^1 \wedge h^2, \\ \omega^- & = & t^1 \wedge t^2 \otimes h^1 \cdot h^2 \text{ is a marked quasi-instanton.} \end{array}$$

Let $\Omega^2_{\mathrm{II}} \subset \Omega^2$ be the subvariety of all such tensors ω. Then

$$\dim \Omega^2_{\mathrm{II}} = 2(n+2).$$

III. Let $\xi_{\varphi_1,\varphi_2}$, $\varphi_i : T \to H^*$, $\ker \varphi_1 = \ker \varphi_2$, $\dim \ker \varphi_i = 2$ be a tensor of rank 1 (see (2.8)), and $\omega = \xi_{\varphi_1,\varphi_2} - \xi^*_{\varphi_1,\varphi_2} = (\xi_{\varphi_1,\varphi_2})^+_- + (\xi_{\varphi_1,\varphi_2})^-_+$ (see (2.7)). Then $T/\ker \varphi_i = W_0$, $(\varphi_1, \varphi_2) : W_0 \otimes I_0 \longrightarrow H^*$, where $I_0 = \mathbb{C}^2$, and $\omega^+ : S^2 T \xrightarrow{S^2(j)} S^2 W_0 \longrightarrow \Lambda^2 H^*$, $\omega^- = \kappa \otimes q$, where $\ker \kappa = \ker \varphi_i$, and $q = \mathbb{P}(W_0) \times \mathbb{P}(I_0)$ is a quadric of rank 4 in $\mathbb{P}(H)$.

Let $\Omega^2_{\mathrm{III}} \subset \Omega^2$ be the subvariety of all such tensors ω. Then

$$\dim \Omega^2_{\mathrm{III}} = 4n.$$

IV. $\omega_\psi, \psi : T \otimes W_0 \longrightarrow H^*$ (see (2.13)), $\dim \psi(T \otimes W_0) = 4$, and for some $\omega_0 \in W_0$ the homomorphism $\psi_{\omega_0} = \psi|_{T \times \omega_0} : T \longrightarrow H^*$ is a monomorphism.

Let $\Omega^2_{\mathrm{IV}} \subset \Omega^2$ be the subvariety of all such tensors. It has the following description. A homomorphism ψ can be interpreted as a pencil of homomorphisms:

$$0 \longrightarrow T \otimes \mathcal{O}_{\mathbb{P}(W_0)}(-1) \xrightarrow{\overline{\psi}} H^* \otimes \mathcal{O}_{\mathbb{P}(W_0)}. \tag{3.1}$$

In the general case, among the homomorphisms $\overline{\psi}_p, p \in \mathbb{P}(W_0)$ there are four degenerate ones whose kernels define four points $\{p_i\}$ in $\mathbb{P}(T)$, $i = 0, \ldots, 3$. Any non-degenerate homomorphism $\overline{\psi}_p$ of the pencil (3.1) maps these points on the four points $\{q_i\} \in \mathbb{P}(H^*)$. In view of this, we have a map:

$$f : \Omega^2_{\mathrm{IV}} \longrightarrow S^4(\mathbb{P}(T) \times \mathbb{P}(H)^*), \tag{3.2}$$

by which the points (p_i, q_i), $i = 0, \ldots, 3$, are attached to a sheaf (3.1).

To describe the fiber of this map it is sufficient to restrict oneself to the case $H^* = \operatorname{im}\psi$. The points $\{p_i\}$ in $\mathbb{P}(T)$ and $\{q_i\}$ in $\mathbb{P}(H^*)$ define the decompositions of the corresponding spaces into a direct sum of 1-spaces:

$$T = \bigoplus_{i=0}^{3} L_i, \quad H^* = \bigoplus_{i=0}^{3} M_i, \quad \dim L_i = \dim M_i = 1.$$

Using the components of these decompositions, we can construct a new 4-vector space:

$$T_{\{p_i,q_i\}} = \bigoplus_{i=0}^{3} L_i \otimes M_i^*. \tag{3.3}$$

Proposition 3.1. *The points of the fiber of the map* (3.2) *are in one-to-one correspondence with the projective lines $L \subset \mathbb{P}T_{\{p_i,q_i\}}$.*

Indeed, a pencil of homomorphisms (3.1) can be decomposed into a direct sum

$$\begin{gathered} 0 \longrightarrow L_i \otimes \mathcal{O}_{\mathbb{P}(W_0)}(-1) \xrightarrow{\overline{\psi}_i} M_i, \\ \overline{\psi} = \bigoplus_{i=0}^{3} \psi_i. \end{gathered} \tag{3.4}$$

Each 1-pencil is defined by a monomorphism $\psi_i : L_i \to W_0^* \otimes M_i$, which we can interpret as monomorphism $\overline{\psi}_i : L_i \otimes M_i^* \longrightarrow W_0^*$.

Adding these monomorphisms, we obtain an epimorphism

$$\tilde{\psi} : \bigoplus_{i=0}^{3} L_i \otimes M_i^* \longrightarrow W_0^* \longrightarrow 0. \tag{3.5}$$

This epimorphism defines a skew-symmetric correlation

$$\begin{gathered} T_{\{p_i,q_i\}} \xrightarrow{\tilde{\psi}} W_0^* \xrightarrow{\kappa_0} W_0 \xrightarrow{\tilde{\psi}^*} T^*_{\{p_i,q_i\}}, \\ \kappa = \tilde{\psi} \cdot \kappa_0 \cdot \tilde{\psi}^*, \end{gathered} \tag{3.6}$$

where κ_0 is a standard skew-symmetric correlation of W_0 (see (2.16)).

Corollary.. *The fiber of the map f (3.2) at $\{p_i, q_i\}$ is given by*

$$f^{-1}(\{p_i, q_i\}) = G \subset \mathbb{P}\Lambda^2 T^*_{\{p_i,q_i\}},$$

where G is the grassmannian variety of lines in $\mathbb{P}T^*_{\{p_i,q_i\}}$.

Remark. A choice of basis $\{h^i\}$ in H^* such that $\mathbb{P}(h^i) = q_i$ defines isomorphisms $M_i \cong \mathbb{C}$ and, by this, an isomorphism $T \cong T_{\{p_i,q_i\}}$. Analogously, a choice of a basis $\{t_i\}$ in T, such that $\mathbb{P}(t_i) = p_i$, defines an isomorphism $T^*_{\{p_i,q_i\}} \cong H^*$. Consequently, if the four points $q_i = q$ coincide, then the isomorphisms $M_i \cong \mathbb{C}$ define an isomorphism $\mathbb{P}(T) \cong \mathbb{P}(T_{\{p_i,q_i\}})$ and a line $L \subset \mathbb{P}(T)$, which is the line corresponding to a half-instanton (2.16).

The components of the correlation (3.6)

$$\begin{array}{ccc} T_{\{p_i,q_i\}} & & T^*_{\{p_i,q_i\}} \\ \| & \xrightarrow{\kappa} & \| \\ \bigoplus\limits_{i=0}^{3} L_i \otimes M_i^* & & \bigoplus\limits_{i=0}^{3} L_i^* \otimes M_i, \end{array}$$

$$\kappa_{ij} : L_i \otimes M_i^* \longrightarrow L_j^* \otimes M_j \tag{3.7}$$

can be interpreted as homomorphisms

$$\tilde{\kappa}_{ij} : L_i \otimes L_j \longrightarrow M_i \otimes M_j. \tag{3.8}$$

They can be collected together to form a homomorphism

$$\begin{array}{ccc} \left(\bigoplus\limits_{i=0}^{3} L_i\right) \otimes \left(\bigoplus\limits_{i=0}^{3} L_i\right) & & \left(\bigoplus\limits_{i=0}^{3} M_i\right) \otimes \left(\bigoplus\limits_{i=0}^{3} M_i\right) \\ \| & \xrightarrow{\tilde{\kappa}} & \| \\ T \otimes T & & H^* \otimes H^*. \end{array} \tag{3.9}$$

We obtain

$$\begin{aligned} \omega^- &= \Lambda^2 T = \bigoplus_{i<j} L_i \wedge L_j \xrightarrow{\tilde{\kappa}^-} \bigoplus_{i\leqslant j} M_i \otimes M_j = S^2 H^*, \\ & \quad L_i \otimes L_j \xrightarrow{\tilde{\kappa}^-_{ij} = \tilde{\kappa}_{ij} - \tilde{\kappa}_{ji}} M_i \otimes M_j, \\ \omega^+ &= S^2 T = \bigoplus_{i\leqslant j} L_i \otimes L_j \xrightarrow{\tilde{\kappa}^+} \bigoplus_{i<j} M_i \wedge M_j = \Lambda^2 H^*, \\ & \quad L_i \otimes L_j \xrightarrow{\tilde{\kappa}^+_{ij} = \tilde{\kappa}_{ij} + \tilde{\kappa}_{ji}} M_i \otimes M_j. \end{aligned} \tag{3.10}$$

Corollary. $\ker \omega^+ = \bigoplus\limits_{i=0}^{3} L_i^2$.

From this follows

Proposition 3.2. *The variety* Ω^2_{IV} *is birationally equivalent to a direct product*:

$$\Omega^2_{\mathrm{IV}} \overset{\mathrm{bir}}{\sim} G \times S^4(\mathbb{P}(T) \times \mathbb{P}(H^*)),$$

and

$$\dim \Omega^2_{\mathrm{IV}} = 4(n+3).$$

If $\omega \in \Omega^2_{IV}$, *then its parameters will be denoted by symbols,* $\{p_i\}_\omega$, $\{q_i\}_\omega$, $T_\omega = T_{\{p_i,q_i\}}$, $L_\omega \subset \mathbb{P}(T_\omega)$.

Let us fix the quadruple of points $\{p_i\}$ in $\mathbb{P}(T)$ and consider the subvariety $\Omega^2_{\{p_i\}} \subset \Omega^2_{\mathrm{IV}}$ defined by

$$\Omega^2_{\{p_i\}} = \left\{ \omega \in \Omega^2_{\mathrm{IV}} \mid \{p_i\}_\omega = \{p_i\} \right\}. \tag{3.11}$$

Then

$$\dim \Omega^2_{\{p_i\}} = 4n.$$

Definition 6. If $D \subset \Omega^2$ is any subvariety of Ω^2, then a formal sum

$$\sum_{i=1}^{n+1} \omega_i, \quad \omega_i \in D, \tag{3.12}$$

defined up to a multiplicative constant, is called a *D-superposition.*

By the symbol $S(D)$ we denote the variety of the all D-superpositions.

From this definition it follows that $S(D)$ is birationally equivalent to a direct product

$$S(D) \overset{\text{bir}}{\sim} \mathbb{P}^n \times S^{n+1}(D), \tag{3.13}$$

where S^{n+1} denote $(n+1)$-th symmetric power of our variety.

Associating the tensor $\alpha = \sum_{i=1}^{n+1} \omega_i \in \mathbb{P}(\Lambda^2(T \otimes H)^*)$ to the formal sum (3.12), we define a map

$$\alpha : S(D) \longrightarrow \mathbb{P}\Lambda^2(T \otimes H)^*, \tag{3.14}$$

and the projections on components α^+_- and α^-_+ (2.7) provide maps:

$$\begin{aligned} S(D) &\xrightarrow{\alpha^+} \mathbb{P}(S^2T^* \otimes \Lambda^2 H^*), \\ S(D) &\xrightarrow{\alpha^-} \mathbb{P}(\Lambda^2 T^* \otimes S^2 H^*). \end{aligned} \tag{3.15}$$

These projections define a subvariety $S^-(D) \subset S(D)$ by

$$S^-(D) = \left\{ \sum_{i=1}^{n+1} \omega_i \in S(D) \,|\, \alpha^+(\sum_{i=1}^{n+1} \omega_i) = 0 \right\}. \tag{3.16}$$

Finally, the image $\alpha^-(S^-(D)) \subset M_{2n+2}$ belongs to the $(2n+2)$-th element of the filtration (2.2). Applying this construction to the orbits I - IV in Ω^2, we obtain the following:

I. $\alpha^-(S^-(M_2)) \subset \overline{M_n(H)}$ is a subvariety of exact marked t'Hooft instantons.

II. Since $\alpha^-(S^-(\Omega^2_{\mathrm{II}})) \supset \alpha^-(S^-(M_2))$,

$$\alpha^-(S^-(\Omega^2_{\mathrm{II}})) \subset \overline{M_n(H)} \tag{3.17}$$

is a subvariety of exact marked instantons which are the superpositions of $(n+1)$ quasi-instantons.

Analogously

$$\begin{aligned} \alpha^-(S^-(\Omega^2_{\mathrm{III}})) &\subset \overline{M_n(H)}, \\ \alpha^-(S^-(\Omega^2_{\mathrm{IV}})) &\subset \overline{M_n(H)}, \\ \alpha^-(S^-(\Omega^2_{\{p_i\}})) &\subset \overline{M_n(H)}. \end{aligned}$$

Proposition 3.3. *The variety $\alpha^-(S^-(\Omega^2_{\{p_i\}}))$ is the component of $\overline{M_n(H)}$ containing t'Hooft instantons. The dimension of the fiber of the map α^- over a point of $\alpha^-(S^-(\Omega^2_{\{p_i\}}))$ is not more than* 4.

