Yuri Tschinkel (Ed.)

Algebraic Groups

Mathematisches Institut
Georg-August-Universität Göttingen
Summer School, 27.06.-13.07.2005

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CONTENTS

ABSTRACTS

We recall various constructions of Kähler and hyperkähler manifolds via Nahm's equations, with an emphasis on semisimple algebraic groups and their adjoint orbits. Some applications, such as the Kostant-Sekiguchi correspondence, are also discussed.

Group cohomology of a finite group contains a quotient which reflects birational geometry of the quotient of a linear space with the linear group action by the group. In this series of lectures I describe some results and conjectures concerning this group cohomology, its analogue for profinite groups and birational invariants of algebraic varieties.

We discuss analysis on symmetric and locally symmetric spaces.

We present the notion of a p-adic family of modular eigenforms and survey briefly how it led from its beginnings in Serre's work on p-adic zeta functions to the construction of the Coleman-Mazur eigencurve. We motivate the two constructions of this eigencurve by briefly presenting how the fibre of the eigencurve over a fixed weight-character can be constructed. Finally we survey some of the progress that has been made beyond the case of $\mathrm{GL}_{2,\mathbb{Q}}$.

We discuss recent developments in Schubert calculus.

In this note we describe three algorithms for calculating the number of closed, reduced paths of a graph and discuss some applications.

In this note, a concise treatment of the representation theory of loop groups of compact Lie groups is given, following mainly the book "Loop groups" of Pressley and Segal. Starting with the basic definitions, we end up with the discussion of central extensions of loop groups, and with the classification of positive energy representations. We always compare with the parallel theory for compact Lie groups, and try to sketch concisely the basic constructions and ingredients of proofs.

In this paper we survey some recent results regarding automorphic forms on the similitude symplectic group of order four. We will also explain recent progress on analytic properties of L-functions associated to such automorphic forms.

INTRODUCTION

This volume contains the Proceedings of the Summer School "Algebraic groups", which took place at the Mathematics Institute of the University of Göttingen in June-July, 2005.

Yuri Tschinkel

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LIE GROUPS, NAHM'S EQUATIONS AND HYPERKÄHLER MANIFOLDS

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***Abstract*.** We recall various constructions of Kähler and hyperkähler manifolds via Nahm's equations, with an emphasis on semisimple algebraic groups and their adjoint orbits. Some applications, such as the Kostant-Sekiguchi correspondence, are also discussed.

1. Introduction

The Nahm equations are a powerful tool for constructing hyperkähler metrics on various algebraic manifolds, e.g., spaces of rational maps, coadjoint orbits, resolutions of Kleinian singularities. In these lectures I shall concentrate on their relation to semisimple algebraic groups and adjoint orbits. The aim is to show that, on the one hand, the Nahm equations provide a powerful tool for constructing geometric structures on these objects and for explaining certain representation-theoretic puzzles (the Kostant-Sekiguchi correspondence [**Ver95**], action of the Weyl group on flag manifolds [**AB02**]), and, on the other hand, to stress just how mysterious these geometric structures remain. This is so particularly for the case of hyperkähler metrics on coadjoint orbits.

There is little in these lecture that is original: perhaps some extensions of known results (e.g., Proposition 4.4 or Corollary 4.5 in literature) and some of the approach. On the other hand, I allowed myself to speculate at some points.

July 2005.

Notes by Sven-S. Porst.

It is a pleasure to thank Victor Pidstrygach and Yuri Tschinkel for the invitation to give these lectures and Sven Porst for taking and typing up the notes.

2. Quaternions and Lie groups

We begin with an unusually complicated construction of a bi-invariant metric on a compact Lie group G using the space of paths in its Lie algebra $\mathfrak{g}$. This approach is extended to the Kähler and hyperkähler reductions in the complexified and quaternionised cases.

Construct a bi-invariant metric. Consider the sets of paths in a compact connected Lie group G and its Lie algebra $\mathfrak{g}$

$$\mathscr{A} = \{T : [0,1] \longrightarrow \mathfrak{g}\} \qquad \text{and} \qquad \mathscr{G} = \{g : [0,1] \longrightarrow G\}.$$

Here $\mathscr{A}$ is an infinite-dimensional vector space and $\mathscr{G}$ is an infinite-dimensional Lie group which can be modelled by a Hilbert space. Denote by $\mathscr{G}_0$ be the subgroup of loops starting at the unit element of G:

$$\mathscr{G}_0 = \{g \in \mathscr{G} \mid g(0) = 1 = g(1)\} \subset \mathscr{G}.$$

Elements $g \in \mathscr{G}$ act on $\mathscr{A}$ by $(g.T)(s) = g(s)T(s)g(s)^{-1} - \dot{g}(s)g^{-1}(s)$ where $\dot{g}$ is the derivative of g at 0. Thus $\mathscr{A}$ is the space of connections on the trivial G-bundle over $[0,1]$ and $\mathscr{G}$ is the gauge group. We shall drop the parameter s from notation whenever possible, stating the action as:

$$g.T = gTg^{-1} - \dot{g}g^{-1}.$$

The action of $\mathscr{G}_0$ on $\mathscr{A}$ is free. For any $T \in \mathscr{A}$ we have the set $\{g \in \mathscr{G} \mid g.T = 0\} \subset \mathscr{G}$. Using its unique element g with $g(0) = 1$ we define the map

$$\mathscr{A} \longrightarrow G \qquad T \longmapsto g(1)$$

which is surjective, has kernel $\mathscr{G}_0$ and thus gives an isomorphism $\mathscr{A}/\mathscr{G}_0 \simeq G$. We observe that the action of $\mathscr{G}$ on $\mathscr{A}$ descends to an action of $\mathscr{G}/\mathscr{G}_0 \simeq G \times G$ on G.

We now view the vector space $\mathscr{A}$ as an infinite-dimensional manifold with a natural flat metric. Using an invariant scalar product $\langle\cdot,\cdot\rangle$ on $\mathfrak{g}$, we have the norm $\|t\|^2 = \int_0^1 \langle t(s), t(s)\rangle ds$ for tangent vectors $t \in T_T\mathscr{A}$. The groups $\mathscr{G}$ and $\mathscr{G}_0$ act isometrically on $\mathscr{A}$, so we get a metric on $\mathscr{A}/\mathscr{G}_0$ as follows: at every point $m \in \mathscr{A}$ the tangent space of $\mathscr{A}$ splits into a subspace tangent to the orbit of the $\mathscr{G}_0$ action and its orthogonal complement, $T_m\mathscr{A} = T_m\mathscr{A}^{\parallel} \oplus T_m\mathscr{A}^{\perp}$. We define the metric at $[m] \in \mathscr{A}/\mathscr{G}_0$ via the identification of $T_{[m]}\mathscr{A}/\mathscr{G}_0$ with $T_m\mathscr{A}^{\perp}$.

As $\mathscr{A}/\mathscr{G}_0 \simeq G$, we now have a metric on G. It is bi-invariant, since the action of $\mathscr{G}$ induces an isometric action of $G \times G$.

Exercise 2.1. Compute the metric directly as the quotient metric.

Complexify. We now complexify the construction of the previous section. This obviously should lead to a complexification of the Lie group G.

First we complexify the space of paths in the Lie algebra $\mathfrak{g}$:

$$\mathscr{A}^{\mathbb{C}} = \{T_0 + iT_1 : [0,1] \longrightarrow \mathfrak{g} \otimes \mathbb{C}\}.$$

$\mathscr{A}^{\mathbb{C}}$ is an infinite dimensional hermitian vector space with the hermitian metric

$$\|(t_0, t_1)\| = \int_0^1 \|t_0(s)\|^2 + \|t_1(s)\|^2 ds. \tag{1}$$

We write the corresponding symplectic form

$$\omega\big((t_0,t_1),(t_0',t_1')\big) = \int_0^1 \langle t_0, t_1'\rangle - \langle t_1, t_0'\rangle$$

as $\omega = \int_0^1 dT_0 \wedge dT_1$.

Extend the action of $\mathscr{G}$ on $\mathscr{A}$ to $\mathscr{A}^{\mathbb{C}}$ as follows:

$$g.(T_0, T_1) = (gT_0g^{-1} - \dot{g}g^{-1}, gT_1g^{-1}). \tag{2}$$

This action again preserves all the structures, i.e., the metric and the symplectic form.

Claim 2.2. The action is *Hamiltonian*, i.e., there is an equivariant map $\mu : \mathscr{A}^{\mathbb{C}} \to (\operatorname{Lie} G)^*$ such that for any $\rho \in \operatorname{Lie}\mathscr{G}_0$ and any $v \in T\mathscr{A}^{\mathbb{C}}$

$$\omega(\rho^*, v) = \langle d\mu(v), \rho\rangle,$$

where ρ^* is the vector field induced by ρ, i.e., $\rho^*\big|_m = \frac{d}{d\varepsilon}\big(\exp(\varepsilon\rho).m\big)\Big|_{\varepsilon=0}$. The map μ is called a *moment map*.

To prove the claim, consider $\rho \in \operatorname{Lie}\mathscr{G}_0$, i.e., a path with $\rho(0) = 0 = \rho(1)$. Then, putting $g = \exp(\varepsilon\rho)$ in (2) and differentiating with respect to ε, we obtain $\rho^* = \big([\rho, T_0] - \dot{\rho}, [\rho, T_1]\big)$. We now compute:

$$\begin{aligned}
\omega\big(\rho^*, (t_0,t_1)\big) &= \int_0^1 \big\langle [\rho, T_0] - \dot{\rho}, t_1\big\rangle - \langle [\rho, T_1], t_0\rangle \\
\text{(integration by parts)} &= -\langle \rho, t_1\rangle\Big|_0^1 + \int_0^1 \langle \rho, \dot{t}_1\rangle + \langle [\rho, T_0], t_1\rangle - \langle [\rho, T_1], t_0\rangle \\
&= \int_0^1 \big\langle \rho, \dot{t}_1 + [T_0, t_1] + [t_0, T_1]\big\rangle \\
&= \int_0^1 \Big\langle \rho, d\underbrace{(\dot{T}_1 + [T_0, T_1])}_{\mu}(t_0, t_1)\Big\rangle.
\end{aligned}$$

Hence there is a moment map $\mu(T_0, T_1) = \dot{T}_1 + [T_0, T_1]$, proving the claim.

Now recall the *Kähler reduction*: let $(M, \langle\cdot,\cdot\rangle, I)$ be a Kähler manifold with its metric and complex structure and let $\mu : M \to \mathfrak{h}^*$ be a moment map for a free isometric and holomorphic action of a Lie group H on M. If $c \in \mathfrak{h}$ is a fixed point of the coadjoint action, then the quotient $\mu^{-1}(c)/H$ is again a Kähler manifold, called the *Kähler quotient* and denoted by $M//_c H$.

We can apply this construction in our infinite-dimensional setting to $M = \mathscr{A}^{\mathbb{C}}$, $H = \mathscr{G}_0$, $c = 0$. Since the action of $\mathscr{G}_0$ extends to a global action of $\mathscr{G}_0^{\mathbb{C}}$, we can identify the Kähler quotient with a quotient of an open subset of $\mathscr{A}^{\mathbb{C}}$ by $\mathscr{G}_0^{\mathbb{C}}$. This open subset is the union of *stable* $\mathscr{G}_0^{\mathbb{C}}$-orbits, i.e., those that meet $\mu^{-1}(0)$. We shall see shortly that all $\mathscr{G}_0^{\mathbb{C}}$-orbits are stable and so the same procedure as in the real case gives:

$$\mathscr{A}^{\mathbb{C}}//_0\mathscr{G}_0 \simeq \mathscr{A}^{\mathbb{C}}/\mathscr{G}_0^{\mathbb{C}} \simeq G^{\mathbb{C}}. \tag{3}$$

What is the symplectic form on $G^{\mathbb{C}}$? Let us analyse the Kähler quotient construction. The level set $\mu^{-1}(0)$ of the moment map is given by the equation:

$$\dot{T}_1 = [T_1, T_0]. \tag{4}$$

As in the previous section, quotienting by $\mathscr{G}_0$ is equivalent to finding a gauge transformation $g \in \mathscr{G}$ with $g(0) = 1$ and sending T_0 to 0. Equation (4) is gauge-invariant and hence g makes T_1 constant. We obtain a map

$$(T_0(s), T_1(s)) \longmapsto \big(g(1), T_1(0)\big) \tag{5}$$

which gives an isomorphism

$$\mu^{-1}(0)/\mathscr{G}_0 \simeq G \times \mathfrak{g} \simeq T^*G,$$

where the last isomorphism identifies $\mathfrak{g}$ with right-invariant 1-forms. T^*G, being a cotangent bundle, has a canonical symplectic form and an easy calculation shows that the symplectic form on $\mu^{-1}/\mathscr{G}_0$ coincides with the one on T^*G.

We have now identified the Kähler manifold $\mu^{-1}(0)/\mathscr{G}_0$ with (possibly an open subset of) $G^{\mathbb{C}}$ as a complex manifold, and with T^*G as a symplectic manifold. To connect these two, we look again at the isomorphism (3), obtained by finding a complex gauge transformation $\tilde{g}$, $\tilde{g}(0) = 1$, sending $T_0 + iT_1$ to 0. If $T_0 + iT_1$ is already in $\mu^{-1}(0)$, i.e., it satisfies (4), then we can find $\tilde{g}$ in two stages as follows. First, we find a real gauge transformation $g(s)$ sending T_0 to 0 (with $g(0) = 1$). This means that $g.(T_0 + iT_1) = iT_1(0)$. This gives the map (5). Now the transformation $\exp(isT_1(0))$ makes $g.(T_0 + iT_1)$ identically zero and hence $\tilde{g}(s) = \exp(isT_1(0))g(s)$. Therefore $\tilde{g}(1) = \exp(iT_1(0))g(1)$ and so the Kähler metric on $G^{\mathbb{C}}$ arises from the identification $G^{\mathbb{C}} \simeq T^*G \simeq G \times \mathfrak{g}$ given by the polar decomposition.

This argument shows also that all $\mathscr{G}_0^{\mathbb{C}}$-orbits are stable, as we have just identified the Kähler quotient with whole $G^{\mathbb{C}}$ and not just with its open subset.

We remark that the passage from G with its bi-invariant metric to $T^*G \simeq G^{\mathbb{C}}$ with its Kähler structure is an example of the *adapted complex structure* construction (cf. [**LS91**]).

Quaternionise. Analogously to complexifying $\mathscr{A}$, we can quaternionise it:

$$\mathscr{A}^{\mathbb{H}} = \{T_0 + iT_1 + jT_2 + kT_3 : [0,1] \longmapsto \mathfrak{g} \otimes \mathbb{H}\}$$

On this space the natural L^2-metric (analogous to (1)) is Kähler for three anti-commuting complex structures I_1, I_2, I_3 given by right multiplication by i, j and k. Thus we get three symplectic forms ω_1, ω_2, ω_3, which can be explicitly computed. E.g., $\omega_1 = \int_0^1 dT_0 \wedge dT_1 - dT_2 \wedge dT_3$[(1)]. In other words, $\mathscr{A}^{\mathbb{H}}$ is *hyperkähler*.

As before $\mathscr{G}$ acts on $\mathscr{A}^{\mathbb{H}}$ preserving the metric and the symplectic forms by extending the actions we had for the real and complexified cases:

$$g.(T_0, T_1, T_2, T_3) = (gT_0g^{-1} - \dot{g}g^{-1}, gT_1g^{-1}, gT_2g^{-1}, gT_3g^{-1}).$$

The action of the subgroup $\mathscr{G}_0$ is free and Hamiltonian for all 3 symplectic forms, giving us *three* moment maps:

$$\begin{aligned} \mu_1 &= \dot{T}_1 + [T_0, T_1] - [T_2, T_3] \\ \mu_2 &= \dot{T}_2 + [T_0, T_2] - [T_3, T_1] \\ \mu_3 &= \dot{T}_3 + [T_0, T_3] - [T_1, T_2]. \end{aligned} \tag{6}$$

We can perform the *hyperkähler reduction*, i.e., consider the quotient

$$\mu_1^{-1}(0) \cap \mu_2^{-1}(0) \cap \mu_3^{-1}(0)/\mathscr{G}_0$$

which, as we shall show in a moment, is a hyperkähler manifold. The equations obtained by setting $\mu_i = 0$ in (6) are known as the *Nahm equations* [**Nah82**].

To see that we obtain a hyperkähler manifold, consider the complex-valued 2-form $\omega_2 + i\omega_3$ which is a holomorphic 2-form for the complex structure I_1. With respect to this complex symplectic form we get a holomorphic complex moment map

$$\mu_2 + i\mu_3 : \mathscr{A}^{\mathbb{H}} \longrightarrow \mathrm{Lie}(\mathscr{G}_0) \otimes \mathbb{C}$$

and so $(\mu_2 + i\mu_3)^{-1}(0)$ is an I_1-complex submanifold of $\mathscr{A}^{\mathbb{H}}$, in particular Kähler. Using this we can identify the hyperkähler quotient $N = \mu_1^{-1}(0) \cap \mu_2^{-1}(0) \cap \mu_3^{-1}(0)/\mathscr{G}_0$ with the Kähler quotient of $(\mu_2 + i\mu_3)^{-1}(0)$ by $\mathscr{G}_0$. Doing this for the complex structures I_2, I_3 (and corresponding holomorphic 2-forms) shows that N is a hyperkähler manifold.

---

(1) Recall that $\int_0^1 dT_0 \wedge dT_1$ is a shorthand for $\omega\big((t_0, t_1), (t_0', t_1')\big) = \int_0^1 \langle t_0, t_1'\rangle - \langle t_1, t_0'\rangle$.

To identify N let us go through this procedure in greater detail. We compute:

$$\begin{aligned}\mu_2 + i\mu_3 &= (\dot{T}_2 + i\dot{T}_3) + [T_0, T_2 + iT_3] - [T_3, T_1] - i[T_1, T_2] \\ &= \frac{d}{ds}(T_2 + iT_3) + [T_0 - iT_1, T_2 + iT_3].\end{aligned}$$

Just as in the real and complex cases, we now use the element $g \in \mathscr{G}^{\mathbb{C}}$ with $g(0) = 1$ which sends the first component $T_0 - iT_1$ to 0 and $T_2 + iT_3$ to the constant $(T_2 + iT_3)(0)$, giving a biholomorphism[(1)]

$$(N, I_1) \longrightarrow G^{\mathbb{C}} \times \overline{\mathfrak{g}^{\mathbb{C}}} \qquad (T_0, T_1, T_2, T_3) \longmapsto \big(g(1), (T_2 + iT_3)(0)\big),$$

where $\overline{\mathfrak{g}^{\mathbb{C}}}$ denotes $\mathfrak{g}^{\mathbb{C}}$ with the opposite complex structure. Thus (N, I_1) is biholomorphic to a cotangent bundle:

$$(N, I_1) \simeq G^{\mathbb{C}} \times \overline{\mathfrak{g}^{\mathbb{C}}} \simeq T^{0,1}G \simeq T^*G^{\mathbb{C}}.$$

The same construction can be done for the other complex structures I_2 and I_3.

Remark 2.3. The quaternionisation procedure of this section does not yield a Lie group for the simple reason that $\mathfrak{g} \otimes \mathbb{H}$ is not a Lie algebra. Nevertheless there is an algebraic structure (Kronecker product) on $\mathfrak{g} \otimes \mathbb{H}$ obtained the same way as for $\mathfrak{g} \otimes \mathbb{C}$: the Lie bracket in the first factor and the field (or skew-field) product in the second factor. The Nahm equations are closely related to this and it has always been my feeling that there should exist an algebraic structure (non-associative product?) on $N \simeq T^*G^{\mathbb{C}}$ reflecting the Kronecker product on $\mathfrak{g} \otimes \mathbb{H}$.

Kähler potentials. First recall that on hyperkähler manifolds we don't just have three complex structures at our disposal but a whole 2-sphere of them which is the unit sphere in the imaginary quaternions. Our hyperkähler manifold N obtained previously as the moduli space of solutions to Nahm's equations on the interval $[0, 1]$ has a special property: all these complex structures are equivalent. In fact there is an isometric $SO(3)$-action on N which induces the $SO(3)$-action on the 2-sphere of complex structures. A matrix $A = [a_{ij}] \in SO(3)$ acts by

$$(T_0, T_1, T_2, T_3) \longmapsto (T_0, A\begin{pmatrix} T_1 \\ T_2 \\ T_3 \end{pmatrix}) = (T_0, \sum a_{1j}T_j, \sum a_{2j}T_j, \sum a_{3j}T_j).$$

(1) Again this requires proving that all $\mathscr{G}_0^{\mathbb{C}}$-orbits on $(\mu_2 + i\mu_3)^{-1}(0)$ are stable. An argument can be found in [**Don84**]: solving the remaining Nahm equation $\mu_1 = 0$ corresponds to finding a path in $G^{\mathbb{C}}/G$ stationary under certain positive Lagrangian. Donaldson's argument also shows that any $\mathscr{G}_0^{\mathbb{C}}$ orbit on $(\mu_2 + i\mu_3)^{-1}(0)$ meets $\mu_1 = 0$ in a *unique* $\mathscr{G}_0$-orbit - something that requires a proof in the infinite-dimensional setting.

Knowing this $SO(3)$-action explicitly (as an action on $T^*G^{\mathbb{C}}$, say) is of course equivalent to knowing the hyperkähler structure and it appears to be beyond reach. Instead, consider the $S^1 \subset SO(3)$ which fixes the complex structure I_1 (and ω_1). It is given by

$$(T_0, T_1, T_2 + iT_3) \longmapsto (T_0, T_1, e^{i\theta}(T_2 + iT_3))$$

and the vector field induced by the action is $\theta^* = (0, 0, -T_3 + iT_2)$. Compute the moment map for ω_1:

$$i_{\theta^*}\omega_1 = \int_0^1 \langle T_3, dT_3\rangle - (-\langle dT_2, T_2\rangle) = \frac{1}{2} d \int_0^1 \|T_2\|^2 + \|T_3\|^2.$$

Hence the moment map for this action of S^1 is $\mu_{S^1} = (\|T_2\|^2 + \|T_3\|^2)/2$. It turns out also to be a *Kähler potential* for the complex structure I_2, i.e., $dI_2 d\mu_{S^1} = \omega_2$:

$$\begin{aligned} dI_2 d\mu_{S^1} &= dI_2 \int_0^1 \langle T_3, dT_3\rangle + \langle T_2, dT_2\rangle \\ &= d\int_0^1 \langle T_3, -dT_1\rangle + \langle T_2, -dT_0\rangle \\ &= \int_0^1 (dT_1 \wedge dT_3 + dT_0 \wedge dT_2) = \omega_2. \end{aligned}$$

This is indeed a general fact: an isometric circle action on a hyperkähler manifold which preserves one of the complex structures and rotates the others will give a Kähler potential for another [**HKLR87**].

Spectral curves. Following Hitchin, we can say something about the Kähler potential found in the previous section in the case of $G = SU(k)$.

Consider again the level set for the complex moment map $\mu_2 + i\mu_3$. It is given by the equation

$$(T_2 \dot{+} iT_3) + [\underbrace{T_0 - iT_1}_{\alpha}, \underbrace{T_2 + iT_3}_{\beta}] = 0 \qquad \text{i.e.} \qquad \dot{\beta} = [\beta, \alpha]$$

which is a *Lax equation* and thus implies that $\beta(t)$ lies in a fixed adjoint orbit of $G^{\mathbb{C}}$.

If $G = SU(k)$ the spectrum of β is constant and consists of k points in $\mathbb{C}$ but it depends on the complex structure we are using. Thus, for any complex structure, i.e., any point in $\mathbb{P}^1$, we get k points in $\mathbb{C}$. This is equivalent to getting k points in $T_\zeta\mathbb{P}^1$ for every $\zeta \in \mathbb{P}^1$, meaning that we are getting a k-fold ramified covering of $\mathbb{P}^1$ in $T\mathbb{P}^1$: an algebraic curve S, called a *spectral curve*. To understand S, notice that Nahm's equations are equivalent to $\dot{\beta}(\zeta) = [\beta(\zeta), \alpha(\zeta)]$ for all $\zeta \neq \infty$, where $\beta(\zeta) = \beta + (\alpha + \alpha^*)\zeta - \beta^*\zeta^2$ and $\alpha(\zeta) = \alpha - \beta^*\zeta$. Therefore S is described as a compactification of the spectrum of $\beta(\zeta)$ for all $\zeta \neq \infty$.

For $G = SU(k)$, we have $S = \{(\eta,\zeta) \mid \det(\eta \cdot 1 - \beta(\zeta)) = 0\}$ where $\zeta \in \mathbb{C}$ is an affine coordinate on $\mathbb{P}^1$ and η is a fibre coordinate of $T\mathbb{P}^1$. Hence

$$S = \{(\zeta,\eta) \mid \eta^k + a_1(\zeta)\eta^{k+1} + \cdots + a_k(\zeta) = 0\}$$

where the a_j are polynomials of degree $2k$.

Next try to get Nahm's equations back from the curve S. A theorem by Beauville [**Bea77**, 1.4] gives that

$$\{A(\zeta) = A_1 + A_2\zeta + A_3\zeta^2; A_i \in \mathfrak{gl}(k,\mathbb{C}), \det(\eta \cdot 1 - A(\zeta)) = 0\}/GL(k,\mathbb{C}) \simeq J^{g-1}(S) - \Theta,$$

where g is the genus of S, $J^{g-1}(S)$ the Jacobian of line bundles of degree $g-1$ on S and Θ the theta-divisor. The idea of the proof is to consider the sequence of sheaves on $T\mathbb{P}^1$

$$0 \longrightarrow \pi^*\mathcal{O}(-2)^k \longrightarrow \pi^*\mathcal{O}^k \longrightarrow E \longrightarrow 0, \tag{7}$$

where the first map is $\eta \cdot 1 - A_1 - A_2\zeta - A_3\zeta^2$ and E is the cokernel. The sheaf E is supported on S and if S is smooth, then E is a line bundle of degree $g+k-1$. After tensoring (7) with $\pi^*\mathcal{O}(-1)$, we conclude that $E(-1)$ is a line bundle of degree $g-1$ with no global sections and, hence, $E(-1) \in J^{g-1}(S) - \Theta$.

Solutions to Nahm's equations for $U(k)$ correspond to a linear flow on $J^{g-1}(S)$ in a tangent direction corresponding to a fixed line bundle L on $T\mathbb{P}^1$ with $c_1(L) = 0$. It has transition functions $e^{\eta/\zeta}$ and induces a line bundle of degree 0 on S and thus gives us a vector field on $J^{g-1}(S)$.

Theorem 2.4. *(Hitchin,* [**Hit98**, Prop 5.2]*) For a smooth S, let ϑ be the Riemann theta function on $J^{g-1}(S) \simeq J(S)$, where the isomorphism is given by $E \mapsto E(-k+2)$. Then the Kähler potential μ_{S^1} for ω_2 on $T^*SL_n(\mathbb{C})$ is given by*

$$\mu_{S^1} = \frac{\vartheta'(a)}{a} - \frac{\vartheta'(b)}{b} + \frac{2\vartheta^{(N+2)}}{(N+1)(N+2)\vartheta^N(0)} - \frac{1}{6}a_2(1),$$

where $a, b \in J^{g-1}(S)$ correspond to the triples $\big(T_1(0), T_2(0), T_3(0)\big)$ and $\big(T_1(1), T_2(1), T_3(1)\big)$, $'$ is the derivative in L direction, $a_2(\zeta)$ is the second coefficient in $\eta^k + a_1(\zeta)\eta^{k-1} + a_2(\zeta)\eta^{k-2} + \cdots$ and $N = k(k^2-1)/6$.

With this we get a Kähler potential in terms of theta functions. It is non-algebraic, but this is only one problem. We now move to a setting where things are algebraic, and yet the hyperkähler structure is hardly better understood.

3. Orbits

Adjoint orbits and the 'baby Nahm equation'. Consider again the 'baby Nahm equation', i.e., the Lax equation

$$\dot{T}_1 = [T_1, T_0] \tag{8}$$

this time on a half-line: $T_0, T_1 : [0, +\infty) \to \mathfrak{g}$. Assume that for $s \to \infty$, $T_0(s)$ converges exponentially fast to 0 and $T_1(s)$ converges exponentially fast to a fixed element $\tau \in \mathfrak{g}$. For simplicity assume that τ is *regular*, i.e., its centraliser $Z[\tau]$ is a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$.

Let $\mathcal{N}_\tau$ be the space of solutions to (8) satisfying these boundary conditions and define

$$\mathcal{G} = \{g : [0,\infty) \longrightarrow G \mid \lim_{t\to\infty} g(s) \in \exp(\mathfrak{h}),\ \text{converging exponentially fast}\}.$$

Also define the subgroup $\mathcal{G}_0 = \{g \in \mathcal{G} \mid g(0) = 1\}$. While these groups and solutions have different boundary conditions from those we used in the previous sections, we can define the same action as in (2):

$$T_0 \longmapsto gT_0g^{-1} - \dot{g}g^{-1} \qquad \text{and} \qquad T_1 \longmapsto gT_1g^{-1}.$$

For every $(T_0, T_1) \in \mathcal{N}_\tau$ we find an element $g \in \mathcal{G}_0$ such that

$$g.(T_0, T_1) = (0, T_1(0)) \qquad \text{where} \qquad T_1(0) = g(\infty)\tau g(\infty)^{-1}.$$

Thus we get that the quotient by the action is the *adjoint orbit* of τ: $\mathcal{N}_\tau/\mathcal{G}_0 \simeq \mathcal{O}_\tau$. As in the previous sections this is a Kähler quotient. Its symplectic form is the Kostant-Kirillov-Souriau form on $\mathcal{O}_\tau$: identifying $T_x\mathcal{O}_\tau$ with $\{[\rho, x] : \rho \in \mathfrak{g}\}$ we have:

$$\omega([\rho, x], [\rho', x]) = \langle [\rho, \rho'], x\rangle.$$

To find out the complex structure of our Kähler quotient, let, again, $\alpha = T_0 - iT_1$ and rewrite the 'baby Nahm equation' as $\dot{\alpha + \alpha^*} = [\alpha^*, \alpha]$, where α converges exponentially fast to $i\tau$ as $t \to \infty$. Hence we can find $g \in \mathcal{G}^{\mathbb{C}}$ such that $\alpha = g(i\tau)g^{-1} - \dot{g}g^{-1}$.