Indeed, by (3.13), $\dim S(\Omega^2_{\{p_i\}}) = 4n^2 + 5n$. By the corollary to (3.10), a subvariety $S^-(\Omega^2_{\{p_i\}}) \subset S(\Omega^2_{\{p_i\}})$ (3.16) is defined by no more than $3n(n-1)$ equations. From this we get

$$\dim S^-(\Omega^2_{\{p_i\}}) \geqslant n^2 + 8n.$$

Careful checking of the second assertion of our proposition concludes the proof.

For small value of n the conditions

$$\sum_{i=1}^{n+1} \omega_i^+ = 0, \quad \omega_i \in \Omega^2_{\{p_i\}}$$

have a simple geometrical meaning. In this situation the direct geometrical constructions provide

Proposition 3.4. The variety $S^-(\Omega^2_{\{p_i\}})$ is unirational for $n < 6$.

This proves the unirationality of M_4 and also of the component of M_5, containing the t'Hooft instantons.

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Delzant models of moduli spaces

For every genus g, we construct a smooth, complete, rational polarized algebraic variety DM_g together with an effective normal crossing divisor $D = \cup D_i$ such that for every moduli space $M_\Sigma(2,0)$ of semistable topologically trivial vector bundles of rank 2 on an algebraic curve Σ of genus g there is a holomorphic isomorphism $f\colon M_\Sigma(2,0) \setminus K_g \longrightarrow DM_g \setminus D$, where K_g is the Kummer variety of the Jacobian of Σ, sending the polarization of DM_g to the theta divisor of the moduli space. This isomorphism induces isomorphisms of the spaces $H^0(M_\Sigma(2,0), \Theta^k)$ and $H^0(DM_g, H^k)$.

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Introduction.

At the last meeting of GAEL,[2] Oxbury asked for a "topological" identification of the moduli space $M_\Sigma(2,0)$ with complex projective 3-space $\mathbb{CP}^3$ for any curve Σ of genus 2. To understand the problem, we recall that, as a real manifold, this moduli space is the space $\mathfrak{RC}(\pi_1(\Sigma))$ of representations classes of the fundamental group $\pi_1(\Sigma)$ in SU(2). The problem is to recognize $\mathbb{CP}^3$ in terms of this space.

By standard arguments of algebraic geometry, every complex structure on a compact Riemann surface Σ of genus 2 induces a complex structure on $\mathfrak{RC}(\pi_1(\Sigma))$ regarded as the moduli space $M_\Sigma(2,0)$ of semistable rank 2 holomorphic vector bundles with trivial determinant. The space $\mathfrak{RC}(\pi_1(\Sigma))$ with this complex structure is precisely $\mathbb{CP}^3$. But we want to identify $\mathfrak{RC}(\pi_1(\Sigma))$ directly with projective 3-space.

In particular, we claim that

1) as an algebraic variety, the moduli space $M_\Sigma(2,0)$ is independent of the complex structure on Σ;

2) the moduli space $M_\Sigma(2,0)$ is rational, and

3) the spaces of conformal blocks (that is, holomorphic sections of the polarization) is independent of the moduli of the curve.

The space $\mathfrak{RC}(\pi_1(\Sigma))$ of representation classes of $\pi_1(\Sigma)$ in SU(2) admits a canonical symplectic form Ω, which is defined in a purely topological way (see [2]). Hence we can apply symplectic arguments or, more precisely, arguments from the theory of Hamiltonian torus actions, that is, *symplectic toric geometry* (see the monograph [8], which is our main reference for technical details).

We recall the set-up of the theory of toric manifolds. Let (M,ω) be a symplectic manifold of dimension $2n$ with a smooth Hamiltonian action of the n-dimensional torus T^n, that is, there is a map $f\colon T^n \longrightarrow \mathrm{Diff} M$ such that

[2] Géométrie algébrique en liberté (9th edition, 19th–23rd March 2001), the school organised by EAGER, EU project contract no. HPRN-CT-2000-00099.

the actions of the images preserve ω. Then the action-angle coordinates define the moment map

$$\pi\colon M \to \Delta \subset \mathbb{R}^n,$$

whose image Δ is a convex polyhedron in Euclidean n-space. This polyhedron contains complete information on the symplectic geometry of (M, ω), that is, Δ determines the manifold, the symplectic form and the T^n-action (see [1]).

Moreover, if (M, ω) is prequantized (see [8]), and M has a Hodge structure with Kähler form ω, then this Hodge structure can also be reconstructed.

The space $\mathfrak{RC}(\pi_1(\Sigma))$ also admits a well-known Hamiltonian action of T^{3g-3} (see, for example, [7]). The differences in properties seem at first sight to be very slight:

1) for $g > 2$ the representation space $\mathfrak{RC}(\pi_1(\Sigma))$ is singular,the singular locus being the Kummer variety

$$\operatorname{Sing} \mathfrak{RC}(\pi_1(\Sigma)) = K_g = R_g^{U(1)}/\pm \mathrm{id},$$

that is, the space of $U(1)$-representations of $\pi_1(C)$ up to $\pm\,\mathrm{id}$, and

2) our $(3g-3)$-torus action is smooth only over interior points of Δ and is continuous everywhere.

For example, if $g = 2$, then the space $\mathfrak{RC}(\pi_1(\Sigma))$ admits an action of T^3, and the image Δ of the moment map of this action is the tetrahedron

$$0 \le t_i \le 1,\ i = 1, 2, \quad |t_1 - t_2| \le t_3 \le \min(t_1 + t_2, 2 - t_1 - t_2) \tag{1.1}$$

in the standard Euclidean space $\mathbb{R}^3$ with coordinates t_1, t_2, t_3. This is a particular case of the Delzant tetrahedron (see [8]). It uniquely determines the Hodge Delzant variety DM_2, which is just $\mathbb{CP}^3$, and the distinguished divisor is $\cup\,\mathbb{CP}^2_i$ for $i = 0, 1, 2, 3$ (four planes in general position). We call this projective space the *Delzant model* of $\mathfrak{RC}(\pi_1(\Sigma))$.

Using the complex structure on $\mathfrak{RC}(\pi_1(\Sigma))$ given by a complex structure on Σ and applying the equivariant Darboux–Weinstein theorem, we get a holomorphic map

$$DM_2 \setminus \bigcup_{i=0}^{4} \mathbb{CP}^2_i \xrightarrow{f} M_\Sigma(2,0) \setminus K_2. \tag{1.2}$$

Although DM_2 and $M_\Sigma(2,0)$ are isomorphic to $\mathbb{P}^3$ as rational algebraic varieties, this map cannot be extended to a holomorphic identification $\mathbb{CP}^3 = DM_2 = M_\Sigma(2,0)$. Instead, we turn to birational (symplectic) geometry.

The purpose of this paper is to construct a Delzant model for any genus with the properties described in the abstract. The case of genus 2 paves the way for this. Our construction also gives a finite chain of elementary "birational" transformations (flips) thast send the Delzant model DM_g to the rational variety $(\mathbb{CP}^3)^{g-1}$ just as for toric varieties in algebraic geometry (see [5]).

The idea for constructing the Delzant models comes from Donaldson [3], where a close cousin of DM_g was constructed in the odd (smooth) case $M_\Sigma(2,1)$

by imitating a moduli space to explain the appearance of Bernoulli numbers in the Verlinde formulae. In the even (non-smooth) case, our construction is close to that of Jeffrey and Hurtubise [12]. It is almost obvious that DM_g coincides with the variety P^D of Hurtubise and Jeffrey but no direct proof has so far been published.

§ 1 The toric structures on $\mathfrak{RC}(\pi_1(\Sigma))$.

Let Σ be a smooth Riemann surface of genus g with fundamental group $\pi_1(\Sigma)$ and let C be a simple closed curve on Σ. We have the so-called *Goldman function* on $\mathfrak{RC}(\pi_1(\Sigma))$:

$$c_C \colon \mathfrak{RC}(\pi_1(\Sigma)) \longrightarrow [\,0\,,\,1\,] \subset \mathbb{R}.$$

It sends a representative $\rho \in \mathfrak{RC}(\pi_1(\Sigma))$ to the number

$$\frac{1}{\pi} \cdot \cos^{-1}\Big(\frac{1}{2}\mathrm{Tr}(\rho([C]))\Big) \in [\,0\,,\,1\,],$$

where $[C]$ is the homotopy class of C. Goldman [2] proved that c_C is a Hamiltonian function of a $U(1)$-action on $\mathfrak{RC}(\pi_1(\Sigma))$ with respect to the canonical symplectic structure Ω. (An exact formula for this action in simple geometric terms is given in [3]. Moreover, if C_1 and C_2 are two disjoint curves, then

$$\{c_{C_1}, c_{C_2}\} = 0,$$

where the Poisson bracket is again taken with respect to Ω. In particular, if $[C_1] \neq [C_2]$, then we obtain a Hamiltonian action of the 2-dimensional torus $T^2 = U(1) \times U(1)$ on the space $\mathfrak{RC}(\pi_1(\Sigma))$. Adding a third curve, we get an action of the 3-dimensional torus, and so on. Of course, this process terminates; it is well known that any maximal set of disjoint homotopy inequivalent circles consists of $3g-3$ curves. We fix one such set,

$$\{C_1, \ldots, C_{3g-3}\}.$$

The isotopy class of such a set of circles is called a *marking* of the Riemann surface. It is easy to see that the complement of this set is the union

$$\Sigma_g \setminus \{C_1, \ldots, C_{3g-3}\} = \bigsqcup_{i=1}^{2g-2} P_i$$

of $2g-2$ trinions P_i, where each trinion is a 2-sphere with 3 disjoint discs deleted:

$$P_i = S^2 \setminus \big(D_1 \cup D_2 \cup D_3\big). \tag{2.1}$$

Such a representation of a Riemann surface is called a *trinion decomposition.*

Thus, a trinion decomposition of Σ is given by the choice of a maximal set of disjoint, non-contractible, pairwise non-isotopic smooth circles on Σ. Such a set consists of $3g-3$ simple closed circles $C_1, \dots, C_{3g-3} \subset \Sigma_g$, and its complement is the union of $2g-2$ trinions P_j. The type of such a decomposition is given by its *trivalent dual graph* $\Gamma(\{C_i\})$, which associates a vertex with each trinion P_i, and an edge between P_i and P_j with every circle C_l such that $C_l \subset \partial P_i \cap \partial P_j$. Hence the isotopy class of a trinion decomposition is given by a trivalent graph Γ of genus g.

On the other hand, every trivalent graph Γ with vertex set $V(\Gamma)$ and edges set $E(\Gamma)$ determines a handlebody $\widetilde{\Gamma}$, that is, a 3-manifold with boundary $\partial\widetilde{\Gamma} = \Sigma_\Gamma$ (a Riemann surface of genus g with a trinion decomposition). This is done by the "pumping trick" (see [4]): pump up the edges and vertices of Γ to tubes and small 2-spheres respectively. We get a Riemann surface Σ_Γ of genus g with a tube $\tilde{e}$ for every $e \in E(\Gamma)$ and a trinion $\tilde{v}$ for every $v \in V(\Gamma)$. The isotopy classes of meridian circles of the tubes define $3g-3$ disjoint, non-contractible, pairwise non-isotopic circles $\{C_e\}, \quad e \in E(\Gamma)$, and a trinion decomposition of Σ.

Thus every Riemann surface with a marking $\{C_i\}$ is completely determined by a trivalent graph Γ and so can be denoted by Σ_Γ. For such a surface, we have a map of the space of classes of representations to Euclidean space,

$$c_\Gamma \colon \mathfrak{RC}(\pi_1(\Sigma)) \longrightarrow \mathbb{R}^{3g-3}, \tag{2.2}$$

with fixed coordinates $(c_1, \dots, c_{3g-3})$ labelled by the elements of $E(\Gamma)$. This map is given by $c_i = c_{C_i}$. The following assertions hold.

(1) The map c_Γ is a real polarization of a dynamic system with phase space $(\mathfrak{RC}(\pi_1(\Sigma)), k \cdot \Omega)$.

(2) c_i are action coordinates for this Hamiltonian system.

(3) c_Γ is a moment map for the Hamiltonian action (not everywhere smooth)

$$\mathfrak{RC}(\pi_1(\Sigma)) \times T^{3g-3} \longrightarrow \mathfrak{RC}(\pi_1(\Sigma)),$$

which is described in [7].

(4) The image of $\mathfrak{RC}(\pi_1(\Sigma))$ under c_Γ is a convex polyhedron $\Delta_\Gamma \subset [\,0\,,\,1\,]^{3g-3}$.

(5) The symplectic volume of $\mathfrak{RC}(\pi_1(\Sigma))$ is equal to the Euclidean volume of Δ_Γ:

$$\int\limits_{\mathfrak{RC}(\pi_1(\Sigma))} \Omega^{3g-3} = \operatorname{Vol}\Delta_\Gamma = \frac{2 \cdot \zeta(2g-2)}{(2\pi)^{g-1}}.$$

The functions c_i are continuous on the whole of $\mathfrak{RC}(\pi_1(\Sigma))$ and smooth over $(0,1)$. We recall that the Hamiltonian torus action on $\mathfrak{RC}(\pi_1(\Sigma))$ is determined by a closed trivalent graph Γ of genus g. Summarizing, we have the following set of geometric objects:

1) the convex polyhedron $\Delta_\Gamma \subset [\,0\,,\,1\,]^{3g-3}$;

2) the part of the boundary $P_r = \partial\Delta_\Gamma \cap \partial[\,0\,,\,1\,]^{3g-3} \subset \partial\Delta_\Gamma$,

3) the part of the boundary of the convex polyhedron $P_K = c_\Gamma(K_g) \subset \Delta_\Gamma$,

4) the open subset $\Delta_\Gamma^0 = \Delta_\Gamma \setminus (P_r \cup P_K) \subset [\,0\,,\,1\,]^{3g-3}$, and

5) the open toric space

$$c_\Gamma^{-1}(\Delta_\Gamma^0) = \mathfrak{RC}(\pi_1(\Sigma))^0 \subset \mathfrak{RC}(\pi_1(\Sigma)),$$

which is relatively compact with respect to the moment map

$$\mathfrak{RC}(\pi_1(\Sigma))^0 \xrightarrow{c_\Gamma} \Delta_\Gamma^0\ .$$

These geometric objects will be described in the next section.

§ 2 Combinatorial constructions.

Here we use the basic set-up of [6]. We recall that any closed trivalent graph Γ is determined by the set $V(\Gamma)$ of vertices along with the "incidence quadratic form"as follows. Let $\mathbb{Z}^{V(\Gamma)}$ be the free $\mathbb{Z}$-module of all formal linear combinations of vertices with coefficients in $\mathbb{Z}$. (Of course, the set of vertices is a distinguished basis of this module.) Then the incidence matrix q_Γ is the symmetric matrix whose entry α_{v_i,v_j} is equal to the number of edges joining the vertices $v_i, v_j \in V(\Gamma)$.

Of course, the symmetric group on $V(\Gamma)$ acts on such matrices by permuting rows and columns.

We recall (see [6]) that a graph Γ is said to be *hyperbolic* if there are subsets $V_+, V_- \subset V(\Gamma)$ such that the subspaces $\mathbb{Z}^{V_\pm}$ are isotropic with respect to q_Γ. The matrix of a hyperbolic graph takes the block form

$$q_\Gamma = \begin{pmatrix} 0 & 0 & * & * \\ 0 & 0 & * & * \\ * & * & 0 & 0 \\ * & * & 0 & 0 \end{pmatrix}, \tag{3.1}$$

where the blocks

$$\begin{pmatrix} * & * \\ * & * \end{pmatrix} \in \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}^{V_+}, \mathbb{Z}^{V_-}) \tag{3.2}$$

give an identification

$$*\colon V_+ \ \leftrightarrow\ V_- \tag{3.3}$$

The edges set $E(\Gamma)$ of a hyperbolic graph can be written as the disjoint union of triples with a common vertex

$$E(\Gamma) = \bigcup_{v \in V_+} E(\Gamma)_v$$

where $E(\Gamma)_v$ is the set of three edges from the vertex $v \in V_+$.

We now regard the Riemann surface Σ_Γ as the result of pumping the graph $E(\Gamma)$ and consider the subset

$$\Sigma_+ = \bigcup_{v \in V_+} \tilde{v} \subset \Sigma_\Gamma = \bigcup_{v \in V_+ \cup V_- = V(\Gamma)} \tilde{v},$$

which is called a *half Riemann surface* Σ_Γ (see [6]).

All these constructions hold for any trivalent graph, not necessarily connected. In particular, we consider the disjoint union

$$\Theta^{g-1} = \Theta \sqcup \cdots \sqcup \Theta.$$

This trivalent graph of genus g determines the Riemann surface

$$\Sigma_{\Theta^{g-1}} = \Sigma_\Theta \sqcup \cdots \sqcup \Sigma_\Theta$$

which is the disjoint union of $g-1$ copies of a Riemann surface of genus 2 with the standard trinion decomposition corresponding to the graph Θ.

We fix one vertex in the trinion decomposition of each copy of Σ_Θ and denote this set of vertices by $V_+ \subset V(\Theta^{g-1})$ and its complement by V_-. These sets generate submodules that are isotropic with respect to $q_{\Theta^{g-1}}$. Hence the graph Θ^{g-1} is hyperbolic with the natural identification $*$ sending a trinion $\tilde{v}$ with $v \in V_+$ to the second trinion of the component Σ_Θ.

The half Riemann surface Θ^{g-1} is the union

$$\Sigma_+ = \bigcup_{v \in V_+} \tilde{v} \subset \Sigma_{\Theta^{g-1}},$$

which precisely coincides with the half Riemann surface Σ_Γ:

$$\Sigma_\Gamma \supset \Sigma_+ \subset \Sigma_{\Theta^{g-1}}.$$

§ 3 Spaces of classes of representations.

The spaces $\mathfrak{RC}(\pi_1(\Sigma))$ and $(\mathfrak{RC}(\pi_1(\Sigma)))^{g-1}$ are symplectic spaces with toric structures defined by the graphs Γ and Θ^{g-1} (see § 1) and moment maps

$$c_\Gamma \colon \mathfrak{RC}(\pi_1(\Sigma)) \longrightarrow \Delta_\Gamma,$$
$$c_{\Theta^{g-1}} \colon (\mathfrak{RC}(\pi_1(\Sigma)))^{g-1} \longrightarrow \Delta_{\Theta^{g-1}}.$$

Proposition 4.1. *The polyhedron $\Delta_{\Theta^{g-1}}$ is the direct product of $g-1$ copies of the tetrahedron Δ_Θ:*

$$\Delta_{\Theta^{g-1}} = \prod_{v \in V_+} \Delta_\Theta.$$

We can say more. Let $\pi_1(\Sigma_+)$ be the fundamental group of the half of $\Sigma_{\Theta^{g-1}}$ and let $\mathfrak{RC}(\pi_1(\Sigma_+))$ be the space of classes of its SU(2)-representations. This space admits the map

$$c_{\partial\Sigma_+} \colon \mathfrak{RC}(\pi_1(\Sigma_+)) \longrightarrow \Delta_{\Theta^{g-1}}.$$

Proposition 4.2. *The map $c_{\partial\Sigma_+}$ is an isomorphism.*

The proof for $g = 2$ is contained in the proof of Proposition 3.1 of [7]. Thus our assertion holds for every component of the disconnected graph and, therefore, for any genus.

Now embedding $\Sigma_+ \hookrightarrow \Sigma_\Gamma$ induces the restriction map

$$r \colon \mathfrak{RC}(\pi_1(\Sigma)) \longrightarrow \mathfrak{RC}(\pi_1(\Sigma_+))$$

and the map c_Γ is the composite

$$c_\Gamma = r \circ c_{\partial\Sigma_+}$$

because $\partial\Sigma_+$ is precisely the set $\{C_e\}$, $e \in E(\Gamma)$. We have the following proposition.

Proposition 4.3. *The polyhedron Δ_Γ is contained in the image of $c_{\partial\Sigma_+}$. Thus, $\Delta_\Gamma \subset (\Delta_\Theta)^{g-1}$.*

It is now easy to check the following well-known statement (see, for example, [9], Proposition 3.3.5).

Proposition 4.4.

The polyhedron Δ_Γ is obtained by taking

1) *the product of all tetrahedra corresponding to the trinions,*

2) *with linear constraints given by the gluing equalities of trinions.*

Corollary 4.5.

1) *Step* 1) of Proposition 4.4 *for Δ_Γ coincides with the corresponding step for* $(\Delta_\Theta)^2$;

2) *We must replace the equalities in Step* 2) of Proposition 4.4 *by the corresponding inequalities.*

To describe the Delzant model, we must transform Δ_Γ into $(\Delta_\Theta)^{g-1}$ using elementary transformations of polyhedra. The combination of these transformations corresponds to an induction over the genus. We carry this out below.

§ 4 Manipulations with moment polyhedra.

We first onsider a special trivalent graph of genus g, the so-called *multi-theta graph* $g\Theta$ (see Figures 1-3 in [6]). This is a vertical oval O crossed by $g-1$ horizontal strings

$$\{e_{g-1}, e_g, \dots, e_{2g-3}\}. \tag{5.1}$$

This graph is symmetric with respect to the vertical axis a_0, and we denote reflection in this axis by $i_0 \colon g\Theta \longrightarrow g\Theta$. There are $g-1$ vertices

$$v_1, \dots, v_{g-1} \tag{5.2}$$

on the left-hand side of the graph, numbered from top to bottom. We choose a half of the set of vertices by putting

$$V_+ = \{v_1, i_0(v_2), v_3, i_0(v_4), \dots\} \tag{5.3}$$

and define the complementary half by $V(g\Theta)$ and $V_- = i_0(V_+)$. Then $g\Theta$ becomes a hyperbolic graph (3.1) with isotropic subspaces $\mathbb{Z}^\pm$ and identification involution $* = i_0$ (3.3). The shape of this graph distinguishes the set of edges on the left-hand side of O:

$$\big\{e_1, e_2, \dots, e_{g-2} \bigm| e_i = \partial(v_i) \cap \partial(i_0(v_{i+1}))\big\}, \tag{5.4}$$

and we see that these are the only edges that give non-trivial combinatorial flips. Every such edge e_i determines a coordinate t_3^i of $\mathbb{R}^3_i$ and a coordinate t_3^{i+1} of $\mathbb{R}^3_{i+1}$.

We now consider the case $g=3$. Then the set of horizontal strings (5.1) is $\{e_1, e_2\}$ and the set of vertices (5.2) is $\{v_1, v_2\}$. The subset (5.3) is equal to

$$V_+ = \{v_1, i_0(v_2), v_3, i_0(v_4)\}$$

and the set (5.4) coincides with $\{e_1\}$, $e_1 = \partial(v_1) \cap \partial(i_0(v_2))$.

We now describe the constraints 2) of Corollary 4.5. Consider the following involutions of $\mathbb{R}^6 = \mathbb{R}^3_1 \times \mathbb{R}^3_2$:

1) interchanging the 3-spaces: $i_{12}(\mathbb{R}^3_1) = \mathbb{R}^3_2$, and

2) interchanging two coordinates in the 3-spaces $\mathbb{R}^3_1$ and $\mathbb{R}^3_2$: $i_{e_1}(t_3^1) = (t_3^2)$.

We recall that the constraints 1) of Proposition 4.4 have already been taken into account:

$$|t_1^i - t_2^i| \leq t_3^i \leq t_1^i + t_1^i, \quad i = 1, 2.$$

We now have to glue the trinions v_1, v_2 along e_1. We easily obtain the following proposition.

Proposition 5.1. *The constraints* 2) *of Corollary* 4.5 *are equivalent to the conditions*

$$|t_1^i - t_2^i| \leq t_3^j \leq t_1^i + t_1^i, \quad i \neq j. \tag{5.5}$$

This immediately yields the following theorem.

THEOREM 5.2. *The moment polytope is given by*

$$\Delta_{3\Theta} = (\Delta_\Theta)^2 \cap i_{e_1}((\Delta_\Theta)^2). \tag{5.6}$$

Indeed, the involution i_{12} preserves the polyhedron $(\Delta_\Theta)^2$. Hence (5.6) is the geometric interpretation of the inequalities (5.5).

We recall that the tetrahedron Δ_Θ is the convex hull of the set S of 4 points in $\mathbb{R}^3$:

$$\Delta_\Theta = \langle (0,0,0), (0,1,1), (1,0,1), (1,1,0) \rangle$$

Hence $(\Delta_\Theta)^2$ is the convex hull of the 16 points $S_1 \times S_2$ in $\mathbb{R}^6 = \mathbb{R}^3_1 \times \mathbb{R}^3_2$.

Proposition 5.3. *The polytope $\Delta_{3\Theta}$ is the convex hull of the* 8 *points*

$$\{(*, *', 0, *, *', 0)\} \cup \{(*, *', 1, *, *', 1)\}. \tag{5.7}$$

Here $*$ means a free choice from the set $* = \{0, 1\}$ and $*' = \{0, 1\}$. Indeed, the inequalities $t_3^1 \neq t_3^2$ induces all the constraints (5.5).

A beautiful description of this situation comes from real algebraic geometry. Let C be a real algebraic curve of genus $g = 2$ with real theta characteristics. Its Kummer surface is a real quartic K_2 with 16 complex-conjugate double singular points $\{p_1, \ldots, p_{16}\}$, called *nodes*, in $\mathbb{CP}^3$. Near the real convex hull of these points, the affine part of $\mathbb{CP}^3$ is represented as $\mathbb{R}^6 = \mathbb{R}^3 \times i\mathbb{R}^3$. Then the convex hull

$$\langle p_1, \ldots, p_{16} \rangle = \Delta_{\Theta^2} = \Delta_\Theta \times \Delta_\Theta \subset \mathbb{R}^6$$

is the Delzant polyhedron of $(\mathbb{CP}^3)^2$ with the natural torus action. We recall that there are 6 lines through every vertex p_i and 6 vertices on each line, as in the classical Kummer configuration 16_6. In these terms, one can see the 8 required vertices of the convex polyhedron $\Delta_{3\Theta}$. It is easy to make these polyhedra integral.

We now argue by induction on g. The strategy is quite simple and natural. We construct a sequence of polyhedra as a sequence of approximations of the polyhedron $\Delta_{g\Theta}$:

1) the first approximation is $(\Delta_\Theta)^{g-1}$;

2) the second approximation is $(\Delta_\Theta)^{g-3} \times \Delta_{3\Theta}$;

3) the ith approximation is $(\Delta_\Theta)^{g-i} \times \Delta_{i\Theta}$, and;

4) the final ($(g-1)$th) approximation is, of course, $\Delta_{g\Theta}$ itself.