Note that g is not unique: Given g satisfying the equation, we can replace it by $g'(t) = g(t)e^{i\tau t}pe^{-i\tau t}$, where p is constant, to give us the same α. However, the resulting $g'(t)$ will converge to an element of the Cartan subalgebra $\mathfrak{h}$ if and only if $\mathrm{ad}(i\tau)$ acts on p with non-negative eigenvalues. That is, if p is in the Borel subgroup B determined by $\mathrm{ad}(i\tau)$.

With this, the map $\alpha \mapsto g(0)$ gives a biholomorphism between $\mathcal{N}_\tau/\mathcal{G}_0$ and $G^{\mathbb{C}}/B$: the complex structure of $\mathcal{O}_\tau$ is that of a generalised flag variety.

Exercise 3.1. Adapt the construction to non-regular τ.

Adjoint orbits and Nahm's equations. We now quaternionise to return to Nahm's equations

$$\begin{aligned}\dot{T}_1 + [T_0, T_1] - [T_2, T_3] &= 0\\ \dot{T}_2 + [T_0, T_2] - [T_3, T_1] &= 0\\ \dot{T}_3 + [T_0, T_3] - [T_1, T_2] &= 0.\end{aligned}$$

While the equations are the same as in section 2, this time our solutions are defined on the half-line $[0, +\infty)$. Assume that we have exponentially fast convergence

$$T_0(s) \longrightarrow 0 \qquad \text{and} \qquad T_i(s) \longrightarrow \tau_i \qquad \text{for} \qquad s \longrightarrow \infty,$$

where the τ_i are fixed. The Nahm equations imply that $[\tau_i, \tau_j] = 0$ for $i, j = 1, 2, 3$. Thus if τ_i are regular for $i = 1, 2, 3$, then they lie in a common Cartan subalgebra $\mathfrak{h}$. Actually we will assume a bit less than regularity for each τ_i, namely that the centraliser of the triple $Z(\tau_1, \tau_2, \tau_3) = Z(\tau_1) \cap Z(\tau_2) \cap Z(\tau_3)$ is a Cartan subalgebra $\mathfrak{h}$.

Let $\mathscr{N}_{\tau_1,\tau_2,\tau_3}$ denote the set of solutions of Nahm's equations with the given boundary conditions. Then the quotient $\mathscr{M}_{\tau_1,\tau_2,\tau_3} = \mathscr{N}_{\tau_1,\tau_2,\tau_3}/\mathscr{G}_0$ is a hyperkähler manifold [**Kro90a**].

To find out what are the complex structures of $\mathscr{M}_{\tau_1,\tau_2,\tau_3}$ we write, as before, $\alpha = T_0 - iT_1$ and $\beta = T_2 + iT_3$. We can write[(1)]

$$(\mathscr{M}_{\tau_1,\tau_2,\tau_3}, I_1) = \{(\alpha, \beta) : \dot{\beta} = [\beta, \alpha]\}/\mathscr{G}_0^{\mathbb{C}}.$$

If $\tau_2 + i\tau_3$ is regular in $\operatorname{Lie} G^{\mathbb{C}}$, then the map $(\alpha, \beta) \longmapsto \beta(0)$ gives a biholomorphism

$$(\mathscr{M}_{\tau_1,\tau_2,\tau_3}, I_1) \longrightarrow \mathscr{O}_{\tau_2+i\tau_3}$$

where $\mathscr{O}_{\tau_2+i\tau_3}$ denotes the adjoint $G^{\mathbb{C}}$-orbit of $\tau_2 + i\tau_3$. The idea for proving this is to find a $g \in \mathscr{G}^{\mathbb{C}}$ such that

$$(\alpha, \beta) = \left(g(i\tau_1)g^{-1} - \dot{g}g^{-1}, g(\tau_2 + i\tau_3)g^{-1}\right).$$

We conclude that the generic complex structure of $\mathscr{M}_{\tau_1,\tau_2,\tau_3}$ is that of a complex adjoint orbit. The other complex structures are those of some holomorphic bundles over generalised flag manifolds:

Example 3.2. If $\tau_2 = \tau_3 = 0$, then τ_1 is regular and we can write (cf. end of the previous subsection)

$$(\alpha, \beta) = \left(g(i\tau_1)g^{-1} - \dot{g}g^{-1}, g(\exp(i\tau_1 t)n\exp(i\tau_1 t))g^{-1}\right),$$

(1) Once again, this isomorphism depends on the fact, proved in [**Kro90a**], that all $\mathscr{G}_0$-orbits are stable.

where n is in the nilradical $\mathfrak{n}$ of $\mathfrak{b}$, the Borel subalgebra given by $\mathrm{ad}(i\tau_1)$. Thus we have a map

$$\left(\mathscr{M}_{\tau_1,0,0}, I_1\right) \longrightarrow G^{\mathbb{C}} \times_B \mathfrak{n} \simeq T^*\left(G^{\mathbb{C}}/B\right) \qquad (\alpha,\beta) \longmapsto g(0)$$

which is biholomorphic.

What about other orbits? Notice that there are other non-constant 'standard' solutions to Nahm's equations. For example, consider $\mathfrak{su}(2)$ with the basis $\{e_1, e_2, e_3\}$ and relations $[e_1, e_2] = -e_3, \ldots$. If $\sigma : \mathfrak{su}(2) \to \mathfrak{g}$ is a homomorphism then

$$T_i(s) = \sigma(e_i)/(s+1) \qquad \text{for } i = 1,2,3 \tag{9}$$

is a solution to Nahm's equations. The Jacobson-Morozov theorem gives us a one-to-one correspondence between nilpotent orbits in $\mathfrak{g}^{\mathbb{C}}$ and homomorphisms $\mathfrak{su}(2) \to \mathrm{Lie}\, G$ up to conjugation. If we consider solutions to Nahm's equations asymptotic to (9), we also get a hyperkähler metric, this time on an appropriate nilpotent orbit [**Kro90b**].

Most generally, we fix $\tau_1, \tau_2, \tau_3 \in \mathfrak{h}$ and $\sigma : \mathfrak{su}(2) \to Z(\tau_1, \tau_2, \tau_3)$ to get the set

$$\mathscr{M}_{\tau_1,\tau_2,\tau_3;\sigma} = \{\text{solutions asymptotic to } \tau_i + \frac{\sigma(e_i)}{s+1} \text{ for } i = 1,2,3\}/\mathscr{G}_0,$$

where both "asymptotic" and $\mathscr{G}_0$ need to be defined carefully [**Biq96, Kov96**].

This is again a hyperkähler manifold. Its generic complex structure is that of the orbit of

$$(\underbrace{\tau_2 + i\tau_3}_{\text{semisimple}}) + (\underbrace{\sigma(e_2) + i\sigma(e_3)}_{\text{nilpotent}})$$

if $Z(\tau_2, \tau_3) = Z(\tau_1, \tau_2, \tau_3)$. The complex-symplectic form $\omega_2 + i\omega_3$ is the Kostant-Kirillov-Souriau form of the complex co-adjoint orbit.

With respect to other complex structures $\mathscr{M}_{\tau_1,\tau_2,\tau_3;\sigma}$ is a bundle over a generalised flag manifold.

Even this is not the most general construction of hyperkähler metric on (co)-adjoint orbits – see [**DSC97**]. Nevertheless the manifolds $\mathscr{M}_{\tau_1,\tau_2,\tau_3;\sigma}$ are probably the most general *algebraic* hyperkähler structures on coadjoint orbits. They are also most general *geodesically complete* hyperkähler structures on coadjoint orbits:

Theorem 3.3 ([Bie97]). *If M is a complete hyperkähler manifold with a hyperkähler action of a compact semisimple Lie group G such that for one complex structure $G^{\mathbb{C}}$ acts locally transitively, then M is equivariantly isomorphic to one of the $\mathscr{M}_{\tau_1,\tau_2,\tau_3}$.*

The idea for proving the theorem is to take the moment maps μ_1, μ_2, μ_3 for the G-action on M and consider the gradient flow maps $m(t)$ of $\|\mu_1\|^2$, where I_1 is a complex structure for which $G^{\mathbb{C}}$ is locally transitive. Then the T_i given by $T_i(t) = \mu_i(m(t))$ satisfy Nahm's equations.

Explicit description of the hyperkähler metrics on adjoint orbits? Biquard [**Biq98**] shows that the hyperkähler structure of $\mathscr{M}_{\tau_1,\tau_2,\tau_3;\sigma}$ is algebraic. Nevertheless, there is no algebraic construction of these structures, except for a few special cases ([**BG98, KS01a, KS01b**]). One would like, at the very least, to relate the hyperkähler structure of $\mathscr{M}_{\tau_1,0,0}$ to the Kähler structure of $\mathscr{O}_{\tau_1}$. This has been done by Biquard and Gauduchon [**BG98**], but only when $\mathscr{O}_{\tau_1}$ is a Hermitian symmetric space.

The other possibility (for $G = SU(k)$) is suggested by Hitchin [**Hit98**], analogous to his approach to $T^*SL(k,\mathbb{C})$ described in section 1.5. For $\mathscr{M}_{\tau_1,\tau_2,\tau_3}$, we have a fixed spectral curve S given by the equation

$$\det\big(\eta\cdot 1-(\tau_2+i\tau_3)-2i\tau_1\zeta+(-\tau_2+i\tau_3)\zeta^2\big)=0. \tag{10}$$

S is reducible, in fact a union of rational curves. When $\tau_2 = \tau_3 = 0$, we have, as in section 1.4, a Kähler potential and once again we can try to describe it in terms of theta functions. This time, however, the theta functions are rational [**Mum84**].

While nobody succeeded yet in describing the hyperkähler structure of $\mathscr{M}_{\tau_1,0,0}$ using this approach, thinking along these lines leads to an amusing observation. Recall that if the curve (10) has only nodes, then $J^{g-1}(S)$ has a canonical compactification $\overline{J^{g-1}(S)}$ obtained by adding invertible sheaves of degree $g'-1$ and semistable multidegree on partial normalisations S' of S. The definition of a semistable multidegree is one which allows us to extend the notion of the theta-divisor to $\overline{J^{g-1}(S)}$: semistable multi-degrees are those for which the usual definition of the theta-divisor actually gives a divisor [**Bea77**].

Observe also that the curve S is invariant under the antiholomorphic involution

$$\zeta\mapsto -1/\bar{\zeta},\quad \eta\mapsto -\eta/\bar{\zeta}^2.$$

This induces an antiholomorphic involution on $H^1(S,\mathscr{O}_S)$ and we can speak of *real* line bundles of degree zero. Similarly we define a line bundle L of degree $g-1$ to be real if $L(-k+2)$ is real (here $G = SU(k)$). This extends to the compactified Jacobian and for any $U\subset\overline{J^{g-1}(S)}$ we write $U_{\mathbb{R}}$ for the corresponding real sheaves.

We have:

Proposition 3.4. *Let $\tau_2+i\tau_3$ be a regular semisimple element of $SL(k,\mathbb{C})$ and let $\mathscr{O}_{\tau_2+i\tau_3}$ denote its adjoint orbit. Choose a τ_1 in the same Cartan subalgebra as τ_2 and τ_3 so that the curve S defined by* (10) *has only nodes as singularities. Then there is a canonical algebraic isomorphism between $\mathscr{O}_{\tau_2+i\tau_3}/SU(k)$ and the closure in $\big(\overline{J^{g-1}(S)}-\Theta\big)_{\mathbb{R}}$ of a connected component of $\big(J^{g-1}(S)-\Theta\big)_{\mathbb{R}}$.*

4. Applications to symmetric pairs and real orbits

Kostant-Sekiguchi correspondence. Let (G, K) be a compact symmetric pair, i.e., there exists an orthogonal decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m} \quad \text{with} \quad [\mathfrak{k}, \mathfrak{m}] \subset \mathfrak{m}, \ [\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{k}.$$

Then $\mathfrak{g}^* = \mathfrak{k} + i\mathfrak{m}$ is also a Lie algebra and if G^* is the corresponding Lie group, then (G^*, K) is the dual symmetric pair.

Example 4.1. $G = SU(n)$ and $K = S(U(p) \times U(q))$ with $p+q = n$. $\mathfrak{m}$ consists of the two off-diagonal $p \times q$- and $q \times p$-blocks in the algebra of skew-hermitian matrices and $SU(n)^* = SU(p, q)$, i.e., the group preserving an indefinite hermitian form.

Example 4.2. $G = SU(n)$ and $K = SO(n)$. Then $\mathfrak{m}$ is the space of imaginary symmetric matrices. It follows that $SU(n)^* = SL(n, \mathbb{R})$.

The Kostant-Sekiguchi correspondence tells us that there is a one-to-one correspondence between nilpotent orbits of G^* and $K^{\mathbb{C}}$-orbits of nilpotent elements in $\mathfrak{m}^{\mathbb{C}} \subset \operatorname{Lie} G^{\mathbb{C}}$. This somewhat mysterious fact has been given a natural interpretation in terms of Nahm's equations by Vergne [**Ver95**]. We shall explain this now.

$(\mathfrak{g}, \mathfrak{k})$**- valued solutions to Nahm's equations.** Let $\mathscr{M}$ be any moduli space of $\mathfrak{g}$ valued solutions to Nahm's equations given as a quotient of the space $\mathscr{N}$ of solutions with fixed boundary conditions by an appropriate gauge group $\mathscr{G}_0$. Following Saksida [**Sak99**], we consider the subset $\mathscr{N}^{(\mathfrak{g}, \mathfrak{k})}$ of $(\mathfrak{g}, \mathfrak{k})$*-valued solutions*, i.e., those solutions in $\mathscr{N}$ with $T_0(s), T_1(s) \in \mathfrak{k}$ and $T_2(s), T_3(s) \in \mathfrak{m}$ for all s. We define their moduli space as

$$\mathscr{M}^{(\mathfrak{g}, \mathfrak{k})} = \mathscr{N}^{(\mathfrak{g}, \mathfrak{k})} / \{g \in \mathscr{G}_0; g(s) \in K \ \text{ for all } s\}.$$

The usual $\mathfrak{g}$-valued solutions to Nahm's equations can be thought of as $\mathscr{M}^{(\mathfrak{g} \oplus \mathfrak{g}, \mathfrak{g})}$.