Thus we can use induction on g. Note that we have the following objects at the last step of the induction.

1) The polyhedron $\Delta_\Theta \times \Delta_{(g-2)\Theta} \subset \mathbb{R}^3 \times \mathbb{R}^{3(g-2)}$, which corresponds to the disjoint union $\Theta \cup (g-2)\Theta$.

2) The trinion v_l in the second component $\Delta_{(g-2)\Theta}$. It is distinguished by the previous inductive step as the lowest trinion of $\Delta_{(g-3)\Theta}$. Thus we have the decomposition $\mathbb{R}^{3(g-2)} = \mathbb{R}^3_l \times \mathbb{R}^{3(g-3)}$.

3) Distinguished edges $e \in E(\Theta)$ and $e \in E_{v_l}((g-3)\Theta)$, along which we glue the Riemann surfaces Σ_Θ and $\Sigma_{(g-1)\Theta}$.

4) Distinguished coordinate axes, the t_3-axis in $\mathbb{R}^3$ and the t^l_3-axis in $\mathbb{R}^3_l$, which correspond to the edge e in different spaces with the standard coordinates (t_1, t_2, t_3) in $\mathbb{R}^3$ and (t^l_1, t^l_2, t^l_3) in $\mathbb{R}^3_l$.

The gluing conditions are exactly the same as in (5.5): in the notation just described,

$$|t_1 - t_2| \le t^l_3 \le t_1 + t_2, \quad |t^l_1 - t^l_2| \le t_3 \le t^l_1 + t^l_1.$$

The same argument as in Proposition 5.1 yields the following proposition.

Proposition 5.4. *The polytope $\Delta_{g\Theta}$ is the convex hull of 2^g points*

$$\{(*, *', 0, *, *', 0, *, \dots, *)\} \cup \{(*, *, 1, *, *, 1, *, \dots, *)\} \subset \mathbb{R}^{3(g-1)},$$

where $*$, $*'$ and other symbols have the same meaning as in (5.7).

A slightly different description of the moment polyhedron as a subpolyhedron of Δ^{g-1}_Θ was given by Florentino [10].

We recall (see, for example, [8]) that a convex polyhedron $\Delta \subset \mathbb{R}^n$ is said to be *Delzant* if, for every vertex v, there is an integral $(n \times n)$-matrix A with determinant ± 1 such that the map

$$t \in \mathbb{R}^n \longrightarrow At - v$$

sends a neighborhood of $v \in \Delta$ onto a neighbourhood of 0 in $\mathbb{R}^n$. In other words, a convex polyhedron $\Delta \subset \mathbb{R}^n$ is Delzant if

1) its 1-skeleton (the union of its edges) is an n-valent graph Γ (a topological condition), and

2) the set $E(\Gamma)_v \subset E(\Gamma)$ of edges containing the vertex $v \in V(\Gamma)$ is a rational basis in $\mathbb{R}^n$.

Of course, a direct product of Delzant polyhedra is again Delzant.

Proposition 5.5. *The polyhedron $\Delta_{g\Theta} \subset \mathbb{R}^{3g-3}$ is Delzant.*

To prove this we again use induction on g. It is actually enough to consider the case $g = 3$. Here we have the unit cube $C = [0\,,\,1]^3$ with 8 vertices. To construct the tetrahedron Δ_2, we consider the origin $(0,0,0)$ and take all vertices at a distance $\sqrt{2}$ from it. The convex hull of these 4 vertices is our tetrahedron Δ_2. We carry out the same procedure for $\mathbb{R}^6 = \mathbb{R}^3 \times \mathbb{R}^3$: choose all vertices of the cube at a distance 2 from the origin and take their convex hull, and so on. Finally, for genus g, the polyhedron $\Delta_{g\Theta} \subset \mathbb{R}^{3g-3}$ is the convex hull of vertices of the unit cube in $\mathbb{R}^{3g-3}$ at a distance $\sqrt{2g-2}$ from the origin. This yields all the polyhedra.

§ 5 The Delzant model.

We now have a precise description of the image of the moment map for the Hamiltonian torus action on $\mathfrak{RC}(\pi_1(\Sigma))$. This polytope turns out to be Delzant. Hence, by the main theorem of the Delzant theory, there is a *smooth* algebraic variety DM_g (the Hodge manifold) with a regular Hamiltonian action of T^{3g-3}.

Definition 6.1. The smooth algebraic variety DM_g with the Hodge metric is called the *Delzant model* of the space $\mathfrak{RC}(\pi_1(\Sigma))$ (or $M_C(2,0)$).

The direct construction of this rational variety is described in many references, but [8] is the best. The following list of its characteristic properties is well known.

1) The smooth algebraic variety DM_g has a canonical polarization H.

2) The dimension of $H^0(DM_g, H^k)$ can be computed as the number of $\frac{1}{2k}$-integer points (that is, points with rational coordinates whose denominators are equal to $2k$) in $\Delta_{g\Theta}$ by the Duistermaat–Heckman formula ([8], Ch. 3). This number is the Verlinde number and equal to the dimension of the space of conformal blocks of level k and genus g.

3) The set of points $(\Delta_{g\Theta})_{2k} = \frac{1}{2k}\mathbb{Z}^{3g-3} \cap \Delta_{g\Theta} = BS_k$ coincides with the set of Bohr–Sommerfeld fibers of the fibration $c_{\Delta_{g\Theta}}$ (2.2) of level k.

In terms of symplectic geometry, the picture is given by two fibrations with Lagrangian fibres over the same base:

$$\mathfrak{RC}(\pi_1(\Sigma)) \xrightarrow{c_{\Delta_{g\Theta}}} \Delta_{g\Theta} \xleftarrow{m} DM_g, \tag{6.1}$$

where m is the moment map of the regular torus action on the Delzant manifold.

It is appropriate to compare this configuration with "mirror fibrations". We recall that the typical (hypothetical) set-up of the SYZ-mirror construction

[11] also consists of two dual Lagrangian fibrations over the same base. We can regard both fibrations as families of Lagrangian cycles with degeneracy.

The left-hand family

$$\mathfrak{RC}(\pi_1(\Sigma)) \xrightarrow{c_{\Delta_{g\Theta}}} \Delta_{g\Theta}$$

has fibres of equal dimensions with singular fibres over the $(3g-4)$-skeleton of the base. The fibres of the right-hand family

$$DM_g \xrightarrow{m} \Delta_{g\Theta}$$

are tori of different dimensions. Namely, let $\mathrm{sk}_i(\Delta_{g\Theta})$ be the i-skeleton of $\partial\Delta_{g\Theta}$. Then

$$p \in \mathrm{sk}_i(\Delta_{g\Theta}) \setminus \mathrm{sk}_{i+1}(\Delta_{g\Theta}) \implies m^{-1}(p) = T^i$$

is an i-dimensional torus. Moreover, every i-dimensional face F_i defines a projective subspace $\mathbb{P}^i(F_i) \subset DM_g$ with an i-torus action, which is itself a Delzant space. Thus the Delzant model DM_g contains the configuration of projective subspaces corresponding to the jump of fibre dimensions. This is typical of the behavior of isotropic fibres for a prequantized dynamical system.

§ 6 Conformal blocks.

Thus, for every complex curve Σ we have two compact complex polarized varieties

$$(M_\Sigma(2,0),\, \Theta) \qquad \text{and} \qquad (DM_g,\, H)$$

with equidimensional spaces of sections

$$H^0(M_\Sigma(2,0),\, \Theta^k) \qquad \text{and} \qquad H^0(DM_g,\, H^k)\ .$$

The following construction canonically relates these spaces. We first state a simple geometric fact.

Proposition 7.1. *The polyhedron $\Delta_{g\Theta}$ has a unique internal barycentre c_0 of central symmetry.*

Clearly, both fibrations (6.1) have regular fibres over this centre. Near the regular fibres $c_{g\Theta}^{-1}(c_0)$ and $m^{-1}(c_0)$ we can identify our toric spaces using equivariant Darboux–Weinstein coordinates. In particular, we identify the fibres

$$c_{g\Theta}^{-1}(c_0) = m^{-1}(c_0) = T^{3g-3}. \tag{7.1}$$

The tori of both families are Lagrangian, and so the restrictions $\Theta_{|c_{g\Theta}^{-1}(c_0)}$ and $H_{|m^{-1}(c_0)}$ are trivial line bundles with flat connections which are gauge equivalent. (The equivariant Darboux–Weinstein lemma can be extended to identify the line bundles with unitary connections under the identification (7.1).)

Summarizing, we have the torus T_0^{3g-3} with a trivial line bundle and a flat connection (L_0, a_0) (a supercycle) and Lagrangian embeddings

$$\mathfrak{RC}(\pi_1(\Sigma)) \supset c_{g\Theta}^{-1}(c_0) \hookleftarrow T_0^{3g-3} \hookrightarrow m^{-1}(c_0) \subset DM_g$$

such that the pre-images of Θ and H are equal to (L, a). Then the restriction maps

$$H^0(M_\Sigma(2,0), \Theta^k) \longrightarrow \Gamma^\infty(L_0) \leftarrow H^0(DM_g, H^k)$$

are embeddings and give the identification of the spaces of holomorphic sections.

Hence the neighbourhoods of non-singular points in the space $\mathfrak{RC}(\pi_1(\Sigma))$ with smooth torus action are modelled by the linear torus action on the complex projective space as predicted by the equivariant Darboux–Weinstein lemma. For singular points, we have to find new local models that generalize the case of $\mathbb{CP}^n$.

Acknowledgments.

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Commentaries

The geometry of moduli of vector bundles.

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This paper is the first introduction in Russian mathematical literature into the geometry of moduli spaces of stable vector bundles on algebraic curves. One of the important achievements in this field is the solution by Andrei Nikolaevich of the global Torelli problem for higher rank vector bundles on a curve - see papers [A.N. Tyurin. *Analogue of Torelli theorem for two dimensional vector bundles on an algebraic curve.* Izvestija AN USSR. Ser. mathem. V. 33, No. 5 (1969), 1149-1170], [*Analogues of Torelli theorem for higher dimensional vector bundles on an algebraic curve.* Izvestija AN USSR. Ser. mathem. V. 34, No. 2 (1970), 338-365] (this problem was simultaneously and independently was solved by D. Mumford and P. Newstead in two dimensional case, and by M. Narasimhan and S. Ramanan in higher dimensional case). Thus it is not accidental that the present paper is devoted mainly to the exposition of geometric ideas related to the proof of Torelli theorem for moduli of higher dimensional vector bundles. An original feature of the paper is the elegant exposition of the specific duality between the curve X and the variety $S = S(r, d)$ of moduli of stable bundles of rank r and degree d with fixed determinant over X; this duality means that, in case when r and d are relatively prime, the universal bundle U on $X \times S$ considered on fibers of the projection $X \times S \to X$, i.e. as a bundle on the variety S, has the variety of moduli isomorphic to the original curve X. The question whether there exists a similar duality in the theory of moduli of stable bundles on higher dimensional varieties is extremely interesting and is waiting for further investigation. Among the other merits of this paper one should mention the nice introduction into the theory of elementary transformations of vector bundles on a curve, which are an important technical tool of the theory. The significance of elementary transformations of sheaves for the geometry of vector bundles, as well as for algebraic geometry and its applications in general, is nowadays well known, and was firstly understood just in the papers of M. Narasimhan – S. Ramanan and A.N. Tyurin on the moduli of vector bundles on curves.

A. S. Tikhomirov

* * *

This article is based on a series of lectures given in Shafarevich's seminar in 1973. It is essentially expository, describing the state of knowledge on the geometry of moduli spaces of vector bundles on algebraic curves, with particular reference to then unpublished results of M. S. Narasimhan and S. Ramanan [6] on deformations of the moduli spaces. The lectures and the article were intended for an audience with some knowledge of algebraic geometry, but not expert in the theory of vector bundles. A. Tyurin therefore outlines his own versions of proofs and omits many technical details. However, as might be expected, the article is full of Tyurin's own ideas and insights, including conjectures and speculations, some of which are still not fully worked out.

The article concerns the moduli spaces $S = S_{n,d}$ of stable bundles of rank n and fixed determinant of degree d with $(n, d) = 1$ over a smooth projective algebraic curve X (there are remarks also on the case $(n, d) \neq 1$). Although A. Tyurin makes no explicit assumption about the genus, it is clear that he has in mind the case $g \geq 2$ as treated in [6] and many other papers. Some of the results fail in genus 1, while others are true for trivial reasons.

Chapter I describes the construction of S and some of its basic properties. The only point requiring comment is the final sentence "It is not much harder to prove the rationality of S (Newstead)". Unfortunately this is not true; the cited reference [9] contains an error, partially but not completely corrected in [10]. Many attempts at a general proof were given, in some cases increasing substantially the set of values of n, d for which the result is known to be true, before the rationality of S was finally established by A. King and A. Schofield [4] in the late 1990s. For $(n, d) \neq 1$, the problem remains open.