Recall that having chosen a complex structure we can write the moduli space of solutions as the quotient of the level set of the complex moment map $\dot{\beta} = [\beta, \alpha]$ by the complexified Lie group. Look again at the $(\mathfrak{g}, \mathfrak{k})$-valued solutions and choose complex structures I_1, I_3. This gives

$$\beta_1 = T_2 + iT_3, \qquad \alpha_1 = T_0 - iT_1, \qquad \beta_3 = T_1 + iT_2, \qquad \alpha_3 = T_0 - iT_3,$$

satisfying $\dot{\beta}_i = [\beta_i, \alpha_i]$ for $i = 1, 3$, and where, for all s, $\alpha_1(s) \in \mathfrak{k}^{\mathbb{C}}, \beta_1(s) \in \mathfrak{m}^{\mathbb{C}}, \alpha_3(s) \in \mathfrak{g}^*, \beta_3(s) \in \mathfrak{g}^*$. Let us also define two subgroups of the gauge group $\mathscr{G}_0^{\mathbb{C}}$:

$$\mathscr{K}_0^{\mathbb{C}} = \{g \in \mathscr{G}_0^{\mathbb{C}}; g(s) \in K^{\mathbb{C}} \ \text{ for all } s\},$$

$$\mathscr{G}_0^* = \{g \in \mathscr{G}_0^{\mathbb{C}}; g(s) \in G^* \ \text{ for all } s\}.$$

Then we have identifications

$$\begin{aligned}\mathscr{M}^{(\mathfrak{g},\mathfrak{k})} &= \mathscr{N}^{(\mathfrak{g},\mathfrak{k})}/\mathscr{K}_0\\ &\overset{\varphi_1}{\simeq} \{\dot\beta_1=[\beta_1,\alpha_1]\mid \alpha_1\in\mathfrak{k}^{\mathbb{C}},\beta_1\in\mathfrak{m}^{\mathbb{C}}\}/\mathscr{K}_0^{\mathbb{C}}\\ &\overset{\varphi_3}{\simeq} \{\dot\beta_3=[\beta_3,\alpha_3]\mid \alpha_3\in\mathfrak{g}^*,\beta_3\in\mathfrak{g}^*\}/\mathscr{G}_0^*.\end{aligned}$$

The proof of these identifications relies on solving the third Nahm equation: this, as for the usual Nahm's equations (see the footnote on p. 5), is equivalent to finding a stationary path for a positive geodesically convex potential in negatively curved spaces $K^{\mathbb{C}}/K$ and G^*/K.

We now consider the spaces $\mathscr{M}^{(\mathfrak{g},\mathfrak{k})}$ for the moduli spaces of Nahm's equations considered in these lectures: i.e., $T^*G^{\mathbb{C}}$ and adjoint orbits.

For the moduli space N defining a hyperkähler structure on $T^*G^{\mathbb{C}}$, $N^{(\mathfrak{g},\mathfrak{k})}$ is isomorphic via φ_1 to $K^{\mathbb{C}}\times\mathfrak{m}^{\mathbb{C}}$ which is a complex submanifold of $G^{\mathbb{C}}\times\overline{\mathfrak{g}^{\mathbb{C}}}$ and via φ_3 to $G^*\times\mathfrak{g}^*\simeq T^*G^*$. With this we get a canonical K-invariant Kähler metric on T^*G^*.

Real orbits. It is more interesting to consider $\mathscr{M}^{(\mathfrak{g},\mathfrak{k})}_{\tau_1,\tau_2,\tau_3;\sigma}$, where $\mathscr{M}_{\tau_1,\tau_2,\tau_3;\sigma}$ are the moduli spaces described in section 2, defining hyperkähler structures on adjoint orbits of $G^{\mathbb{C}}$. Assume that I_1, I_3 are generic so that

$$\begin{aligned}(\mathscr{M}_{\tau_1,\tau_2,\tau_3;\sigma}, I_1)\simeq\mathscr{O}_{v_1} &\qquad\text{with}\qquad v_1=\tau_2+i\tau_3+\sigma(e_2)+i\sigma(e_3)\\ (\mathscr{M}_{\tau_1,\tau_2,\tau_3;\sigma}, I_3)\simeq\mathscr{O}_{v_3} &\qquad\text{with}\qquad v_2=\tau_1+i\tau_2+\sigma(e_1)+i\sigma(e_2).\end{aligned}$$

We assume that the restricted moduli space $\mathscr{M}^{(\mathfrak{g},\mathfrak{k})}_{\tau_1,\tau_2,\tau_3;\sigma}$ is not empty, i.e., $\tau_1\in\mathfrak{k}$ and $\tau_2,\tau_3\in\mathfrak{m}$. Then, restricting the isomorphism $(\alpha,\beta)\mapsto\beta(0)$ gives, as in [**Kro90a, Biq96, Kov96**]:

$$\varphi_1:\mathscr{M}^{(\mathfrak{g},\mathfrak{k})}_{\tau_1,\tau_2,\tau_3;\sigma}\xrightarrow{\sim}\mathscr{O}_{v_1}\cap\mathfrak{m}^{\mathbb{C}}\qquad\text{and}\qquad \varphi_3:\mathscr{M}^{(\mathfrak{g},\mathfrak{k})}_{\tau_1,\tau_2,\tau_3;\sigma}\xrightarrow{\sim}\mathscr{O}_{v_3}\cap\mathfrak{g}^*.$$

Thus we have a diffeomorphism

$$\mathscr{O}_{v_1}\cap\mathfrak{m}^{\mathbb{C}}\xrightarrow{\sim}\mathscr{O}_{v_3}\cap\mathfrak{g}^*. \tag{11}$$

In the case $\tau_1=\tau_2=\tau_3=0$ this is the Vergne isomorphism [**Ver95**].

Observe that $\mathscr{O}_{v_3}\cap\mathfrak{g}^*$ is a finite union of G^*-orbits and $\mathscr{O}_{v_1}\cap\mathfrak{m}^{\mathbb{C}}$ is $K^{\mathbb{C}}$-invariant. To show that $\mathscr{O}_{v_1}\cap\mathfrak{m}^{\mathbb{C}}$ is a union of $K^{\mathbb{C}}$-orbits we need

Lemma 4.3. *(cf.* [**Bry98**]*) $K^{\mathbb{C}}$ acts transitively on connected components of $\mathscr{O}_{v_1}\cap\mathfrak{m}^{\mathbb{C}}$.*

*Proof***.** We need to show that the action of $K^{\mathbb{C}}$ on $\mathscr{O}_{v_1}\cap\mathfrak{m}^{\mathbb{C}}$ is infinitesimally transitive. Let $x\in\mathscr{O}_{v_1}\cap\mathfrak{m}^{\mathbb{C}}$ and consider

$$T_x\mathscr{O}_{v_1}=\{[\rho,x]:\rho\in\mathfrak{g}^{\mathbb{C}}\}\simeq[\mathfrak{k}^{\mathbb{C}},x]\oplus[\mathfrak{m}^{\mathbb{C}},x]\subset\mathfrak{m}^{\mathbb{C}}\oplus\mathfrak{k}^{\mathbb{C}}.$$

With this we have that $T_x\mathcal{O}_{v_1}\cap\mathfrak{m}^{\mathbb{C}}\simeq[\mathfrak{k}^{\mathbb{C}},x]$ and so $K^{\mathbb{C}}$ acts locally transitively as required. □

We can extend the Vergne diffeomorphism and the Kostant-Sekiguchi correspondence to non-nilpotent orbits as follows:

Proposition 4.4. *Let (G,K), $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{m}$, be a compact symmetric pair. Let $\tau_1\in\mathfrak{k}$, $\tau_2,\tau_3\in\mathfrak{m}$ be such that their $\mathfrak{g}$-centralisers satisfy:*

$$Z(\tau_1)\supset Z(\tau_2)\cap Z(\tau_3),\; Z(\tau_3)\supset Z(\tau_1)\cap Z(\tau_2).$$

Then there is a one-to-one correspondence between G^*-orbits whose closure contains the orbit of $\tau_1+i\tau_2$ and $K^{\mathbb{C}}$-orbits in $\mathfrak{m}^{\mathbb{C}}$ whose closure contains the $K^{\mathbb{C}}$-orbit of $\tau_2+i\tau_3$. Moreover, for each pair of corresponding orbits, there exists a canonical K-equivariant diffeomorphism between them.

In particular, setting $\tau_1=\tau_3=0$, we obtain:

Corollary 4.5. *Let (G,K), $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{m}$, be a compact symmetric pair. There is a $1-1$ correspondence between adjoint G^*-orbits of elements whose semisimple part lies in $i\mathfrak{m}$ and $K^{\mathbb{C}}$-orbits of elements of $\mathfrak{m}^{\mathbb{C}}$ whose semisimple part lies in $\mathfrak{m}$. Moreover, for each pair of corresponding orbits, there exists a canonical K-equivariant diffeomorphism between them.*

We finish with a simple example of the Vergne diffeomorphism. Let $\mathcal{O}$ be the non-zero nilpotent orbit in $\mathfrak{sl}_2(\mathbb{C})$. $\mathcal{O}$ can be identified with $\big(\mathbb{C}^2-0\big)/\mathbb{Z}_2$ using the map

$$(u,v)\longmapsto\begin{pmatrix} uv & u^2\\ -v^2 & -uv\end{pmatrix}. \tag{12}$$

We have the usual identification of $\mathbb{C}^2$ with $\mathbb{H}=\{x_0+ix_1+jx_2+kx_3\}$ (for the complex structure i) by letting $u=x_0+ix_1$ and $v=x_2+ix_3$. The hyperkähler structure on $\mathcal{O}$ is then that of $\big(\mathbb{H}-0\big)/\mathbb{Z}_2$. With respect to the complex structure corresponding to j, the map (12) is

$$(u,v)\longmapsto\begin{pmatrix} (u-i\bar v)(v+i\bar u) & (u-i\bar v)^2\\ -(v+i\bar u)^2 & (i\bar v-u)(v+i\bar u)\end{pmatrix}. \tag{13}$$

Consider now $\mathcal{O}\cap\mathfrak{sl}_2(\mathbb{R})=\{u^2,v^2,uv\in\mathbb{R}\}$. It consists of two orbits $\mathcal{O}_+$ and $\mathcal{O}_-$ of $\mathfrak{sl}_2(\mathbb{R})$ given respectively by $u^2+v^2>0$ and $u^2+v^2<0$. The condition $u^2+v^2>0$ is equivalent to $x_1=0$ and $x_3=0$, while $u^2+v^2<0$ is equivalent to $x_0=0$ and $x_2=0$. A direct computation show that the map (13) sends $\mathcal{O}_+$ and $\mathcal{O}_-$ to the sets of matrices of the form

$$\begin{pmatrix} ib & b\\ b & -ib\end{pmatrix}\text{ for }\mathcal{O}_+,\qquad \begin{pmatrix} -ib & b\\ b & +ib\end{pmatrix}\text{ for }\mathcal{O}_-.$$

These are precisely the two non-zero nilpotent orbits of $K^{\mathbb{C}} = SO(2,\mathbb{C})$ on $\mathfrak{m}^{\mathbb{C}} =$ {symmetric matrices}.

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STABLE COHOMOLOGY OF FINITE AND PROFINITE GROUPS

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***Abstract*.** Group cohomology of a finite group contains a quotient which reflects birational geometry of the quotient of a linear space with the linear group action by the group. In this series of lectures I describe some results and conjectures concerning this group cohomology, its analogue for profinite groups and birational invariants of algebraic varieties.

1. Introduction

This article is an expanded version of material presented at the Summer school "Algebraic groups", June 2005, at the University of Göttingen. It includes basic results on group cohomology with applications to algebraic geometry.

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2. Group Cohomology: algebraic approach

In the following we are mostly interested in finite groups, though most of the results remain valid for affine algebraic groups and profinite groups.

Let G be a finite group and F a finite G-module. In particular, F is an abelian group. Then $H^i(G,F)$ is defined via the *group ring*

$$\mathbb{Z}[G] = \{\sum a_i g_i \mid \text{ finite sums }\},$$

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so that the set of all cohomology groups with respect to all finite modules form an associative ring, with product corresponding to the product of cohomology coming from the tensor product of G-modules. Consider a free resolution S of $\mathbb{Z}$

$$1 \leftarrow \mathbb{Z} \xleftarrow{i_0} \mathbb{Z}[G] \xleftarrow{i_1} F_1 \xleftarrow{i_2} F_2 \leftarrow \dots,$$

where i_0 is the *augmentation map* (i.e., $i_0(g) = 1$ for all $g \in G$). This is an exact sequence of free $\mathbb{Z}[G]$-modules. Define

$$H^i(G,F) = H_i(\mathrm{Hom}(S,F)) = \ker(i_j^*)/\mathrm{im}(i_{j-1}^*).$$

For a finite group G there is a standard resolution:

$$1 \leftarrow \mathbb{Z} \xleftarrow{i_0} \mathbb{Z}[G] \xleftarrow{i_1} \mathbb{Z}[G]^{\#G} \xleftarrow{i_2} \mathbb{Z}[G]^{\cdots} \leftarrow \dots$$

with the boundary homomorphism

$$\begin{aligned} \partial^r[g_1,\dots,g_r] &= g_1[g_2,\dots,g_r] + \sum(-1)^i[g_1,\dots,(g_i g_{i+1}),\dots g_r] \\ &\quad +(-1)^r[g_1,\dots,g_r] \end{aligned}$$

This complex is isomorphic to the simplicial chain complex of a simplicial space BG with r-simplices indexed by $[g_1,\dots,g_r]$. The space BG is in fact a quotient EG/G of another space EG represented by a simplicial complex with a free action of G. Any G-module F induces a sheaf on EG/G (denoted by $\tilde{F}$).

The derivative in the chain complex for BG looks asymmetric, but it becomes nicer if we consider EG and the simplices $[g_1,\dots,g_r] = G^r$ as a quotient G^{r+1}/G by the diagonal.

The r-simplices in EG are indexed by G^{r+1} (before taking the quotient by G) with simplices of dimension n given by ordered sets of vertices $(g_1,g_2,\dots,g_{n+1})$. We define a natural boundary operator on these simplices via identification with a standard simplex in $\mathbb{R}^{n+1}$ given by equations $x_i \geqslant 0$ and $\sum x_i = 1$ with vertices g_i given by equations $x_j = 0, j \neq i, x_i = 1$ with induced orientation. The space EG is contractible. In order to describe the contraction we notice first that for the group $G = e$ (group with only one element) the corresponding complex $Ee = e \in [e,e] \in \dots[e,e,\dots.e] \in \dots$ contains one simplex of each dimension with :

$\dots\partial[e,\dots,e](2n\text{-times}) = [e,\dots,e],(2n-1\text{-times})$. $\dots\partial(e,\dots,e)(2n-1\text{-times}) = 0$. This complex is a cellular decomposition of an infinite-dimensional ball, hence contractible. For an arbitrary G, the complex EG contains complexes $Eg, g \in G$ isomorphic to Ee and we can contract all of them independently $Eg \to g \in EG$ where g is a 0-dimensional subcomplex $G \subset EG$. Now we can define a contraction of EG to the simplex $\Delta = (g_1,\dots,g_n)$, where g_i runs through G once without repetitions. Any simplex $(g_1,\dots,g_N)$ (now g_i may repeat) will contract to a subsimplex of Δ which is a linear combination of different vertices of $(g_1,\dots,g_N)$. Thus, we have natural contraction of EG to a simplex whose dimension is $\#G-1$. A similar construction works for infinite groups as well (see, for example, [**ML63**]).

We have $H^i(G,F) = H^i(BG,\tilde{F})$, by the definition of cohomology for simplicial spaces. The sheaf $\tilde{F}$ is defined using the representation of BG as a quotient of EG by G, so that $\tilde{F}$ is a locally constant sheaf on BG corresponding to F.