Chapter II contains the main results of the article. According to the description given in Chapter I, there exists a universal bundle U on $X \times S$, and, for each $x \in X$, we can consider the restriction U_x of U to $\{x\} \times S$ as a bundle on S. (Tyurin calls the bundle U_x a Poincaré bundle; nowadays this term is usually used for U itself.) The bundle U is regarded as a family of bundles on X parametrised by S, but can also be viewed in this way as a family of bundles on S parametrised by X. This idea goes back a long time in the context of line bundles, while for vector bundles it is at least implicit in [5] and explicit in [6]. Tyurin's statement (Theorem 1) is as follows: for any $x \in X$, the curve X is the variety of moduli $S(U_x)$ for the bundle U_x.

This statement can be broken into two parts, firstly that $U_x \ncong U_{x'}$ if $x \neq x'$ (Theorem 2), secondly that all small deformations of U_x have the form U_y for some $y \in X$. The second part is proved in [6] and the proof is outlined here in Chapter V. Theorem 2, on the other hand, is stated but not proved in [6], so Tyurin's proof (given in Chapter IV) would seem to be the only one in the literature.

A. Tyurin sees it as a type of "inversion theorem" comparable to (but definitely different from) a standard theorem for line bundles. The proof itself is not completely convincing and indeed does not seem to work in the case $n = 2$ (the construction yields bundles of trivial determinant rather than determinant of odd degree), so the result must perhaps remain as a conjecture. What is certainly true is that X is an étale covering of $S(U_x)$. If X is a general curve of genus $g \geq 2$, this étale covering must be trivial; thus Theorem 2 (hence also Theorem 1) holds for the general curve of any genus. It would be interesting to construct a totally convincing proof for an arbitrary curve on the lines proposed by A. Tyurin.

Another important result from [NR1] is what A. Tyurin calls the étale theorem: $H^1(X, \Theta X) \cong H^1(S, \Theta S)$ (where ΘX and ΘS are the tangent bundles of X and S). This theorem implies the local Torelli theorem for $X \mapsto S$; in fact the global version of the Torelli theorem, asserting that non-isomorphic X give rise to non-isomorphic S, had already been proved by both Mumford and Newstead [5] and Tyurin [12] in rank 2, and by Tyurin [13] in arbitrary rank. Two further significant corollaries are that the group of biregular automorphisms of S is finite and Seshadri's theorem that $\operatorname{Pic} S \cong \mathbf{Z}$ (see [11, Proposition 3.4]).

The final section of Chapter II contains some problems and conjectures. Problem I concerns the case $(n,d) \neq 1$. In this case S is not complete; let $\overline{S}$ denote the Mumford-Seshadri completion (using semistable bundles) and let $\widetilde{S}$ be a desingularisation of $\overline{S}$. (It should be noted that there are several possible desingularisations and partial desingularisations of $\overline{S}$ which can be used for $\widetilde{S}$.) Moreover U does not exist, but the adjoint bundle $\operatorname{ad} U$ does exist on S (but not on $\overline{S}$ and possibly not on $\widetilde{S}$). Tyurin asks whether the theorems of the previous section hold for $\widetilde{S}$. In fact most of the results have no obvious meaning on $\widetilde{S}$, but the local results make sense on S and in general may be expected to remain true. Moreover one important global result does continue to hold for S and $\overline{S}$, namely the theorem that $\operatorname{Pic} S \cong \mathbf{Z}$. This is true for $(n,d) \neq 1$ except when $n = g = 2$, and $\operatorname{Pic} \overline{S} \cong \mathbf{Z}$ always; this was proved by J.-M. Drezet and Narasimhan [2], who showed further that $\overline{S}$ is factorial.

Returning now to the case $(n,d) = 1$, Problem II concerns the interpretation of the direct images on S and X of iterated adjoints of the Poincaré bundle. Although some of these sheaves have been used and investigated, there seems to have been no geometric interpretation of them.

This problem is followed by a very interesting conjecture, namely that there should be an exact sequence

$$0 \longrightarrow J_n(X) \longrightarrow \operatorname{Aut} S \longrightarrow \operatorname{Aut} X \longrightarrow 0,$$

where $J_n(X)$ is the group of n-torsion points of the Jacobian of X. This has recently been proved by A. Kouvidakis and T. Pantev [3, Theorem B], who have obtained also a version valid for $(n,d) \neq 1$. The proof involves Higgs bundles and the Hitchin map.

Finally Problem III concerns the Hodge numbers $h^{p,q}(S)$. For the case $n = 2$, they were in essence already known. In fact, an additive basis for all the cohomology groups of S was given in [8]; if care is taken over the choice of basis for $H^1(X)$ which is used in constructing this basis, the basis elements all belong to some $H^{p,q}(S)$. The simplest way of stating the formula for the $h^{p,q}$ in this case is in the form of the Poincaré-Hodge polynomial

$$\sum h^{p,q}(S)x^p y^q = \frac{(1+x^2y)^g(1+xy^2)^g - x^g y^g(1+x)^g(1+y)^g}{(1-xy)(1-x^2y^2)}.$$

A version of this for motivic cohomology has been obtained recently by S. del Baño [1]; his paper contains also a formula for $h^{p,q}(\widetilde{S})$ when $n = 2$ and d is even, where $\widetilde{S}$ is the Seshadri desingularisation of $\overline{S}$. For general rank, the standard cohomology generators can again all be taken to be of pure Hodge type and there is no difficulty in principle about calculating $h^{p,q}(S)$. Tyurin's conjecture that $H^{i+q}(X, \Omega^q) \cong H^{i+p}(S, \Omega^p)$ for any p, q is clearly wrong. Almost certainly, there is a typographical error here, but it is not clear what Tyurin intended to conjecture. What is true and is in the spirit of the conjecture is that the "Hodge diamond" of S is relatively thin; for example, when $n = 2$, $h^{p,q}(S) = 0$ whenever $|p - q| > \frac{1}{3}(p + q)$.

Chapters III to V are mainly concerned with the proofs of the results stated in Chapter II; these follow the same lines as those in [6]. In particular, in Chapter III, Tyurin develops the theory of elementary transformations; this is very much in the spirit of the work of Narasimhan and Ramanan on Hecke transformations, which was to be further developed in [7]. These transformations, which are also related to parabolic structures on bundles, have been used in many deformation problems. In this chapter, Tyurin attempts to construct a general bundle from the trivial bundle using elementary transformations; this method is quite sound in principle, but the conjecture in section 2 that the map φ he constructs there is a birational isomorphism is wrong. Indeed φ is not even dominant. This can already be illustrated by the case $g = 2$, $n = 2$, which he discusses immediately after stating the conjecture. If V (or strictly speaking V^*) is obtained from I_2 by elementary transformations, then I_2 is a subsheaf of V. Hence V has two independent sections; on the other hand V_{gen}, as correctly stated by Tyurin, has only one independent section. However the proof of rationality of $S_{2,1}$ is correct and generalises to $S_{n,d}$ whenever $d \equiv -1 \bmod n$. The general question of rationality has already been discussed.

The study of the geometry of moduli spaces of vector bundles, and of many related moduli spaces, remains a very active area of research. Unfortunately, there is no recent survey of this area to which the reader can be referred.

P. Newstead

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On the classification of 2-dimensional vector bundles on an algebraic curve of any genus.

Izv. AN SSSR. Ser. mathem. 1964. V. 28. no. 1

This was one of Andrey Tyurin's first publications, and initiated his long series of works on the theory of vector bundles. In fact this subject is present in different flavors in most of his works. With this paper, Tyurin entered the circle of leading experts in the theory of algebraic vector bundles, and he remained in this position until his untimely death, while the theory developed into a gigantic domain with many links to other areas of mathematics and physics.

In this article Tyurin considers moduli problems in the theory of vector bundles on complex projective curves of genus > 1. A remarkable feature of the objects of algebraic geometry is that their deformations are themselves also parametrised by algebraic varieties – so called moduli varieties. Thus "derived objects" of algebraic geometry remain within the framework of algebraic geometry, and its methods apply in turn to them. This closed character of algebraic geometry dramatically extends the area of application of its methods and ideas. It is because of this that algebraic geometric approaches have proved so effective in several other areas of mathematics and physics.

A paradigm case is the theory of rank one vector bundles on a curve: bundles of the same topological type (that is, with given degree or first Chern class) are parametrised by the points of a commutative algebraic group, the Jacobian of the curve. For vector bundles of higher rank the situation is more complicated, and to obtain a natural algebraic parametrisation we have first to modify the question. ANT showed in this article that if we add some rigidity to an algebraic vector bundle of rank 2 on the curve, the resulting objects is parametrised by an algebraic variety. He proposed to consider pairs consisting of a rank 2 vector bundles with an exceptional subbundle of rank one, and proved that these "pairs" have an algebraic parametrisation. We have to fix a point on the curve to define *an exceptional subbundle of rank one*, which is then simply a subbundle of rank one "of maximal degree with respect to this point". Since for every rank 2 vector bundle there are only a finite number of exceptional subbundles (bounded by $2g$, where g is the genus of the curve), ANT obtains in this way an algebraic parametrisation of all rank 2 vector bundles with "an exceptional subbundle" of a given degree. Note that this approach is somewhat analogous to the introduction of a "level" in the moduli theory of elliptic curves and Abelian varieties. In the latter case introducing some rigidity (fixing certain points of finite order) allows us to construct a moduli variety.

In addition, ANT shows that the moduli variety of rank 2 vector bundles of a given topological type with an exceptional subbundle of maximal degree can be identified with an open subvariety of a projective space. Thus ANT obtained a complete solution to the moduli problem for rank 2 vector bundles using the approach of classical algebraic geometry.

Tyurin's paper slightly preceded the work of Mumford, Newstead, Narasimhan and Seshadri, who took a different approach to the construction of moduli space, based on the notion of stability of a vector bundle, analogous to the notion of stability of a point in the theory of algebraic group actions. Namely if we consider from the outset only stable vector bundles (roughly speaking, the vector bundles that do not contain subbundles of large degree) then there is a natural algebraic parametrisation of such bundles without any additional structure. In this way we can also obtain a very natural compactification of the corresponding moduli spaces.

Formally speaking, these two approaches give different but somewhat similar result in the case of rank 2 vector bundles. However, the main advantage of the second approach lies outside purely classical algebraic geometry. Namely Narasimhan and Seshadri showed that stable vector bundles with trivial first Chern class are exactly those constructed via irreducible unitary representations of the fundamental group of the corresponding complex projective curve. Due to this remarkable coincidence and the depth of the notion of stability, the second approach has dominated the theory of vector bundles ever since it was introduced.

Later, ANT obtained a number of remarkable results based on the notion of stability. A detailed survey of all these results is contained in the preceding article of this volume.

Fedor Bogomolov

* * *

This is Tyurin's first paper, written at a time when interest was developing in the classification of vector bundles on algebraic curves. A.Grothendieck [5] had described bundles on the projective line in 1954 (although the basic elements of the description go back to the 19th century), and M. F. Atiyah [2] had done the same for bundles on an elliptic curve in 1957. With hindsight one can see that these successes, while of major importance, were somewhat deceptive. It had of course been well understood for many years that curves of genus ≥ 2 behave in a fundamentally different way from those of genus 0 or 1, but it had probably not been realized that essentially new techniques would be needed for classifying bundles on curves of higher genus. (In modern terms we can describe the distinction by saying that the problem is of finite type for genus 0, tame for genus 1 and wild for genus ≥ 2.) Atiyah [1] had attempted a classification of ruled surfaces in arbitrary genus and obtained partial results in genus 2, but it was clear that the problem was already complicated.

Tyurin's attack on the problem was based on an idea of A. Weil [17], namely that of matrix divisors. For line bundles, there is a very nice correspondence with divisors in the classical sense: isomorphism classes of line bundles correspond bijectively to divisor classes. Weil's idea was to generalize this by considering matrix divisors. The problem is that the equivalence relation on matrix divisors corresponding to isomorphism of vector bundles is very badly behaved, and one must try to simplify this by imposing conditions on the matrix divisors. This is what Tyurin does for bundles of rank 2 through his concepts of exceptional subbundles and quasibundles. He fixes a point $P \in X$ and defines a line bundle L to be of height h if h is the smallest integer such that $L(hP)$ has a non-zero section. He defines a subbundle L of E to be exceptional if $\dim H^0(X, E \otimes L^*) = 1$, and then proves that every indecomposable bundle E of rank 2 has at least one and at most $2g$ exceptional subbundles of minimal height. He calls the pairs (E, L), or the corresponding extensions

$$0 \longrightarrow L \longrightarrow E \longrightarrow L' \longrightarrow 0,$$

quasibundles. These are easy to classify since there is a good classification of extensions. Using this approach, Tyurin classifies quasibundles for fixed $\det E$ and shows that there is a unique "component of maximal dimension" $3g - 3$ (Theorems 10, 11). Moreover there exist universal objects. When $d = \deg E$ is odd, he shows further that distinct quasibundles give distinct bundles, so in modern terms he has constructed a dense open subset of the moduli space.

In fact, one can describe this as follows. By tensoring our bundles E by some fixed line bundle, we can suppose that $d = 2g - 1$; the component of maximal dimension then consists of quasibundles of the form

$$0 \longrightarrow I \longrightarrow E \longrightarrow \det E \longrightarrow 0,$$

where I is the trivial line bundle and $\dim H^0(X, E) = 1$. These extensions are classified by an open subset of $\mathbf{P}(H^1((\det E)^*))$, which by Riemann-Roch has dimension

$$(2g - 1) + (g - 1) - 1 = 3g - 3.$$

There are two problems with A. Tyurin's approach. One is that, in even degree, even on the component of maximal dimension, a given bundle will correspond to more than one quasibundle. The second, more fundamental, problem is that a bundle in the component of maximal dimension can degenerate to a bundle in one of the other components. In other words, the "components", when considered in terms of bundles rather than quasibundles, are not really separate from one another; they should more properly be thought of as strata in some more global object.