3. Topological approach

The cohomology groups constructed above depend only on the homotopy type of the space BG. In particular, other models of the space BG give the same groups.

In general, let E be a contractible topological space with a free action of G, then E/G has the homotopy type of BG above. The space BG (modulo homotopy equivalence) can be characterized by the structure of its homotopy groups. Namely, it is a connected space, which means that $\pi_0(BG)=1$, with $G=\pi_1(BG)$ and $\pi_i(BG)=0$, for $i \geqslant 2$. The universal covering EG of BG is connected and has trivial homotopy groups, hence is contractible. Thus the action of G on EG introduced above is the action of the fundamental group of the quotient space BG on its universal covering. Any G-module F defines a locally constant sheaf of groups $\mathscr{F}$ on BG. We define:

$$H^i(G,F) = H^i(BG,\mathscr{F}).$$

Unfortunately, BG is always an infinite complex for a finite group G, so the group G has non-trivial cohomology in arbitrary high dimensions.

Theorem 3.1. *The space BG is unique, up to homotopy equivalence. If E_1, E_2 are two different contractible spaces with a free G-action, then the following diagram commutes:*

$$\begin{tikzcd}
(E_1 \times E_2)/G \arrow[r] & E_1/G \arrow[r, "\sim"] & E_2/G \\
E_1 \times E_2 \arrow[r] \arrow[u] & E_1 \arrow[r] \arrow[u] & E_2 \arrow[u]
\end{tikzcd}$$

Proof. The homotopy equivalence is proven by the following diagram:

$$\begin{tikzcd} & (E_1 \times E_2)/G \arrow[dl, "E_2, f_2"'] \arrow[dr, "E_1, f_1"] & \\ E_1/G & & E_2/G. \end{tikzcd}$$

Here, the fibre of the map f_2(respectively f_1) is E_2 (respectively E_1), which is contractible. Fibrations with contractible fibres always have sections, so composing a section with the projection onto the second factor we get a homotopy equivalence. $\square$

The same reasoning shows that, if X is any space with a free group action of G, there is a homotopically unique map

$$\begin{tikzcd} X \arrow[r] \arrow[d, "\pi"] & E \arrow[d] \\ X/G \arrow[r, "f"'] & BG \end{tikzcd}$$

(again, the projection $(X \times E)/G \to X/G$ is a homotopy equivalence).

On cohomology this map induces a homomorphism $f^* : H^i(G, F) \to H^i(X/G, F)$, so BG is an analog of a "point" (the final object) in the category of spaces with free actions of G:

$$\begin{tikzcd} X \arrow[d] & \tilde{X} \arrow[d] \\ pt. & X/G \arrow[d] \\ & BG \end{tikzcd}$$

For example, if we have a map of G-coverings, we get a diagram:

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & Y \arrow[d] \\
X/G \arrow[r] \arrow[d] & Y/G \arrow[dl] \\
BG. &
\end{tikzcd}$$

We see that the map $H^*(BG, F) \to H^*(X/G, F)$ factors through $H^*(Y/G, F)$, giving an obstruction for the existence of a $G-$equivariant map from X to Y.

Conclusion: the following diagram shows that for spaces with the additional structure of a G-covering, we have a homotopically unique map $X/G \to BG$, because of the following diagram of maps:

$$\begin{tikzcd}
& (X \times E)/G \arrow[dl, "\sim"'] \arrow[dr] & \\
X/G \arrow[rr, dotted] & & E/G = BG
\end{tikzcd}$$

(a) The image $H^i(BG, F) \to H^i(X/G, F)$ defines an ordering on spaces with G-action and provides with natural obstruction to the existence of G-maps between spaces with G-actions.

(b) We could also replace $Hom(*, F)$ by $\otimes F$ in the beginning and get group homology. However, the cohomology groups have several advantages:

a) cup product

b) existence of the universal object. (In homology we would only get a map into $H_i(BG, F)$).

Recall that all cohomology classes are the images of certain universal classes: for any positive integers $n, l > 1$ there is a space $K(n; \mathbb{Z}/l)$ and a cohomology class $b \in H^n(K(n; \mathbb{Z}/l), \mathbb{Z}/l)$, such that any cohomology class $a \in H^n(X, \mathbb{Z}/l)$ is induced from $b \in H^n(K(n; \mathbb{Z}/l), \mathbb{Z}/l)$ by a homotopically unique map $f_a : X \to K(n; \mathbb{Z}/l)$ with $f_a^*(b) = a$. The existence of a special geometric model for the universal space $K(n; \mathbb{Z}/l)$ provides an opportunity to study proprties of the given cohomology group for an arbitary topological space (see, for example, [**ML63**] ,[**AM04**]).

Example. Complex projective space $\mathbb{CP}^\infty = K(2,\mathbb{Z})$ and since $\mathbb{CP}^\infty$ is also $BU(1)$-space with a natural complex one-dimensional vector bundle any cohomology class $H^2(X,Z)$ is also a characteristic class of a one-dimensional complex vector bundle on X.

Cohomology groups of low degree were discovered in a different disguise while solving natural problems in group theory (see for example [**ML63**], [**AM04**]):

(a) $H^1(G,N)$ appeared in the extension problem: For finite G-module N the group $H^1(G,N)$ classifies extensions

$$1 \to N \to \tilde{G} \to G \to 1$$

which have a section $s : G \to \tilde{G}$, modulo isomorphisms identical on N, G. Two such extensions are considered isomorphic if there is a diagram:

$$\begin{tikzcd}
1 \arrow[r] & N \arrow[r] \arrow[d, "id"'] & \tilde{G}' \arrow[r, shift left] \arrow[d, "iso"'] & G \arrow[l, shift left, "s"] \arrow[r] \arrow[d, "id"] & 1 \\
1 \arrow[r] & N \arrow[r] & \tilde{G} \arrow[r] & G \arrow[r] & 1
\end{tikzcd}$$

Note that the corresponding groups may be isomorphic when there is no isomorphism as above. The added rigidity in this definition provides the set of extensions with a natural additive structure which first cohomology group has.

(b) To any extension of G by N we can associate naturally a cohomology class in $H^2(G,N)$, which is non-trivial precisely when the corresponding extension has no section (this is an analogue of the Brauer class in algebraic geometry). To any such extension we can also associate a natural extension of G by N with a section (the analogue of Jacobian fibration in algebraic geometry). The group $H^2(G,N)$ parametrizes all extensions of G by N with the same "jacobian" extension, modulo isomorphisms which are identical on both N and G. Again, the set of extensions associated to the same extension with a section becomes naturally an abelian group only after introducing additional rigidity into the definition of an isomorphism.

(c) $H^3(G,N)$ appears as an obstruction group in the problem of constructing extensions (see, for example, [**AM04**],[**ML63**]).

Higher dimensional cohomology groups appear only as “obstruction functors”. Basically, each cohomology group measures an obstruction to the solution of some topological (algebraic) problem in the cohomology of smaller dimension.

4. Geometric approach

Any finite group has nontrivial cohomology in arbitary high dimensions (see for example Quillen-Venkov theorem below [**AM04**]). Thus, any model of BG is always infinite-dimensional.

How can one find good models for BG? The idea is to try to build BG as an inductive limit using nice geometric objects which represent BG up to some finite dimension.

If we consider $BG^{(i)} \subset BG$ such that $BG^{(i)}$ has the same i-skeleton as BG, then for any simplicial space Σ^{i-1} of dimension $\leqslant i-1$ the maps Σ^{i-1} to BG factor through $BG^{(i)}$:

$$\begin{tikzcd} \Sigma^{i-1} \arrow[rr] \arrow[dr] & & BG \\ & BG^{(i)} \arrow[ur] & \end{tikzcd}$$

We also have the cohomological property $H^k(BG^{(i)}, F) = H^k(BG, F)$, for $k \leqslant i-1$.

Example. (a) $\mathbb{CP}^\infty = B\mathbb{C}^*$ where $\mathbb{C}^*$ is considered as a topological group. The space $\mathbb{CP}^\infty$ contains copies of $\mathbb{CP}^n$ for all positive integers n and they have the same i-skeleton as $\mathbb{CP}^\infty$ for $i \leqslant 2n-1$.

(b)

$$B\mathbb{Z}/l = \mathbb{C}^\infty \setminus \{0\}/(\mathbb{Z}/l)$$

because $\mathbb{C}^\infty \setminus \{0\}$ is a contractible space with a free action of $\mathbb{Z}/l$.

Naturally, $\mathbb{C}^\infty \setminus \{0\}) \supset \dots \mathbb{C}^n \setminus \{0\}$ for all possible $n > 0$. The space $\mathbb{C}^n \setminus \{0\}$ is contractible up to dimension $2n-2$, i.e., for $i \leqslant 2n-2$ any map $f : S^i \to \mathbb{C}^n \setminus \{0\} \rightsquigarrow S^{2n-1}$ is contractible ($\rightsquigarrow$ means “homotopy equivalence”), hence $\mathbb{C}^n - \setminus\{0\}/(\mathbb{Z}/l)$ has the same i-skeleton as $B\mathbb{Z}/l$ for $0 \leqslant i \leqslant 2n-2$.

(c) Generalizing the previous example, let M be the space of real matrices $A = \begin{pmatrix} \lambda_{11} & \dots & \lambda_{n1} \\ \vdots & & \vdots \\ \lambda_{1k} & \dots & \lambda_{nk} \end{pmatrix}$ with $\operatorname{rk} A = k$. This space is contractible up to dimension $nk - k + 1$ because M is contained in the vectorspace V of all matrices, and its complement S is :

$$V \setminus M = S = \{A | \operatorname{rk}(A) \leqslant k-1\}$$

Notice, that if $\dim(S) < r$, then $V \setminus S$ is contractible up to dimension $n - r - 2$. Here S is the space of matrices of rank $\leqslant k-1$, of dimension $\dim(S) = n(k-1) + k - 1$

(we can choose $k-1$ rows arbitrarily and the last one must be linearly dependent, which leaves a $(k-1)$-dimensional choice) then $\operatorname{codim}(S) = n-k+1$.

Application: Let G be a group with a faithful action on a complex linear space $V_k = \begin{pmatrix} \lambda_{k1} \\ \vdots \\ \lambda_{kn} \end{pmatrix}$, i.e., g acts as the identity only if $g = e$. The space $M_n \subset V_k^{\oplus n}$ is contractible up to dimension $n-k-1$. So for large n the space M_n/G is homotopy equivalent to $BG^{(i)}$. For example, if $G = \mathrm{GL}(k,\mathbb{C})$ the previous construction gives a Grassmannian variety.

$$G(k,n) \subset G(k,n+1) \subset \dots$$

In particular, we obtain that $Gr(k,\infty)$ (considered as an inductive limit of finite-dimensional Grassmannians induced by the embeddings of the corresponding linear spaces) is $BGL(k,\mathbb{C})$ viewed as an algebraic group.

In general, suppose we consider the following sequence of spaces:

$$M_n/G \subset M_{n+1}/G \subset \dots$$

The construction above gives finite-dimensional approximations to BG for any finite group G and provides a geometric model for BG, which is an inductive limit of the spaces M_n/G.

Thus, we have a model of BG which is an inductive limit of algebraic varieties.

More specifically, we can approximate BG by finite-dimensional quotients V_n^L/G, $V_n^L \subset V^{\oplus n}$, where V is an arbitrary faithful representation of G.

An open subvariety V_n^L can be explicitly described. For any $g \in G$ denote by $V_n^g := \{x \in V^{\oplus n} | gx = x\}$. Then $V_n^L = V^{\oplus n} \setminus \bigcup_{g \in G\setminus\{\mathrm{id}\}} V_n^g$. The assumption that the action of G on V is faithful (for a finite group G) means that $V^g \neq V$ for any g. Hence on $V^L = V \setminus \bigcup_{g \in G\setminus\{\mathrm{id}\}} V^g$ the action of G is free and the space V^L is nonempty. If $\operatorname{codim} V^g > k$ in V then $\operatorname{codim} V_n^g > kn$ in $V^{\oplus n}$. Thus the codimension of points in $V^{\oplus n}$, where the action of G is not free, is $> kn$. Therefore, $H^i(V_n^L/G, \mathscr{F}) = H^i(G,F)$ for $i < 2kn-1$, where $\mathscr{F}$ is the sheaf over V_n^L/G induced from F.

5. Separation of primes

Group cohomology forms a contravariant functor under maps of groups. Any group homomorphism $f : H \to G$ induces a homotopically unique map $f_* : BH \to BG$. Now we have a model of the map f_* for finite groups as a finite map between two sequences of algebraic varieties.

Indeed, if $H \subset G$, then H acts freely on V^L as well and we get a map $V^L/H \xrightarrow{\pi} V^L/G$, which is an étale covering of degree $|G/H|$. This induces maps on cohomology:

$$\pi^* : H^*(V^L/G) \to H^*(V^L/H)$$

$$\pi_* : H^*(V^L/H) \to H^*(V^L/G)$$

and $\pi_*\pi^*\alpha = \#|G/H| \cdot \alpha$ is the multiplication by the degree of the covering.

If $H = \mathrm{Syl}_p(G)$ is a $p-$Sylow subgroup, then $|G/H|$ is prime to p. In this case $\pi_*\pi^*\alpha$ is a multiplication by a number, which is prime to p, so $\pi_*\pi^*$ is invertible in $\mathbb{Z}/p$, hence the map $H^*(G,F) \to H^*(\mathrm{Syl}_p(G,F))$ induces an embedding of the p-primary part of the cohomology group $H^*(G,F)$.

If F is a finite $G-$module, $F = \oplus_p F_{\{p\}}$ where $F_{\{p\}}$ is the p-primary component of the finite abelian group F. We have a direct decomposition:
$H^*(G,F) = \sum_p H^*(G,F_{\{p\}})$, and $H^*(G,F_{\{p\}}) \to H^*(\mathrm{Syl}_p(G),F_{\{p\}})$ is an injection.

The above discussion shows that for a finite group G the "primes" are "separate". Moreover, the same argument applied to the trivial subgroup $e \in G$ shows that the cohomology with coefficients prime to the order of G are trivial.

The group $\mathrm{Syl}_p(G)$ is a p-group and hence it is built out of groups $\mathbb{Z}/p$ used as elementary blocks.

Let us denote $G_p = \mathrm{Syl}_p(G)$. Then there is a filtration of G_p by normal subgroups:

$$0 \subset G^n \subset \cdots \subset G^1 \subset G^0 = G_p,$$

such that $G^i/G^{i-1} = \mathbb{Z}/p$ and $G^i/G^{i-1} \subset G_p/G^{i-1}$ is in the center of G_p/G^{i-1}. In particular, we get exact sequences:

(a) $1 \to G^1 \to G_p \to \mathbb{Z}/p \to 1$,
(b) $1 \to \mathbb{Z}/p \to G_p \to G_1 \to 1$.

Using induction on the order of G_p and the knowledge of the cohomology of $\mathbb{Z}/p\mathbb{Z}$ one can get results on the cohomology of G_p. The most general result in this direction is:

Theorem 5.1 (Quillen-Venkov).

(a) *$H^*(G_p,\mathbb{Z}/p)$ is a noetherian ring.*
(b) *A subgroup $EA \subset G$ with $EA \cong \mathbb{Z}/p^{\oplus r}$, is called an elementary abelian subgroup. Consider:*

$$ker\left(H^*(G,\mathbb{Z}/p) \xrightarrow{\sigma} \sum_{all\, EA(i)} H^*(EA_{(i)},\mathbb{Z}/p)\right)$$

where the sum runs through all elementary abelian p-subgroups of G. This kernel is contained in the nil-ideal of $H^*(G,\mathbb{Z}/p)$.

For an arbitary finite group G the ring $H^*(G,\mathbb{Z}/p)$ can be described expicitly as a subring of $H^*(\mathrm{Syl}_p(G))$. Consider the action of the normalizer of a $p-$Sylow subgroup $N(\mathrm{Syl}_p(G))$ in G on $H^*(\mathrm{Syl}_p(G))$.Then $H^*(G,\mathbb{Z}/p) = H^*(\mathrm{Syl}_p(G))^{N(\mathrm{Syl}_p(G))}$ (see [**AM04**]).

6. Applications to birational geometry

From now on we fix some algebraically closed field k which has characteristic prime to p and consider linear spaces and varieties over k. We will always assume that all the modules F have orders coprime to the characteristics of k. We assume that $k=\mathbb{C}$ in some of the arguments below in order to apply geometric intuition.

There is a natural map from $H^*(G,F)$ to $H^*(V^L/G,\tilde{F})$ where V is faithful linear representation of G.

The group $H^*(V^L/G,\tilde{F})$ contains the image of $H^*(G,F)$. It follows from the above discussion that there is an equality $H^k(V^L/G,\tilde{F}) = H^k(G,F)$, if $k \leq \mathrm{codim} V^g - 1, g \neq e$. Note that V^L/G is an algebraic variety. Below we show that $H^*(V^L/G,\tilde{F})$ has a quotient, which is independent of the initial faithful representation V of G.

Definition 6.1. Consider the image of $H^*(G,F)$ in the cohomology of the complement of "all divisors" in V^L/G:

$$r^*: H^*(G,F) \to H^*(V^L/G,\tilde{F}) \to H^*((V^L/G)\setminus D,\mathbb{Z}/p),$$

where D are the divisors. Denote by $H^*_S(G,\tilde{F})_V$ the quotient $H^*(G,F)/Ker(r^*)$. It is isomorphic to the image of $r^*(H^*(G,F))$ in $H^*(V^L/G,\tilde{F})$.

Since the cohomology groups $H^i(V^L/G,\tilde{F})$ are finite for any i, it suffices to remove a finite number of divisors from V^L/G to obtain $H^*_S(G,\tilde{F})_V$ for any V and F.

Remark 6.2. The above definition depends on V but we will see further that in fact $H^*_S(G,\tilde{F})_V$ does not depend on the initial faithful representation V.

Below we consider a purely group theoretic version of this definition which is similar, but not equivalent to Serre's (Serre's primary objects are negligible elements - the ones which lie in the kernel of a map similar to the stabilization map above, see [**Ser**],[**Ser94**], [**GMS03**]). It involves the Galois group of an algebraic closure of the field of functions $k(V^L/G)$.

Definition 6.3. Denote by $Gal(K)$ the Galois group of an algebraic closure $\bar{K}$ over K.

The group $Gal(K)$ is a profinite topological group which surjects onto any Galois groups of a finite algebraic extensions of K. These surjections generate a natural topology on $Gal(K)$. Surjections onto finite groups are continuous on $Gal(K)$ with respect to the above topology, and we are going to consider only continuous maps and continuous cochains on the $Gal(K)$. Since $V^L \to V^L/G$ corresponds to a finite Galois extension $k(V^L) : k(V^L/G)$ with G as a Galois group, we have a natural surjection $p_G : Gal(K) \to G$ for $K = k(V^L/G)$.

Definition 6.4. Consider the map $p_G^* : H^*(G,F) \to H^*(\mathrm{Gal}(k(V^L/G),F)$ induced by surjection p_G. Define $H_S^*(G,\tilde{F})_V = H^*(G,F)/Ker(p_G^*)$. It is isomorphic to the image of $p_G^* : H^*(G,F) \to H^*(\mathrm{Gal}(k(V^L/G),F)$.

Thus we have two definitions of the same object. Below I sketch an argument showing they are equivalent.

Lemma 6.5. *The kernels of p_G^* and r^* coincide.*

Proof. Assume first that $p_G^* a = 0$ for some $a \in H^*(G,F)$. Then there is a finite quotient G_a of $Gal(K)$ with a chain of surjections $Gal(K) \to G_a, p_a : G_a \to G$ and $p_a^*(a) = 0$. Thus, there is a finite chain of Galois extensions $k(V^L/G) \subset k(V^L) \subset k(X)$ with Galois groups G, G_a respectively, and $p_a : G_a \to G$ is the corresponding homomorphism between the Galois groups. There is a Zariski open subvariety $W = V^L/G \setminus D \subset V^L/G$ such that the extension $k(X) : k(V^L/G)$ is nonramified over W. Then there is a Zariski open subvariety $X' \subset X$ with a free action of G_a such that $W = X'/G_a$. Since the element a on V^L/G is induced from G, its restriction to W is induced from G_a, but $p_a^* a = 0$ and hence a restricts to 0 on $W = V^L/G \setminus D$ which means that $r^* a = 0$. Thus $Ker(p_G^*) \subset Ker(r^*)$.

Now assume that $a \in Ker(r^*)$ and that the restriction of a is trivial on an open subvariety $W \subset V^L/G$. In order to prove that $a \in Ker(p_G^*)$, it is sufficient to find an open subvariety $W' \subset W$ with the property that $W' = K(\pi_1(W'),1)$ and $\pi_1(W')$ imbeds into the profinite completion $\hat{\pi}_1(W')$. Then $\hat{\pi}_1(W')$ is a quotient of $Gal(K)$ and, since a is trivial in the cohomology of $\hat{\pi}_1(W')$, the result follows.

Thus the lemma is reduced to the geometric result below. □

Lemma 6.6. *Let W' be an arbitrary smooth quasi-projective variety. There is an open subvariety $W \subset W'$ with $W = K(\pi_1(W),1)$ and $\pi_1(W) \to \hat{\pi}_1(W)$ is an imbedding.*

Proof. This is a well-known result in algebraic geometry. We provide a proof for varieties over $k = \mathbb{C}$ but the same argument modulo minor technical details works for any algebraically closed field k. The proof proceeds by induction on the dimension. The statement is clearly true for curves since any smooth irreducible curve apart from $\mathbb{P}^1$ satisfies the lemma.

Any variety W' contains an open affine subvariety W with a finite étale map onto an open subvariety $U^n \subset V^n \subset \mathbb{P}^n$ (here V^n is an affine subspace). Thus the existence of the required open subvariety in U^n will imply the lemma for W. Consider a projection $\pi_{n-1} : U^n \to V^{n-1}$ induced from the standard projection $V^n \to V^{n-1}$.

After removing a proper subvariety from U^n we can assume that U^n is fibered over an open subvariety $U^{n-1} \subset \mathbb{P}^{n-1}$ with $\pi_{n-1} : U^n \to U^{n-1}$ having 1-dimensional generic fiber which is connected and topologically constant. By induction we can assume that U^{n-1} is $K(\pi_1(U^{n-1}), 1)$ variety and hence that U^n is also $K(\pi_1(U^n), 1)$. We can assume in addition that U^n is an open subvariety of $\mathbb{A}^1 \times U^{n-1}$ with the following properties

(a) $U^n = \mathbb{A}^1 \times Y \setminus D$ where $Y \subset U^{n-1}$ satisfies the lemma

(b) $D = \bigcup D_i$ is a divisor with irreducible components D_i having trivial intersection in U^n.

(c) the projection map $\pi_{n-1} : D \to Y$ is a finite étale covering.

This may be achieved by first removing from U^{n-1} the images of the ramification subvarieties of π_{n-1} in D_i and intersections of different components D_i of D. Then, by applying the lemma to the resulting open subvariety of U^{n-1} we obtain Y.

Consider finite unramified coverings $\pi^i_{n-1} : D_i \to Y$ and let $\tilde{D}$ be the fiber product of D_i over Y. The variety $\tilde{D}$ consists of a finite number of isomorphic irreducible varieties.Let $\tilde{Y} = Y \times_Y \tilde{D}$ and denote by $h : \tilde{Y} \to Y$ the projection of $\tilde{Y}$ onto Y which is also a finite unramified covering. Consider the induced map $h_A : \mathbb{A}^1 \times \tilde{Y} \to \mathbb{A}^1 \times Y$. Let $U^n = \mathbb{A}^1 \times Y \setminus D \subset \mathbb{A}^1 \times Y$. Consider the premage $h_A^{-1}(U^n) \subset \mathbb{A}^1 \times \tilde{Y}$. Then $h_A^{-1}(U^n) = \mathbb{A}^1 \times \tilde{Y} \setminus \tilde{D}$ where $\tilde{D}$ consists of a finite number of sections over $\tilde{Y}$. Indeed, the product $\tilde{Y} \times D$ is equal to the union of the products $\tilde{Y} \times D_i$ and the latter consists of $deg(\pi^i_{n-1})$ number of copies of $\tilde{Y}$.

Note that $\pi_1(U^n)$ contains $\pi_1(\tilde{U}^n)$ as subgroup of finite index, but the latter is a direct product $\pi_1(\tilde{Y}) \times F$, where F is a free group. Since $\pi_1(\tilde{Y}) \times F = \pi_1(\tilde{U})$ imbeds into the profinite completion (by inductive assumption), the same holds for $\pi_1(U)$ and $\pi_1(W)$ □

Remark 6.7. In fact, we have proved a stronger statement. We have shown that any algebraic subvariety $W, dimW = n$ (over $\mathbb{C}$) contains an open subvariety W' which is $K(\pi_1(W'), 1)$ and $\pi_1(W')$ is commensurable to a product of n free groups.

The last statement means that $\pi_1(W')$ has a subgroup of finite index which is also a subgroup of finite index in the product of n- finitely generated free groups. For other fields there is a profinite version of this result.

Theorem 6.8. *The groups $H^*_S(G,F)_V$ do not depend on the choice of a faithful representation V.*

Proof. Let V,W be two faithful representations of G. Assume that $H^*_s(G,F)_V$ is a subgroup of $H^*(V^L/G\setminus D,F)$ and $H^*_s(G,F)_W \subset H^*(W^L/G\setminus D',F)$ for some D,D'.

Let us show the existence of a surjective map $H^*_s(G,F)_W \to H^*_s(G,F)_V$. It suffices to find a regular G–map $\varphi : V^L \to W^L$, (i.e., $\varphi(gx) = g(\varphi(x))$ for all $x \in V^L$ and all $g \in G$) with the property that under the induced map $\varphi_G : V^L/G \to W^L/G$ the image $\varphi_G(V^L/G)$ is not contained in D'. Then, by taking $D = \varphi_G^{-1}(D')$, we get a diagram:

$$\begin{tikzcd}
V^L \arrow[r, "\varphi"] \arrow[d] & W^L \arrow[d] \\
V^L/G \arrow[r] & W^L/G \\
(V^L/G)\setminus D \arrow[r] \arrow[u, hook] & (W^L/G)\setminus D' \arrow[u, hook]
\end{tikzcd}$$

Passing to cohomology we get:

$$H^*(G,F) \to H^*(W^L/G\setminus D',F) = H^*(G,F)_W \to H^*(V^L/G\setminus D,F).$$

Therefore, there is a surjection $H^*_S(G,F)_W \to H^*_S(G,F)_V$. Interchanging V and W concludes the proof.

To find φ we have to find a subspace $R \subset \mathbb{C}[V^L]$ (where $\mathbb{C}[V^L]$ is the ring of polynomial functions on V^L) which is isomorphic to W as a G-module.

Since G is finite, the polynomial ring $\mathbb{C}[V^L]$ is a direct sum of finite dimensional representations of G. Furthermore, for any orbit $Gx \subset V^L$ the ring of functions on the orbit Gx is the regular representation R_G of G. Therefore, for any finite set of orbits $Gx_1, Gx_2 \dots Gx_n \subset V^L$ we have a surjection $\mathbb{C}[V^L] \to \sum_{i=1}^n R_G$ (and again, since G is finite, any such surjection has a section).

Thus we can find arbitrary finite sums $R_G^{\oplus n} \subset \mathbb{C}[V^L]$ and any finite dimensional W is contained in some $R_G^{\oplus n}$. □

Corollary 6.9. *The ring $H^*_S(G,F)_V$ does not depend on a faithful representation V and we are going to denote it simply by $H^*_S(G,F)$ from now on.*

Remark 6.10. The above proof uses geometric definition of $H^*_S(G,F)_V$, but there is also a proof based on algebraic defintion.

Example. The ring $H^*_S((\mathbb{Z}/p)^n, \mathbb{Z}/p) = E\langle\sigma_1, \dots, \sigma_n\rangle$ (exterior algebra over $\mathbb{Z}/p$ generated by one-dimensional generators σ_i).

More generally, if A is a finite abelian group and $\tau : \mathbb{Z}^n \to A$ is a surjection then $H^*_S(A,F) = \tau^*(H^*(A,F))$ for any finite A-module F ([**Bog85**], [**Bog86**]).

In treating this example we can use either of the definitions of stable cohomology suggested above. Namely, consider a special homomorphism $\tau : \mathbb{Z}^n \to A = \prod \mathbb{Z}_{m_i}$ which is given as a sum of individual surjective maps $\mathbb{Z} \to \mathbb{Z}_{m_i}$. If we take $V = \Sigma\mathbb{C}_i$, where $\mathbb{C}_i$ is a faithful one-dimensional representation of $\mathbb{Z}_{m_i}$, then V^L contains a torus $T = \prod(\mathbb{C}_i - 0)$ with $\mathbb{Z}^n = \pi_1(T/A)$. Thus $\tau^*(H^*(A,F))$ surjects onto $H^*_S(A,F)$. We can prove the isomorphism directly by a geometric argument (see [**Bog85**], [**Bog92**]), but the algebraic approach provides an argument which works for a linear representation V over arbitrary field. Namely, the field $k(T/A)$ has a valuation ν with values in $\mathbb{Z}^n$ which has a chain of smooth subvarieties:

$$Y_n \subset Y_{n-1} \cdots \subset Y_0 = V/AY_i = \Sigma\mathbb{C}_i/A, i > k, k = 0,1,\dots n$$

as the center. The Galois group of the completion K_ν, contained in $Gal(k(T/A))$, is equal to $(\hat{\mathbb{Z}})^n$ and surjects onto A under the map $p_A : Gal(k(T/A)) \to A$. The surjection p_A is a completion $\hat{\tau}$ of τ and the maps τ^* and $\hat{\tau}^*$ have the same kernel for finite modules F. This implies (and explains) the equality.

Remark 6.11. The above construction can be extended to an arbitrary finite group G. Any finite group G has a faithful representation V with the following properties:

(a) There are coordinates $x_1, \dots, x_n$ in $V, dimV = n$ with $g(x_i) = \lambda(g,i,j)x_{j(g)}$ for any $g \in G$ (the elements of G permute coordinates x_i with additional multiplication by constants)

(b) The action of G is free on a subvariety $T \subset V, x_i \neq 0$ for any i.