These issues were in the process of being resolved by the introduction of the concept of stability by D. Mumford [8]. By the time A. Tyurin wrote his second paper [13] (see also [14]), published in 1965, this time on bundles of arbitrary rank, Mumford's results had become available and he was able to use them. None the less this second paper is still largely concerned with the use of matrix divisors. This now presents even more difficulties than it did for bundles of rank 2.

However A. Tyurin did succeed in calculating dimensions and showed that, for $g \geq 2$, the "number of moduli" of unstable bundles (incidentally by "stable" in this paper Tyurin means what we now call "semistable") is strictly less than the corresponding number for stable bundles [13, Theorem 2.5.1]. This has often been taken as an obvious consequence of Mumford's theory, but does in fact require proof, and A. Tyurin was the first to give this proof. In the course of the proof [13, Lemma 2.5.1], he introduces what is now known as the Harder-Narasimhan filtration of an arbitrary vector bundle on X [6].

Following A. Tyurin's second paper, this line of development came to a halt because of the work of M. S. Narasimhan and C. S. Seshadri. Before Mumford's concept of stability had become well known, Narasimhan and Seshadri had already followed up another suggestion from Weil's paper [17], namely to consider bundles arising from unitary representations of the fundamental group [9]. This handles only bundles of degree 0, but they subsequently developed the ideas to cover bundles of any degree (the construction is again implicit in Weil's paper) [10] and showed that their unitary bundles are precisely those that can be expressed as direct sums of stable bundles, all of the same slope (the slope of a bundle is the rational number $\frac{\deg E}{\operatorname{rank} E}$). In particular, the irreducible unitary bundles are precisely the stable bundles. This gives a representation-theoretic construction for the moduli spaces of stable bundles, which has proved extremely useful and has been developed in many ways. Subsequently a purely algebraic description, following Mumford's approach, was completed by Seshadri (see [12, Première partie] for an outline).

There are further early results of A. Tyurin on the structure of the moduli spaces (see, for example, [15, 16] and the survey article "The geometry of moduli of vector bundles" reprinted in this volume). Seshadri's notes [12] constitute, among other things, a survey of results known up to 1980. Shortly after this the introduction of methods inspired by physics played a major rôle; see particularly [3]. This inspiration has continued to the present day.

One particular line of study seems to relate well to A. Tyurin's ideas. We have already remarked that one can view Tyurin's "components", when restricted to stable bundles, as providing stratifications of the moduli spaces. A variation of this idea leads to the consideration of other stratifications, especially those associated with the name of Corrado Segre. These stratify the moduli spaces by the degrees of subbundles of maximal degree. In the context of ruled surfaces, they go back to an 1889 paper of Segre, and their properties are established in [7], [4] and [11]. The advantage of Tyurin's approach is that every bundle arises from a finite number of quasibundles, while bundles can possess infinitely many subbundles of maximal degree. However the Segre stratifications are in some sense more natural and maximal subbundles have become of particular interest recently because of connections with Gromov-Witten theory.

These papers of Tyurin certainly contain errors (most notably his claim in [13] to have proved rationality of the moduli spaces), but they also contain many excellent ideas which were ahead of their time.

P. Newstead

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Vector bundles of finite rank over infinite varieties.

Math. USSR Izvestija. **10** (1976), no. 6.

In this pioneering article, Tyurin solves a number of problems in algebraic geometry using the language of infinite dimensional manifolds. One of his main results is a theorem that any vector bundle of finite rank on a smooth algebraic subvariety $X \subset \mathbb{P}^\infty$ of finite codimension in infinite dimensional projective space is a direct sum of line bundles $\mathcal{O}(i)$, where the $\mathcal{O}(i)$ are the powers of the standard line bundle on $\mathbb{P}^\infty$ restricted to X. This is a significant generalization of Grothendieck's result for vector bundles on $\mathbb{P}^1$. The corresponding statement for $X = \mathbb{P}^n$ is called Schwarzenberger's conjecture, and was proved by W. Barth and A. Van de Ven for rank 2 vector bundles[3].

Parallel to the theory of vector bundles, Tyurin developed the theory of extensions of algebraic subvarieties in finite dimensional projective spaces to finite codimension subvarieties in infinite dimensional projective space. This approach of ANT substantially clarified many previous results on extensions of algebraic varieties and vector bundles. The analysis of such manifolds is significantly simpler due to their rich geometry: for example, any two points of a such a smooth subavariety X in $\mathbb{P}^\infty$ are joined by a chain of two lines contained in X, and all such chains are parametrised by an infinite dimensional variety.

ANT's proof is based on estimates for vector bundles on the ruled surface obtained by blowing up a point on $\mathbb{P}^2$, and corollaries of these estimates for vector bundles on other ruled varieties. The interested reader should look at the english translation of ANT's article[4] since the translator Miles Reid made some corrections.

Somewhat similar results were also obtained by E. Sato[5] soon after ANT. Later on Sato found a generalization of these results to the Grassmann varieties $\mathrm{Gr}(k,\infty)$. He also obtained similar results for corresponding orthogonal and symplectic Grassmannians[6]. The final picture is slighly more complex in these cases since if $k > 1$ the tautological k-dimensional vector bundle on an infinite Grassmannian is also indecomposable. Sato showed that any vector bundle of finite rank on $\mathrm{Gr}(k,\infty)$ is decomposable into a direct sum of irreducible components of the tensor algebra $T(S_k \oplus S_k^*)$, where S_k is the tautological k-dimensional vector bundle on the infinite Grassmann variety $\mathrm{Gr}(k,\infty)$ and S_k^* its dual.

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[3]Barth W., Van de Ven A. *On the geometry in codimension* 2 *of Grassmannian manifolds.* In Classification of algebraic Varieties and Compact Complex Manifolds Lecture Notes. Mathematics 412. Springer, Berlin, 1974. P. 1 – 35

[4]Math. USSR Izvestia, 1976. Vol. 10. no. 6. P. 1187 – 1204

[5]Sato E. *On decomposition of infinitely extendable vector bundles on projective spaces and Grassmannian varieties.* J. Math. Kyoto Univ. 1977. 17. no. 1. P. 127 – 150

[6]Sato E. *On infinitely extendable vector bundles on G/P.* J. Math. Kyoto Univ. 1979. 19. no. 1. P. 171 –189

The works of Tyurin and Sato from the late 1970s remained out of the mainstream of algebraic geometry for a period. Recently, however, Donin and Penkov[7] considered the more general question of vector bundles on an ind-Grassmann variety. The latter is defined as the inductive limit of smooth embeddings of finite dimensional Grassmannians. They showed in particular that Tyurin's theorem also holds for a Grassmannian $\mathrm{Gr}(H, \infty)$ of subspaces in $\mathbb{C}^\infty$) which are commensurable with a given subspace H of the space $\mathbb{C}^\infty$; in this case H has both infinite dimension and infinite codimension in $\mathbb{C}^\infty$. They also proved that every vector bundle is trivial on a "twisted" infinite dimensional projective space obtained as an inductive limit of finite dimensional projective spaces defined by sequence of smooth embeddings of degree > 1.

I. Penkov

* * *

In this paper A. Tyurin proves a conjecture of R. L. E. Schwarzenberger that vector bundles of finite rank on an infinite dimensional projective space $\mathbf{P}^\infty$ are decomposable as direct sums of line bundles. He also extends this to bundles on a non-singular infinite projective variety; by this he means an infinite chain

$$X_0 \subset X_1 \subset \ldots \subset X_n \subset \ldots$$

of non-singular projective varieties such that each X_n is a hyperplane section of X_{n+1}. Note further that all line bundles on a non-singular infinite projective variety have the form $\mathcal{O}(k)$ for some integer k. There are also finite versions of these results. For these, we define an infinitely extendable (or absolute) projective variety to be any variety that sits in a chain of the above type, with a similar definition for vector bundles. Then every infinitely extendable vector bundle of finite rank on an infinitely extendable non-singular projective variety is a direct sum of line bundles, each of the form $\mathcal{O}(k)$ for some k.

The problem originated from a result of Schwarzenberger [8] (Theorem 22.4.2), who showed that any infinitely extendable vector bundle of finite rank on $\mathbf{P}^n$ has the same Chern classes as a direct sum of line bundles; this is a consequence of the Grothendieck-Riemann-Roch Theorem, and is valid for topological vector bundles. The rank 2 case of Tyurin's main result was first proved by W. Barth and A. Van de Ven [1], while the result for arbitrary rank was proved independently by E. Sato [5, 6]. Sato also obtained results for bundles on Grassmannians [5] and more generally on homogeneous spaces [7], but one should note that these varieties are not infinitely extendable as subvarieties of $\mathbf{P}^n$; of course, for homogeneous spaces, one must allow certain tautological vector bundles as building blocks as well as line bundles.

[7] Donin J., Penkov I. *Finite rank vector bundles on inductive limits of Grassmannians.* IMRN (International Math. Res. Notices) 2003. no. 34. P. 1871 – 1887

The problem is closely related to that of the existence of subvarieties of given codimension in $\mathbf{P}^n$. The corresponding result in this context is that any infinitely extendable subvariety of $\mathbf{P}^n$ is a complete intersection. For subvarieties of codimension 2 in $\mathbf{P}^n$ with $n \geq 6$, there is a precise link with vector bundles of rank 2 via a construction of Serre [9] (see, for example, [4, 1, 3, 2]); in particular, there exists an indecomposable vector bundle of rank 2 on $\mathbf{P}^n$ if and only if there exists a non-singular subvariety of codimension 2 in $\mathbf{P}^n$ which is not a complete intersection. For higher codimension and rank, no such precise relationship is known.

Sato [5, 6] gives a bound N, depending on the invariants of the vector bundle E on $\mathbf{P}^n$, such that E is a direct sum of line bundles whenever it is extendable to $\mathbf{P}^{n'}$ with $n' \geq N$. One can ask whether it is possible to find N independent of the invariants of E. It is in fact conjectured that every vector bundle of rank 2 on $\mathbf{P}^n$ with $n \geq 5$ (or $n \geq 6$ in characteristic 2) is decomposable. It may be noted that there is a subvariety of $\mathbf{P}^5$ of codimension 2 which is not a complete intersection (the Segre embedding of $\mathbf{P}^1 \times \mathbf{P}^2$), but this does not invalidate the conjecture. In the mid 1970s, it was widely hoped that an answer to this conjecture would soon be found, but these hopes have still not been fulfilled and the problem remains open.

For complete intersections of arbitrary codimension, Hartshorne [3] has conjectured that a non-singular subvariety of $\mathbf{P}^n$ of dimension d with $d > \frac{2}{3}n$ is a complete intersection.

P. Newstead

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Symplectic structures on the varieties of moduli of vector bundles on algebraic surfaces with $p_g > 0$.

Math. USSR Izv. Ser. Mat. (1) **33** (1989)

The paper is one of the first works in a new direction of modern algebraic geometry – the theory of symplectic algebraic manifolds. The importance of this class of varieties is based on the fact that irreducible varieties with holomorphic symplectic structure constitute one of the "building blocks", together with complex tori and Calabi–Yau manifolds, of Bogomolov's decomposition of compact Kähler manifolds with torsion class c_1. The intensive study of symplectic algebraic manifolds was started by Mukai[8], where he constructed a holomorphic symplectic structure on the moduli space of simple sheaves on Abelian and K3 surfaces.

The present work is a systematic generalization of Mukai's construction to the case of surfaces with $p_g > 0$ and to higher dimensional varieties; at the same time, it treats Poisson algebraic structures. The paper starts with a detailed treatment of the formalism of Mukai lattices and structures and a description of properties of simple, exceptional and modular sheaves. Then it gives a general procedure constructing a holomorphic symplectic structure (respectively, a Poisson structure) on components of the moduli space of simple sheaves on an arbitrary smooth regular surface, and simultaneously local invariants of these structures. An important geometric examples of the structures discussed in the paper is given by nondegenerate symplectic structures on the moduli spaces of ideal sheaves of zero dimensional subschemes of a surface, i.e., the Hilbert schemes.

The key idea of the paper is Tyurin's introduction of the modular operations on the moduli components of simple sheaves on a surface. The notion of modular operation is based on the theory of mutations of exceptional bundles and helices of bundles developed in the mid-1980s by the Rudakov–Tyurin seminar on exceptional and stable bundles. Namely, the operations of universal

[8]S. Mukai Symplectic structure of the moduli space on an Abelian or K3 surface Invent. math. 77 (1984), 101–116

extension and universal division enable one, starting from a given moduli component of simple sheaves and some additional data including the existence of an appropriate exceptional bundle on a surface, to construct an extensive series of new moduli components birational to the original one. At the same time a symplectic structure carries over to all the new moduli components. This gives rise to a whole hierarchy of moduli varieties of simple modular sheaves, whose properties are studied in the paper.

It is worth mentioning that the paper is conceptually related to another important paper of the author[9], not included into the present collection, that gives ample geometric material illustrating the constructions of the present work. Some of these constructions have been developed in recent research on moduli of sheaves on K3 surfaces[10].