It is clear that V^L contains T which is isomorphic to the torus and T is G-invariant (see, for example, [**Bog95a**]). Thus $T/G \subset V^L/G$ is a $K(\pi_1, 1)$-space with $\pi_1((T/G))$ being an extension of G by a finitely generated abelian group $\pi_1(T)$. The group $\pi_1((T/G))$ has no torsion elements and $\pi_1((T/G)) = \hat{\pi}_1((T/G))$.

There is a natural surjective map $\tau : \pi_1(T/G) \to G$, where $\pi_1(T/G)$ is an abelian extension of G, and hence the image $\tau^*(H^*(G,F))$ surjects onto $H^*_S(G,F)$. The group $\pi_1(T/G)$ has the additional property that for any subgroup $H \subset G$ the map $H^2(H, \mathbb{Q}/\mathbb{Z}) \to H^2(\tau^{-1}(H), \mathbb{Q}/\mathbb{Z})$ is an imbedding as it follows from lemma 8.2 below.

Unfortunately, contrary to the conjecture formulated in [**Bog92**], [**Bog95a**] these rings are different in general apart from dimensions 2 and 3 (see [**Pey99**],[**Pey93**] where the cohomology of dimension 3 were treated).

We can simlutaneously consider the ideal of elements in $H^*(G,F)$ which map to 0 in $H_s^*(G,F)$.

Definition 6.12. We will say that $x \in H^*(G,F)$ is negligible if it maps to zero under the stabilization homomorphism $r^* : H^*(G,F) \to H_s^*(G,F)$. All negligible elements form an ideal in $H^*(G,F)$ with natural functoriality properties under group homomorphism and module maps.

Remark 6.13. The term "negligible" was coined by J. P. Serre. In his lecture notes from 1990 [**Ser**] (and later in [**Ser94**][**GMS03**]) he studied a notion similar to the one we use. In his definition, however, an element $a \in H^*(G,F)$ is negligible if a is contained in the kernel of $f^* : H^*(G,F) \to H^*(Gal(\bar{L}/L),F)$ for a continuous surjective homomorphism $f : Gal(\bar{L}/L) \to G$ where L is an arbitrary field, not necessarily of type $k(X)$ with k algebraically closed. The class of negligible elements in his definition is therefore strictly smaller. More specifically, the difference occurs for elements whose exponent is a power of 2. Thus we are going to use Serre's term negligible for a slightly different set of cohomology elements. Most of his results for negligible elements of odd order are valid also for negligible elements in our sense. Serre's defintion is better adapted for the study of birational invariants of algebraic varieties over nonclosed fields. Our definition formally depends on the ground field k but in fact it is independent of k if $char k \neq exp F$ (which is a non-trivial result but we are not discussing it here). Note that Z. Reichstein (see, for example, [**CGR06**]) introduced a notion of essential dimension, which basically equals to $max_F i > 0, |H_S^i(G,F) \neq 0$ where the maximum is taken over the set of all finite G-modules F.

7. Stable cohomology of algebraic varieties

In this chapter we extend the defintion of stable cohomology to arbitrary algebraic varieties.

Definition 7.1. Let X be an algebraic variety over an algebraically closed field k and $\tilde{F}$ be a finite sheaf corresponding to a finite module F over $Gal(k(X))$. Assume that the exponent of F is coprime to the the characteristic of k. There is a natural map $p_X^* : H^*(X,\tilde{F}) \to H^*(Gal(k(X),F)$. We define $H_s^*(X,\tilde{F})$ as the quotient $H^*(X,\tilde{F})/Ker(p_X^*)$. It is isomorphic to the image of $H^*(X,\tilde{F})$ in $H^*(Gal(k(X),F))$.

In fact, this new object is closely related to stable group cohomology, so that the space V^L/G and the cohomology groups $H_S^*(G,F)$ play the role of universal objects in this theory.

Lemma 7.2. *Let $X,\tilde{F}$ be as above. Then there exists an open subvariety $X' \subset X$ and regular map $X' \to V^L/G$ for some finite group G, such that a sheaf $\tilde{F}$ is induced from a finite module F over G and the image of $H_s^*(X,\tilde{F}) \subset H^*(Gal(k(X),F)$ is contained in the image of $H_s^*(G,F) \subset H^*(Gal(k(X),F))$.*

Proof. By the definition of $\tilde{F}$, there is an open affine subvariety $X_1 \subset X$ with the property that $\tilde{F}$ is induced from the representation of $\pi_1(X_1)$ in the finite monodromy group of F. The existence of X' with above properties follows from Lemma 6.6. □

Thus any stable cohomology element of an algebraic variety can be obtained from some stable cohomology element of some finite group. In particuclar, the ideal of negligible elements in the ring $H^*(G,F)$ always maps to zero in the cohomology of a Galois group of the field of functions $k(X)$. More precisely, let X be an algebraic variety and consider $a \in H^*(X,\tilde{F})$ where $\tilde{F}$ is a finite sheaf. There are a finite group G, a G-module F' and some element $b \in H^*(G,F')$ with the following properties:

There is an open subvariety $i : X' \subset X$ with a regular map $f : X' \to V^L/G$ such that $i^*F = f^*\tilde{F}', i^*a = f^*\hat{b}$ where i^* is the restriction map and $\hat{b}$ is the image of b in $H^*(V^L/G,\tilde{F}')$.

The spaces $V^L/G, H_s^*(G,F)$ form a category of universal spaces and universal classes for stable cohomology of algebraic varieties with values in finite locally constant sheaves of abelian groups.

Remark 7.3. We have a simple, although infinite, set of universal spaces in algebraic geometry, contrary to the situation in topology where the universal space and the class are homotopically unique for the fixed dimension and coefficient sheaf. Conjecturally this set of spaces can be reduced considerably. In particular by the Bloch-Kato conjecture (see [**Blo80**], [**BK86**],[**Bog91b**]) abelian groups A and spaces V^L/A are sufficient to induce any cohomology element with finite constant coefficients at the generic point of an arbitrary algebraic variety(see [**Bog92**]). Note that Voevodsky ([**Voe03b**],[**Voe03a**]) proved the Bloch-Kato conjecture for the coefficient sheaves $\mathbb{Z}/2^n$.

Lemma 7.4. *Let $i : Y^{n-m} \subset X^n$ be a closed smooth subvariety with a generic point of Y smooth in X. Consider $a \in H^i(X,F)$ and assume that $i^*a \neq 0 \in H_s^*(Y,F)$ then $a \neq 0 \in H_s^i(X,F)$*

Proof. Here we have to use the Galois-theoretic definition. Consider a $\mathbb{Z}^m$ valuation μ on $k(X^n)$ which has the sequence $Y \subset Y^{n-m+1} \subset ... \subset X^n$ as a center. Then $Gal(k(X)_\mu) \subset Gal((k(X))$ contains $Gal(k(Y))$ as a subgroup. Since $i^* a \neq 0 \in H_s^*(Y, F)$ means that the restriction of a to $Gal(k(Y))$ is non zero, it also implies that the image of a in the cohomology of $Gal(k(X))$ is nonzero, which is equivalent to $a \neq 0 \in H_s^i(X, F)$. □

This result has the following application to the birational classification of the actions of finite groups on algebraic varieties.

Lemma 7.5. *Let G be a finite group which acts effectively on a smooth algebraic variety X over* $\mathbb{C}$ *(a closed field of characteristic* 0*) with an invariant point:* $x_0 \in X, gx_0 = x_0$ *for any* $g \in G$*. Let* $X^L \subset X$ *be a subset where the action of G is free.Then the map* $H_s^*(G, F) \to H^*(X^L/G, \tilde{F})$ *is injective.*

Proof. Let us extend the action on X to the action on the tangent bundle $T(X)$. Note that $H_s^*(G, F) \to H^*(T(X)^L/G, \tilde{F})$ is an embedding by the previous lemma, since $T(X)^L/G$ contains the quotient $T_{x_0}^L/G$. Note that the action of G is faithful on T_{x_0} (this assumption is true since the ground field has characteristics 0) and the generic point of $T_{x_0}^L/G$ is smooth in $T(X)^L/G$. There is a G-surjection $T(X) \to X$ and hence any element of the cohomology of G which vanishes on $Gal(k(X)^G)$ will also vanish on $Gal(k(T(X)))^G$. Hence the result. □

Remark 7.6. The existence of a G-invariant point on a smooth projective variety X over k with $char\, k = 0$ and effective G-action depends, in general, on the projective model X of the field $k(X)$. Indeed, after a finite sequence of G-invariant blow ups of X we obtain an effective action with abelian stabilizers of points(see [**Bog92**] for details). However, it is a birationally invariant property, if G is abelian, as the next lemma shows.

Lemma 7.7. *(see* [**Bog92**], [**CGR06**]*) Let A be an abelian group with an effective action on a smooth irreducible variety X over k with* $char\, k$ *coprime to the order of A. If A has a smooth invariant point* $x_0 \in X$*, then there is an A-invariant point on any projective A-equivariant model* X' *of* $k(X)$*.*

Proof. Note that the assumption on A ($char\, k$ coprime to the order of A) implies that the action of A at x_0 is diagonalizable and hence there is an A-invariant sequence $x_0 \subset C^1 \subset C^2 \ldots \subset C^{n-1} \subset C^n = X$ where C^i is an irreducible subvariety of X which is smooth at x_0. This sequence defines a $\mathbb{Z}^n$-valuation on $k(X)$ invariant under A. Thus the center of this valuation on any birational model of $k(X)$ is an invariant flag of irreducible subvarieties of dimensions $0, 1, \ldots, n-1$ and the corresponding zero-dimensional irreducible subvariety (which is a point) is also invariant. $\square$

8. Unramified group cohomology

This notion proved to be important in finding non-trivial obstructions to rationality for algebraic varieties.

The notion of unramified cohomolgy is defined for the stable cohomology of any irreducible variety X. We restrict ourselves to the case of constant coefficients. Any divisorial valuation ν_D of the field $k(X)$ corresponds to a divisor D on a normal variety X_D with $k(X_D) = k(X)$. The divisor D contains a smooth point of X_D. There is an open subvariety $i : U \subset X$, such that the birational isomorphism $k(X_D) = k(X)$ identifies U with an open subvariety of X_D, which we also denote by $U, j : U \subset X_D$. An element $a \in H^*_S(X, \mathbb{Z}/p)$ is nonramified with respect to the valuation ν_D if there exist X_D and $b \in H^*_S(X_D, \mathbb{Z}/p))$ such that $j^* b = i^* a$.

Definition 8.1. The element $a \in H^*_{nr}(X, \mathbb{Z}/p)$ is called unramified if it is unramified with respect to any divisorial valuation of $k(X)$.

This definition can be extended to the cohomology of the entire Galois group.

Definition 8.2. Consider $a \in H^*_S(X, \mathbb{Z}/p)$. The element a is called unramified if for every divisorial valuation ν_D the image of a in the decomposition group $Gal_{\nu_D}(K)$ is induced from the quotient group $Gal_{\nu_D}(K)/I_{\nu_D}$ where I_{ν_D} is a topologically cyclic inertia subgroup.

This gives a definition of the subgroup $H^*_{nr}(Gal(K), \mathbb{Z}/p) \subset H^*(Gal(K), \mathbb{Z}/p)$.

Lemma 8.3. *The unramified elements constitute a subring in the ring* $\Sigma H^k(Gal(K), \mathbb{Z}/p), k \geqslant 0$.

Proof. For a divisorial valuation ν_D the group $Gal_{\nu_D}(K)$ is a product $Gal_{\nu_D}(K)/I_{\nu_D} \times I_{\nu_D}$, where $I_{\nu_D} = \hat{\mathbb{Z}}$ is a canonically defined inertia subgroup of $Gal_{\nu_D}(K)$. Therefore:

$$H^k(Gal_{\nu_D}(K), \mathbb{Z}/p) = H^k(Gal_{\nu_D}(K)/I_{\nu_D}, \mathbb{Z}/p) + H^{k-1}(Gal_{\nu_D}(K)/I_{\nu_D}, \mathbb{Z}/p)$$

with a canonically defined projection

$$d : H^k(Gal_{\nu_D}(K), \mathbb{Z}/p) \to H^{k-1}(Gal_{\nu_D}(K)/I_{\nu_D}.$$

This projection satisfies the Leibniz rule

$$d(a \wedge b) = da \wedge b + (-1)^{dega, degb} a \wedge db.$$

Hence $d(a \wedge b) = 0$ if $da = db = 0$. □

The notion of unramified element is functorial with respect to rational maps.

Proposition 8.4. *Let $f : X \to Y$ be a map between irreducible algebraic varieties and consider $a \in H^*_{nr}(Gal(k(Y), \mathbb{Z}/p)$. Assume that the image $f(X)$ contains a smooth point of Y. Then the element f^*a is well-defined and $f^*a \in H^*_{nr}(Gal(k(X), \mathbb{Z}/p)$.*

Proof. Consider first the case when f is surjective. Then f defines the embedding of fields $f^* : k(Y) \subset k(X)$ and hence a map between Galois groups $f_* : Gal(k(X)) \to Gal(k(Y)$. Thus the element $f^*(a)$ is well defined in $H^*(Gal(k(X))$. Any divisorial valuation ν_D of $k(X)$ defines either trivial or a divisorial valuation $\nu_{f_*(D)}$ on $k(Y)$. In the first case the inertia group I_{ν_D} maps to zero subgroup in $Gal(k(Y)$, hence f^*a is unramified with respect to ν_D. In the second case the image $f_*(I_{\nu_D})$ is contained in $I_{\nu_{f_*(D)}}$ and $f_*(Gal_{\nu_D}) \subset Gal_{\nu_{f_*(D)}}$, so again f^*a is unramified. Assume now that $X \subset Y$ is an embedding of irreducible varieties over k and assume that generic point of X is smooth in Y. Using induction we can assume that X is a divisor on Y corresponding to valuation ν_X. Then the decomposition group $Gal_\nu(k(Y)) = Gal(k(X)) \times I_\nu$ where I_ν is a topologically cyclic group. Thus any element $a \in H^*_{nr}(Gal(k(Y), \mathbb{Z}/p)$ restricts to an element of $H^*(Gal(k(X), \mathbb{Z}/p) \subset H^*(Gal_\nu(k(Y))$ and hence the map $f^* : H^*_{nr}(Gal(k(Y)) \to H^*(Gal(k(X), \mathbb{Z}/p)$ is well-defined. Any divisorial valuation of $k(X)$ extends to a divisorial valuation of $k(Y)$, hence $f^*(H^*_{nr}(Gal(k(Y)) \subset H^*_{nr}(Gal(k(X))$. □

We define similarly the subgroup $H^*_{nr}(G, \mathbb{Z}/p) \subset H^*_S(G, \mathbb{Z}/p)$ of unramified elements. This unramified cohomology is defined as the set of all elements inside $H^*_S(G, \mathbb{Z}/p)$ which extend thorough a generic point of any divisorial valuations on $k(V^L/G)$.

Definition 8.5. Consider a subring of elements in $H^*_S(G, \mathbb{Z}/p)$ which map into $H^*_{nr}(V^L/G, \mathbb{Z}/p)$ and denote it as $H^*_{nr}(G, \mathbb{Z}/p)$.