A. S. Tikhomirov

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This article is concerned principally with the existence of symplectic and Poisson structures on the moduli spaces of vector bundles on algebraic surfaces. In this context A. Tyurin defines a symplectic structure on a smooth projective variety B to be a non-zero skew-symmetric homomorphism $\omega : TB \to T^*B$ from the tangent bundle of B to the cotangent bundle. In a similar way he defines a Poisson structure to be a non-zero skew-symmetric homomorphism $\alpha : T^*B \to TB$. When B is a surface S, a symplectic structure is just a non-zero section of the canonical bundle K_S, while a Poisson structure is a non-zero section of the dual bundle K_S^*. Although superficially the two definitions are very similar, they give rise to very different structures except when K_S is trivial (in this case, if we assume further that the surface S is regular, then S is a K3-surface). In fact, a regular surface with a Poisson structure with K_S non-trivial is rational, while (by definition) a surface with a symplectic structure has geometric genus $p_g > 0$

In the first chapter, A. Tyurin shows how to construct symplectic (Poisson) structures on components of moduli spaces of bundles on a regular surface S from a symplectic (Poisson) structure on S. In Chapter II, he introduces modular operations which establish isomorphisms between certain components of the moduli spaces; these are analogous to, but richer than, the operation of tensoring by a line bundle. These operations are used to construct infinite series of moduli spaces of bundles on a surface S of general type which are birationally equivalent to symmetric powers of S.

In Chapter III, A. Tyurin discusses the principal differences in the classification theory of bundles between surfaces with symplectic structure and surfaces

[9] A.N. Tyurin Cycles, curves and vector bundles on K3 surfaces Duke Math. J. V.54. No.1, 1–26

[10] see: E. Markman. On the monodromy of moduli spaces of sheaves on K3 surfaces – I, II. `arXiv: math.AG/0305042`, `math.AG/0305043`

with Poisson structure. In particular, on a surface with symplectic structure, techniques such as the use of monads, resolution of the diagonal and helices do not work. On the other hand, a bundle on a surface with symplectic structure is almost uniquely determined by its second Chern class in the Chow group $CH^2(S)$.

P. Newstead

The moduli spaces of vector bundles on threefolds, surfaces and curves. I.

Preprint. Erlangen. 1990.

The description of the natural restriction maps from moduli spaces of stable sheaves on a variety to its subvarieties (e.g., hyperplane sections) is one of the basic methods of studying these spaces. The present work is an introduction to this method based on the important geometric case when the base variety is one of the varieties of an embedded triple "curve, surface, threefold (solid)". One of Tyurin's first observations is that if the threefold is a Fano variety, and the surface a K3, then the moduli component of stable vector bundles on the threefold is under certain conditions embedded by the restriction map as a Lagrangian submanifold of the corresponding moduli component of stable bundles on a K3 surface; this component has a natural holomorphic symplectic structure. Then the paper considers in detail the next step of the operation of restriction of stable bundles and, more generally, coherent torsion free sheaves when passing from the surface to the curve. The well known theorem of Mehta–Ramanathan states that the restriction map of moduli of semistable sheaves (under the natural condition of ampleness of the curve on the surface) is a rational map. The present paper generalizes this theorem: under some additional conditions on the curve, this map is birational onto its image, thus giving as a corollary a birational embedding of moduli components of stable bundles on the surface (and of their Gieseker closures) into the space of conformal blocks of the curve. In particular, this enables to obtain the Donaldson's line bundle (in the sense of Le Potier) on the moduli space of sheaves on the surface as a rational multiple of the standard Hopf bundle on the space of conformal blocks. This gives rise to a potentially new method of computing the Donaldson polynomials of the original surface.

In the final part of the paper the above ideology is transferred to the classical case when one takes the projective plane as the original surface and considers rank 2 vector bundles on it with trivial determinant, which one then restricts to lines in the plane. Here the role of conformal blocks is played by the space of curves of jumping lines of the bundle. The work is apparently the first to give a formula relating Donaldson's constants of the projective plane to the degree of the variety of curves of jumping lines, and this degree is computed in the first nontrivial case, when the second Chern class of the bundle on the surface is 4. In this case the variety of curves of jumping lines is the hypersurface of Lüroth quartics in the space of plane quartics and, as follows from a classical result of invariant theory (F. Morley 1918), this degree is conjecturally equals to 54. The present paper gives an outline of the algebraic geometric proof of this result using the method of Barth's nets of quadrics. Other variants of the proof of this result (and also of the injectivity of Barth's map of the moduli of bundles on the plane into the space of jumping lines) were independently given by Le Potier[11]. The most detailed exposition of these results and of the injectivity of the Barth's map for the case of the second Chern class ≥ 4 is presented in the paper: J. Le Potier and A.S.Tikhomirov. *Sur le morphisme de Barth.* Ann. Scient. Éc. Norm. Sup., 4^e série, t.34 (2001), 523-629.

A. S. Tikhomirov

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This expository article is a considerably extended version of a talk given by A. Tyurin in Bayreuth in 1990. It is concerned with relating moduli spaces of stable vector bundles on a flag of varieties $X \supset S \supset C$, where X is a Fano threefold, S is a K3-surface in the anticanonical divisor class and C is a curve on S. A. Tyurin considers components M_X, M_S, M_C of the moduli spaces with fixed first Chern class such that $E \in M_X \Rightarrow E|S \in M_S \Rightarrow E|C \in M_C$, so that there exist restriction morphisms $M_X \to M_S \to M_C$.

Tyurin considers here the second of these morphisms in a case where S is a smooth regular surface (not necessarily K3) and $M_S = M_H(2, c_1, c_2)$ is the moduli space of H-stable bundles of rank 2 on S with Chern classes $c_1 \in \operatorname{Pic} S$ and $c_2 \in \mathbf{Z}$. (Here H is an ample line bundle on S and $C \in |dH|$ is a generic smooth curve such that $\deg c_1 \cdot C$ is even.) Tensoring by a suitable line bundle on C, we can take $M_C = M_C(2, \mathcal{O})$, the moduli space of stable bundles of rank 2 on C with trivial determinant. One shows that, for sufficiently large d (depending on k), the restriction morphism $\operatorname{res}_C : M_H(2, c_1, k) \longrightarrow M_C(2, \mathcal{O})$ is an embedding (in fact Tyurin proves a slightly stronger theorem (Theorem 1.1), which includes also the case where $\deg c_1 \cdot C$ is odd).

The next step (Theorem 2.1) is to extend this map to a map

$$\overline{\operatorname{res}}_C : \overline{M}_H(2, c_1, k) \longrightarrow \overline{M}_C(2, \mathcal{O}),$$

where $\overline{M}_H(2, c_1, k)$ is the Gieseker compactification of $M_H(2, c_1, k)$ [6]. Now

$$\operatorname{Pic} \overline{M}_C(2, \mathcal{O}) \cong \mathbf{Z}$$

[11] J. Le Potier. *Fibrés stables sur le plan projectif et quartiques de Lüroth.* Exposé donné a Jussieu le 30.11.1989

J. Le Potier J. *Faisceaux semi-stables et systémes cohérents.* Vector bundles in algebraic geometry (Durham, 1993), London Math. Soc. Lect. Note Ser. **208** (1995), 179-239 (see [4]); let L_0 be a positive generator of this Picard group defining a linear system $|\Delta|$. By a theorem of Beauville [2], $|\Delta|$ is base-point free and the induced morphism

$$f_0 : \overline{M}_C(2, \mathcal{O}) \longrightarrow \mathbf{P}H^0(L_0)^\vee$$

is finite and has degree 2 if C is hyperelliptic and 1 otherwise. Moreover $H^0(L_0)$ can be identified with $H^0(J(C), \mathcal{O}(2\Theta))$ and has dimension $2^{g(C)} - 1$. Now by conformal field theory the projective space $\mathbf{P}H^0(L_0)$ is independent of C for small variations in complex structure. Denoting this projective space by $\mathbf{P}\mathcal{H}$ and composing with the restriction map, we obtain a family of morphisms ((3.40))

$$f_{C\subset S} : \overline{M}_H(2, c_1, k) \longrightarrow \mathbf{P}\mathcal{H}^\vee.$$

Tyurin conjectures (Conjecture 3.1) that $f_{C\subset S}$ is independent of C for small variations.

Independent of this conjecture, however, A. Tyurin raises the question of computing the degree of $f_{C\subset S}$ in the case where $c_1 = \mathcal{O}$ and $M_H(2, \mathcal{O}, k)$ has the "right" dimension, namely $\dim M_H(2, \mathcal{O}, k) = 4k - 3(p_g + 1)$. This degree is clearly independent of C and is in fact given by a Donaldson polynomial [3]. These are difficult to compute, but, for K3-surfaces, the problem has been solved (see (4.18)) by Friedman and Morgan [5] and independently by K. O'Grady. The article finishes with a calculation for $S = \mathbf{P}^2$, in which case the degree has the form $c_k(\deg C)^{4k-3}$ ((4.19)), where c_k is an absolute constant. To compute c_k, we can take C to be a line. Now restrictions of bundles on $\mathbf{P}^2$ to lines have been extensively studied, most notably by W. Barth [1]. It is easy to see that $c_2 = 1$ and $c_3 = 3$, and Tyurin shows, by combining Barth's arguments with classical results of Lüroth and Clebsch and an idea of A. S. Tikhomirov, that $c_4 = 54$. (In fact, Tyurin notes in the article "The classical geometry of vector bundles" contained in this volume that this number was first obtained by F. Morley in 1918.)

P. Newstead

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The classical geometry of vector bundles.

Algebraic geometry (Ankara, 1995).
Lecture Notes in Pure and Appl. Math. **193**, 347–378.
New York: Marcel Dekker, 1997

The paper is a summary of a small course of lectures given by the author on the summer school at Ankara (Turkey) in 1995. It is devoted to the introduction into the geometry of vector bundles on algebraic varieties in relation to applications to enumerative geometry and topology of smooth four-dimensional manifolds. In this paper there is considered a number of important geometric objects closely related to vector bundles – nets of quadrics, Darboux configurations and Hilbert schemes. The significance of nets of quadrics in modern algebraic geometry and, in particular, in birational geometry of threefolds and geometric invariant theory was clearized in 70ies - 80ies of the last century in works of A.N.Tyurin, A.Beauville, C.T.C.Wall and other authors – see, e.g., papers [A.N.Tyurin. *On intersection of quadrics.* UMN, V. 30, No. 6 (1975), p.51-99] e [A.N.Tyurin. *The variety of pairs of commuting pencils of symmetric matrices.* Izvestija AN USSR. Ser. mathem. V. 46, No. 2 (1982), 409-430] not included into the present collection. Further interesting geometric examples of interaction of all these objects were found by S.Mukai [S.Mukai. *Fano 3-folds.* London Math. Soc. Lect. Notes Series, 179 (1992), 255-263], and applications to enumerative geometry were considered by G.Ellingsrud and S.A.Strømme. *Bott's formula and enumerative geometry.* Journ. Amer. Math. Soc. 9 (1996), 175-193]. The present paper is on one hand an original introduction into the above papers, and on the other hand it illuminates applications of the technique of vector bundles and Hilbert schemes to smooth topology of algebraic surfaces, in particular, to computing of Donaldson's polynomials. In more detail these algebro-geometric constructions are considered in the paper of J.Le Potier [J.Le Potier. *Systèmes cohérents et polynômes de Donaldson.* Lecture Notes Pure Appl. Math. Math. Vol. 179, 1996, pp. 103-128.]

A special part of the paper is devoted to the discussion of questions related to computing of Donaldson's constants of the smooth structure of the complex projective plane. It is worthwhile to note that the problem of computing of Donaldson's constants of the projective plane had stimulated a big number of works in this direction, among which one should mention the paper by the author and A.Tikhomirov[12] not included into the present collection, and the sub-

[12] A.Tikhomirov, A.Tyurin. *"Application of the geometric approximation procedure to computing the Donaldson's polynomials for CP_2."* Mathematica Goettingensis, Sonderforsschungsbereichs "Geometrie und Analysis", Heft 12 (1994), 1-71

sequent papers by G.Ellingsrud, L.Göttsche, J.Le Potier and S.A.Strømme[13]. In the present work there is given an interesting interpretation (based on the results of Ellingsrud and Göttsche and one conjecture of Kotschick and Morgan) of Donaldson's constants of the projective plane as its potential homotopy invariants.

A. S. Tikchomirov

* * *

This is the text of a talk given by A. Tyurin at a summer school in Ankara in 1995. Its themes are essentially the unity of geometry and the ubiquity of algebraic geometry in this more general setting. In particular, many results of enumerative geometry remain true not only in symplectic geometry but also in differential geometry and even in topology.

The story starts in the 19th century with Clebsch and Lüroth and runs through moduli spaces to Donaldson invariants. The whole is told in Tyurin's inimitable style. The only recommendation is: read and enjoy, and then think what it all means.

P. Newstead

The Weil – Petersson metric on the moduli space of stable vector bundles and sheaves on an algebraic surface.

Math. USSR Izvestiya. (3) **38** (1992)

Tyurin dedicated this article to the memory of his older sister Galina Nikolaevna Tyurina, who died tragically at the age of 32 during a canoeing trip to the Prepolar Urals. Galina Nikolaevna was also a student of I.R. Shafarevich, who obtained several well known results in algebraic geometry. Most of her work relate to the theory of K3 surfaces and singularity theory.

Weil–Peterson metrics arise naturally on moduli varieties, that is, on varieties that parametrize deformations of some other algebraic varieties. In this case the tangent space to the moduli variety has a natural description in terms of the fibre, the algebraic variety corresponding to the given point in the moduli space. Thus if the fibre has a natural metric, there is also a natural metric on the tangent space of the moduli space. The metric on the moduli space

[13]G.Ellingsrud, J.Le Potier and S.A.Strømme*Some Donaldson invariants of* $\mathbb{P}_2(C)$. Lecture Notes Pure Appl. Math. Vol. 179, 1996, pp. 33-38
G.Ellingsrud, L.Göttsche. *Variation of moduli spaces and Donaldson invariants under change of polarization.* J. reine angew. Math. 467 (1995), 1-49

that arises in this way is called the Weil–Peterson metric; it is a Kähler metric under rather mild conditions on the moduli space.

The Weil–Peterson metric reflects the structure of the variations of the fibre rather concisely. For example, within this framework, representing a Riemannian surface of genus > 1 as a quotient of the standard disc by a discrete subgroup provides a Weil–Peterson metric on the moduli variety of complex curves of genus > 1.

Tyurin considers the Weil–Peterson metric on the moduli space of stable vector bundles on a compact projective (Kähler) surface. In this case every stable vector bundle has a so called Hermitian Einstein connection, which provides a metric on the projectivisation of the vector bundle. This allows us to define Weil–Peterson metric on the moduli variety of stable vector bundles. Note that this moduli variety has an infinite number of components of different dimensions (stable bundles with different topological invariants clearly belong to different components of the moduli space). The components of these moduli varieties are also noncompact in general. Tyurin described a natural compactification of the moduli of stable vector bundles on surfaces by torsion free sheaves. Moreover he proved the existence of components that are smooth varieties after this natural compactification (he calls these *thin* components). In this case, he proves the existence of extension of the Weil–Peterson metric to a Kähler metric on the corresponding compact manifold.

He considers the case of vector bundles on K3 surfaces in more detail. These surfaces have a so called hyper-Kähler structure, that is, a two-dimensional family of Kähler structures related by a quaternionic rotation. Hyper-Kähler structures have many remarkable properties, and the list of known compact hyper-Kähler manifolds is rather small. The main observation of the article is that the extension of the Weil–Peterson metric provides a hyper-Kähler structure on an infinite series of thin components of moduli spaces of vector bundles on K3 surfaces. Thus it is possible in principle to find new examples of compact hyper-Kähler manifolds by considering thin components of moduli spaces of vector bundles. The above construction can also be applied to Abelian surfaces. This remarkable idea was further developed by a number of authors and provided several new examples of compact hyper-Kähler manifolds[14].

Tyurin's idea is also related to Ron Donagi and Yuval Markman construction of a natural hyper-Kähler structure on the family of intermediate Jacobians of Calabi–Yau threefolds over the moduli space[15].

F. Bogomolov

[14]see for example: O'Grady, Kieran G. *A new six-dimensional irreducible symplectic variety.* J. Algebraic Geom. 12 (2003). no. 3. P. 435 – 505; O'Grady, Kieran G. *Desingularized moduli spaces of sheaves on a* $K3$. J. Reine Angew. Math. 512 (1999). P. 49 – 117; where new examples of compact hyper-Kähler manifolds were constructed using moduli varieties of vector bundles on curves of genus 2 and K3 surfaces

[15]Donagi R., Markman E. *Cubics, integrable systems and Calabi - Yau threefolds.* Proceedings of the Hirzebruch 65 Conference on Algebraic Geometry (Ramat Gan, 1993). p.199 – 221. Israel Math. Conf. Proc. 9

On the superpositions of mathematical instantons.

Progr. Math. 1983. **36**, 433–450

Mathematical instantons considered in this paper constitute the important class of stable algebraic vector bundles on the projective three-space. The description of the variety of moduli M_n of mathematical instantons (here the natural number n is the second Chern class of the instanton, which is its basic discrete invariant) began in 70-s of the last century and since that time they permanently attract attention of algebraic geometers. The specific property of instantons is their determination by monads or, equivalently, by the so called hypernets of quadrics in n-dimensional vector space H. These hypernets, understood as vectors in the space with additional tensor structure, satisfy the so called Barth's conditions which determine in the space of hypernets of quadrics the locally closed subset $M_n(H)$, and the problem of smoothness, irreducibility and (uni)rationality of the space M_n is reduced to the similar problem for $M_n(H)$. In the present paper Andrei Nikolaevich systematically develops the idea of the set $M_n(H)$ via a representation of its vectors as linear combinations (superpositions) of special decomposable tensors of the space of hypernets called half-instantons. By this method there are obtained two main results of the paper - the proofs of unirationality of the space M_4 and of the main component of the space M_5 containing instantons of t'Hooft. The first of these results is still the best achieved in the field, and there is a conjecture that M_4 is rational. Remark that a little earlier than this paper had appeared there was proved irreducibility (W.Barth, 1981) and smoothness (J.Le Potier, 1981) of M_4. (Similar results for the case $n \leq 3$ were stated earlier: $n = 1$ - by W.Barth, 1977; $n = 2$ - by R.Hartshorne, 1978; $n = 3$ - by G.Ellingsrud and Strømme, 1981.) The second result was improved only much later by P.Katsylo (1993) who proved the rationality of the main component of the space M_5. Very recently I.Coandă, A.Tikhomirov and G.Trautmann (2003) proved smoothness and irreducibility of M_5 in the paper [I.Coandă, A.Tikhomirov, G.Trautmann. *Irreducibility and smoothness of the moduli space of mathematical 5-instantons over* P_3. Intern. J. Math., V. 14, No.1 (2003), 1-45]. It is curious that in that paper the authors use certain technical results on superpositions of instantons obtained in the present work.

It should be mentioned that another version of a representation of instanton hypernets by superpositions of decomposable tensors is worked out in the paper by Andrei Nikolaevich [A.N.Tyurin. *The instanton equations for* $(n+1)-$*superpositions of marked* $adTP^3 \oplus adTP^3$ J. reine angew. Math., B. 341 (1983), 131-146] not included in the present collection. In that paper the author proceeds from hypernets of quadrics to special hypernets of homomorphisms and studies these last hypernets by means of their geometric invariants - the curves of Hesse and Steiner. A general view on the geometry of these curves as spectral invariants of hypernets was elaborated in the paper by Andrei Nikolaevich [A.N.Tyurin. *The variety of pairs of commuting pencils of symmetric* *matrices.* Izvestija AN USSR. Ser. mathem. V. 46, No. 2 (1982), 409-430], also not included in the present collection. That paper is related to the work of W.Barth [W.Barth. *Irreducibility of the space of mathematical instanton vector bundles with rank 2 and* $n_2 = 4$. Math. Ann., 258 (1981), 81-106] devoted to the proof of the above mentioned result on irreducibility of M_4 in which Barth points out the relation stated by A.N.Tyurin between the description of degenerations of instanton hypernets and the enumeration of components of Hilbert schemes of Steiner space curves supplied with theta-characteristics. These components in the case of reducible curves are enumerated by Andrei Nikolaevich for small values of n in the above mentioned paper on commuting pencils of symmetric matrices.

A. S. Tikhomirov

Delzant models of moduli spaces.

Izv. Mathematics. 2003. (2) **67**

Developed by A. N. Tyurin during the last years of his life was the Abelian Lagrangian Algebraic Geometry (ALAG) that is some universal algebraic viewpoint on the geometric quantization area. Besides some other fruitful things, ALAG allows to compare the outputs of Berezin –Toeplitz and Bohr – Sommerfeld quantization procedures applied to a given symplectic manifold M as soon M does admit simultaneously an integrable Kähler structure and some completely integrable real polarization. In this paper A. N. Tyurin carries through such the comparison in a very hardly studied case when $M = M_\Sigma(2,0)$ is the moduli space of holomorphic structures on the topologically trivial rank 2 vector bundle E over a curve Σ of genus $g \geq 2$.

A Kähler structure on M is provided by the algebraic geometric construction of this moduli space and is prescribed by a choice of a complex structure on Σ. An integrable real polarization on M comes from Narasimhan – Seshadri identification of M with the moduli space of gauge classes of flat SU(2)-connections on E or, equivantly, with the space $\mathfrak{RC}$, of isomorphism classes of unitary representations of the fundamental group $\pi_1(\Sigma) \xrightarrow{\varrho} \mathrm{SU}(2)$. The symplectic structure[16] on $\mathfrak{RC}$ is induced by 2-form

$$\int_\Sigma K(\sigma_1 \wedge \sigma_2)\,, \quad \text{where } \sigma_1, \sigma_2 \in \Omega^1(\Sigma) \otimes \mathfrak{su}_2\,,$$

$$K \in S^2\mathfrak{su}_2^* \text{ is the Killing form}\,,$$

[16]it has singularities for $g > 2$ and should be considered in orbifold's terms

on the whole space of all SU(2)-connections on E. Each 'pants decomposition' of Σ by $3g-3$ simple pairwise non-crossing and non-isotopic loops $\gamma \subset \Sigma$ gives a complete system of $3g-3$ commuting Hamiltonians:

$$c_\gamma \,:\, \mathfrak{RC} \to [\,0\,,\,1\,] \,:\, \varrho \longmapsto \frac{1}{\pi}\arccos\Bigl(\mathrm{tr}\bigl(\varrho(\gamma)\bigr)/2\Bigr)\,.$$

This provides $\mathfrak{RC}$ with a structure of Lagrangian fibration

$$\mathfrak{RC} \xrightarrow{c} \Delta \tag{0.1}$$

over some convex polyhedron[17] $\Delta \subset [\,0\,,\,1\,]^{3g-3}$. Goldman has shown[18] that the fibres of (0.1) over interior points of Δ are the real $(3g-3)$-dimensional Lagrangian tori. So, $\mathfrak{RC}$ has got a symplectic toric variety structure[19].

In this paper, A. N. Tyurin gives precise effective description for Δ and constructs very explicitly a chain of flips leading from $\mathfrak{RC}$ to $\left(\mathbb{CP}^3\right)^{g-1}$ (for $g > 2$). This gives an effective precise description for the combinatoric topology of $\mathfrak{RC}$ in a purely real stup, that is without any references to the identification $M \xrightarrow{\sim} \mathfrak{RC}$.

In fact, this result does much more than close a gap which defied the real topologists over quite a long time. It implies at once that 'the conformal block spaces'[20] $H^0(M, \vartheta^{\otimes N})$, which are the Hilbert spaces for the Kähler quantization of M, are naturally isomorphic to the Hilbert spaces coming from the real polarization of $\mathfrak{RC}$. This offers a big challenge, because the Kähler conformal blocks depend only on a complex structure on Σ (and are not related to a pants decomposition of Σ) but the real conformal blocks[21], quite the contrary, depend only on the Lagrangian fibration (0.1) (and have no concern with a complex structure).

A. N. Tyurin evolves this beautiful remark in his last book[22] 'Quantisation, Classical and Quantum Field Theory and Theta - Functions', where he uses the Bortwick – Paul – Uribe map[23] to attach a section $\sigma_S \in H^0(M, \vartheta^{\otimes N})$ to each Bohr – Sommerfeld torus S in the fibration (0.1) and to show that these sections σ_S form the standard base in the space of (non Abelian) theta-functions. On one side, this construction immediately gives a flat projective connection on the bundle of holomorphic conformal blocks over the moduli space of complex structures on Σ. On the other side, we get a collection of transition matrices

[17] it is called the Delzant polyhedron

[18] see *Goldman W. The symplectic nature of fundamental groups of surfaces.* Adv. in Math. **54** (1984) pp. 200–225 and *Goldman W. Invariant functions on Lie groups and Hamiltonian flows of surface group representations.* Invent. Math., 85, 1986, pp. 263–302

[19] in the sense of *Guillemin V. Moment maps and combinatorial invariants of Hamiltonian T^n-spaces.* Birkhäuser (Progress in Mathematics 122), 1994.

[20] i. e. the spaces of global sections of the natural ϑ-bundle on M

[21] i. e. the spaces spanned by Bohr –Sommerfeld fibres of the projection (0.1)

[22] its preliminary version is available at `math.AG/0210466`

[23] see *Borthwick D., Paul T. and Uribe A. Legendrian distributions with applications to the non-vanishing of Poincaré series of large weight.* Invent. math, **122** (1995), pp. 359–402 or `hep-th/9406036`

between the Bohr – Sommerfeld bases in $H^0(M, \vartheta^{\otimes N})$ coming from the different pants decompositions of Σ and these transition matrices, clearly, form 'a rational conformal field theory' over the graph of all pants decompositions.

So, this short elegant paper displays once again the staggering talent of Andrey Nikolaevich for assembling a lot of distinct complicated technical details coming from distant branches of mathematics into strikingly clear geometric picture.

A. L. Gorodentsev

This is the first volume of a three volume collection of Andrey Nikolaevich Tyurin's Selected Works. It includes his most interesting articles in the field of classical algebraic geometry, written during his whole career from the 1960s. Most of these papers treat different problems of the theory of vector bundles on curves and higher dimensional algebraic varieties, a theory which is central to algebraic geometry and most of its applications.

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