If V, W are two different faithful representations of G then $(V^L \times W^L)/G$ surjects on both V^L/G and W^L/G . It is birationally isomorphic to $V^L \times W$ and $W^L/G \times V$, hence the above definition does not depend on the initial faithful representation of G ([**CTO89**],[**CTO92**]).

There is a purely group theoretic characterization of unramified elements coming from G for a variety V^L/G which does not depend on V.

Namely, for any subgroup $H \subset G$ denote by $Z(H)$ the center of H

$$H^*_S(G, \mathbb{Z}/p) \xrightarrow{|_H} H^*_S(H, \mathbb{Z}/p) \leftarrow H^*_S(H/Z(H), \mathbb{Z}/p).$$

A sufficient condition for $\alpha \in H^*_S(G,\mathbb{Z}/p)$ to be contained in $H^*_{nr}(G,\mathbb{Z}/p)$ is that $\alpha|_H$ is induced from $H/Z(H)$ for all proper subgroups $H \subset G$.

Consider $x \in G$ and let $Z(x)$ be the centralizer of x in G. Let V_x be a faithful representation of $Z(x)$ in which x acts as a non-trivial scalar. Let H_x be an infinite hyperplane in the projective space $\bar{V}_x$. Consider a divisor $H_x/Z(x) \subset \bar{V}_x/Z(x)$.

Definition 8.6. We will say that $a \in H^*_S(G,\mathbb{Z}/p)$ is x-unramified if a is unramified with respect to $H_x/Z(x) \subset \bar{V}_x/Z(x)$ for some V_x as above.

Lemma 8.7. *An element $a \in H^*_S(G,\mathbb{Z}/p)$ which is x-nonramified for every x in G is contained in $H^*_{nr}(G,\mathbb{Z}/p)$.*

*Proof***.** The condition is clearly necessary. Let us prove that it is also sufficient. Assume that there is an irreducible divisor D' in some completion $\tilde{V}^L/G$ of V^L/G and that the element a is ramified on D'. We assume that a generic point of D' is smooth in $\tilde{V}^L/G$. Consider the corresponding covering $\tilde{V}^L$ of $\tilde{V}^L/G$ and the preimage D of D'. The map $\tilde{V}^L \to \tilde{V}^L/G$ is ramified along D'. Otherwise the action of G would be free at the generic point of D and then all the elements of $H^*_S(G,\mathbb{Z}/p)$ would extend to $D' = D/G$. Consider an irreducible component D_1 of D. There is a cyclic subgroup $< x >$ of G which acts identically on D_1 and has x as the generator, D_1 is invariant exactly under the subgroup $Z(x) \subset G$. Consider an open G-invariant subvariety $W \subset \tilde{V}^L$ which contains a generic point of every component D_i of D and such that the action of $Z(x)/ < x >$ is free on $D_i = D_i \bigcap W$. The element a extends to D' if it extends to $D_1/(Z(x)/ < x >) = D' \bigcap W/G$. Thus we have reduced the action of G to the action of $Z(x)$. Let V_x be a linear representation of $Z(x)$ described above. Then there is a $Z(x)$ equivariant map of the pair $h : W, D_1 \to \bar{V}_x, H_x$ and hence the fact that a extends to the generic point of $H_x/(Z(x)/ < x >)$ implies that a extends to D'.

□

The following result shows that unramified cohomology provides an obstruction to stable rationality and stable rational equivalence:

Theorem 8.8. *If X is a smooth variety, then*

$$H^*_{nr}(X,\mathbb{Z}/p) \quad = \quad H^*_{nr}(X \times V,\mathbb{Z}/p).$$

For a proof see [**CTO89**]. In particular,

$$H^*_{nr}(X,\mathbb{Z}/p) \quad = \quad 0,$$

if X is a stably rational variety.

Thus, for any faithful complex linear representation V of G the quotient space V/G is not rational if $H^i_{nr}(G,\mathbb{Z}/p) \neq 0$ for some i. Moreover, the same result holds for any quotient variety X/G defined over a field k, $char\,k = 0$, if G has a smooth G-invariant point on X and the action of G on X is effective, as follows from Lemma 7.7.

Example. Let A be an abelian group. For any complex linear representation V the quotient space V/A is rational and hence $H^i_{nr}(A,\mathbb{Z}/p) = 0$ for any i. Therefore, for any group G and $a \in H^*_{nr}(X,\mathbb{Z}/p)$ the restriction of a to $H^i_s(A,\mathbb{Z}/p)$ is zero for any abelian subgroup $A \subset G$.

All the elements of $H^1_s(G,\mathbb{Z}/p)$ are ramified since V^L/G is unirational and its smooth completion is a simply-connected variety.

Example. Consider the group $H^2(G,\mathbb{Z}/p)$ and define the stabilization map: $H^2(G,\mathbb{Z}/p) \to H^2_S(G,\mathbb{Z}/p)$. The group $H^2(G,\mathbb{Z}/p)$ describes central $\mathbb{Z}/p$ extensions of G. Consider the map:

$$H^2(G,\mathbb{Z}/p) \xrightarrow{i^*} H^2(G,\mathbb{Q}/\mathbb{Z})$$

corresponding to the imbedding $i : \mathbb{Z}/p \to \mathbb{Q}/\mathbb{Z}$. Its image coincides with a subgroup $H^2(G,\mathbb{Q}/\mathbb{Z})_p \subset H^2(G,\mathbb{Q}/\mathbb{Z})$ of elements of order p.

Lemma 8.9. *$H^2(G,\mathbb{Q}/\mathbb{Z}) = H^2_S(G,\mathbb{Q}/\mathbb{Z})$ and the kernel of the map and the stabilization map coincide on $H^2(G,\mathbb{Z}/p)$. In particular the kernel of the stabilization map on $H^2(G,\mathbb{Z}/p)$ coincides with the image of $H^1(G,\mathbb{Z}/p)$ in $H^2(G,\mathbb{Z}/p)$ under Bockstein operation (See the proof in* [**Bog89**],[**Sha90**], [**CTS**]*)*

Proof. There are several proofs of this result but I will sketch below a topological argument showing why the cohomology in dimension 2 is so special. Consider the exact sequence of coefficients

$$\mathbb{Z}/p \xrightarrow{i} \mathbb{Q}/\mathbb{Z} \xrightarrow{p} \mathbb{Q}/\mathbb{Z}$$

and the corresponding exact cohomology sequence. The Bockstein operation is defined as a connecting homomorphism of the cohomological exact sequence: $\beta : H^1(G,\mathbb{Z}/p) \to H^2(G,\mathbb{Z}/p)$ and the quotient $H^2(G,\mathbb{Z}/p)/\beta(H^1(G,\mathbb{Z}/p))$ coincides with the image of $H^2(G,\mathbb{Z}/p)$ in $H^2(G,Q/(\mathbb{Z})$ since $H^2(G,\mathbb{Q}) = 0$. Removing divisors means killing the elements of $\mathrm{Pic}(V^L/G)$ (see [**Bog89**]). The Picard group of V^L/G is $\mathrm{Hom}(G,\mathbb{Q}/\mathbb{Z})$ and its subgroup of exponent p is equal to $H^1(G,\mathbb{Z}/p)$. The only elements in $H^2(G,\mathbb{Z}/p)$ we can kill under stabilization belong to $\beta H^1(G,\mathbb{Z}/p)$.

Topologically this can be seen as follows. Let V be a faithful complex linear representation of the group G. The group $H^2((V^L/G),\mathbb{Z}/p)$ is represented by (infinite, real) cycles of codimension 2 in (V^L/G). On a smooth variety (V^L/G) closed real codimension 2 subvarieties are dual to compact cycles of real dimension two - surfaces.

Any two-cycle C which is orthogonal (via intersection) to the group $\mathrm{Pic}(V^L/G)$ is homologous to a cycle which does not intersect a given divisor D. Thus, if $a \in H^2((V^L/G), \mathbb{Z}/p)$ has a non-trivial value on a cycle C as above, then a does not vanish in $H^2_s((V^L/G), \mathbb{Z}/p)$. In particular, the only negligible elements of $H^2((V^L/G), \mathbb{Z}/p)$ are the codimension two cycles represented by divisors and hence belong to $\beta H^1(G, \mathbb{Z}/p)$. $\square$

The group $H^2_{nr}(G) = B_0(G)$ can be defined inside $H^2(G, \mathbb{Q}/\mathbb{Z})$ as a subgroup consisting of elements restricting to 0 on any abelian subgroup $A \subset G$ with two generators. (see [**Bog89**], [**Bog87**],[**Sal84**]). Note that an element $H^i_{nr}(G, \mathbb{Z}/p)$ restricts trivially to $H^i_s(H, \mathbb{Z}/p)$ for any subgroup $H \in G$ if W/H is stably rational for some faithful representation W of H. In particular, the restriction of $H^i_{nr}(G, \mathbb{Z}/p)$ to abelian subgroups is always trivial. The fact that for $a \in H^2(G, \mathbb{Q}/\mathbb{Z})$ it is sufficient to check only abelian subgroups with two generators follows from the argument which is specific for two-dimensional cohomology (see [**Bog89**], [**Bog87**]).

If we consider an element $h \in H^2(G, \mathbb{Z}/p)$, the fact that the image of h belongs to $B_0(G) \subset H^2(G, \mathbb{Q}/\mathbb{Z})$ can be described directly via a property of a central $\mathbb{Z}/p$-extension G_h of G corresponding to h: for any two commuting elements $a, b \in G$, any two premages $a', b' \in G_h$ of a and b respectively also commute (see [**BMP04**]).

Thus, if G is a group with $B_0(G) \neq 0$ (there are such groups with order p^7 see [**Sal84**],[**Bog89**],[**Bog87**]), V/G is never stably rational for a faithful complex representation V of such G.

There is a similar description of the group $H^2_{nr}(X, \mathbb{Z}/p)$ for an arbitrary algebraic variety (see [**Bog91a**] and chapter 10).

9. Stable cohomology of the Galois groups of function fields

For any topological profinite group G we can define stable and unramified cohomology.

Definition 9.1. Let G be a compact profinite group with a countable fundamental system of open subgroups given as kernels of surjective homomorphsims $f_i : G \to G_i$, where each G_i is a finite group. Define $H^*_s(G, F)$ for a finite module F as an inductive limit of $H^*_s(G_i, F)$ over finite quotient groups G_i.

Definition 9.2. Define $H^*_{nr}(G, \mathbb{Z}/p)$ as an inductive limit of groups $H^*_{nr}(G_i, \mathbb{Z}/p)$.

Example. Let $G = \mathbb{Z}_p$. Then $H^i_s(\mathbb{Z}_p, F) = 0, i > 1, H^1_s(\mathbb{Z}_p, F) = H^1(\mathbb{Z}_p, F)$ for any finite module F. Thus in this case group cohomology coincides with stable cohomology.

The Galois group of a countable function field is a profinite topological group, hence we can define its stable and unramified cohomology as above.

In particular stable cohomology of the group $Gal(K), K = k(X)$ with coefficients in a finite topological module is equal to the inductive limit of the stable cohomology of the finite quotients of $Gal(K)$.

Galois groups have special property with respect to the stabilization map among all profinite groups.

Lemma 9.3. *For any finite $Gal(K)$-module F one has*

$$H^*(Gal(K), F) = H^*_s(Gal(K), F).$$

Proof. In a way, the lemma becomes a tautology after careful analysis of definitions. As we have shown before, $H^*(Gal(K), F)$ is also an inductive limit of the cohomology of open subvarieties of $X, k(X) = K$. Any element $a \in H^*(Gal(K), F)$ is induced from a finite quotient group G_a under a surjective continuous map $f : Gal(K) \to G$, or, equivalently, from the cohomology of a sheaf $\tilde{F}$ on an open subvariety $X_a \subset X$. We want to show that if $a \in H^*(Gal(K), F)$ vanishes in $H^*_s(Gal(K), F)$ then a also vanishes on some open subvariety $X_{va} \in X_a$. Since a has finite order, its vanishing in $H^*_s(Gal(K), F)$ is equivalent to the existence of a finite group G' and surjective (continuous) maps $h : Gal(K) \to G'$ and $g : G' \to G$ with $gh = f$ such that $h^*(a) = 0 \in H^*_s(G', F)$. It means that there exists a divisor $D \in V^L/G'$ such that the restriction of $h^*(a)$ on $V^L/G' \setminus D$ is 0. Note that group G' is thus a quotient of $\pi_1(X')$ for some open subvariety $X' \subset X_a$.

Now we can use the same argument as in the proof of Theorem 6.8 to find a map $\varphi : X' \to V^L/G'$ with a point $p \in X'$, such that $\varphi(p) \notin D$. Thus, if we consider $X_{va} = X' \setminus \varphi^{-1}(D)$ then the image of a on X_{va} is trivial. In particular, we have found an open subvariety in X where a vanishes and hence a defines a trivial element in $H^*(Gal(K), F)$. □

The lemma shows that $BGal(K)$ has two different topological models which are realised by a system of algebraic varieties.

One of them is X minus divisors - “generic point” of X- the projective system of open subvarieties $X_i \subset X$, where the maps are embeddings, and the other one is obtained as a tower of V^L_i/G_i where G_i is a projective exhausting system of finite quotients of the Galois group $Gal(K)$.

Proposition 9.4. *Let H be a closed subgroup of the Galois group $Gal(K)$. Then $H^*(H, F) = H^*_s(H, F)$.*

*Proof***.** Any element $a \in H^*(H,F)$ which is trivial in $H_s^*(H,F)$ is induced from a finite quotient $p_a : H \to H'$ such that a is trivial in $H_s^*(H',F)$. Since H is a closed subgroup of $Gal(K)$ the map p_a extends to a surjective homomorphism of the pair $p'_a : (Gal(K),H) \to (G,H')$ where $G,H' \subset G$ is a finite quotient of $Gal(K)$. Consider a complete preimage $g_a^{-1}(H') \subset Gal(K)$. This group is equal to $Gal(K')$ $g_a : Gal(K') \to H'$ for a finite extension $K' : K$. The group $Gal(K')$ contains H and the map $g_a : Gal(K') \to H'$, which is a restriction of p'_a on $Gal(K')$, coincides with p_a on H. By the previous lemma, $H^*(Gal(K'),F) = H_S^*(Gal(K'),F)$ and hence $g_a^*(a) = 0$ which implies $p_a^*(a) = 0$. □

Corollary 9.5. *This result generalizes the previous lemma for the Galois groups $Gal(\bar{k}(X)/k(X))$ to an arbirary Galois group $Gal(\bar{F}/F)$ where the field F is obtained as infinite algebraic extension of $k(X)$ for some algebraically closed field k.*

Remark 9.6. It is an interesting question whether the equality between stable and usual cohomology characterizes the topological profinite groups which can be realized as Galois groups $Gal(\bar{F}/F)$.

Similarly, we can define the groups $H_{nr}^*(Gal(K),\mathbb{Z}/p)$ as inductive limits of the groups $H_{nr}^*(G_i,\mathbb{Z}/p)$ for finite quotients G_i of $Gal(K)$. If $I \subset Gal(K)$ is a closed abelian subgroup and $C(I) \subset Gal(K)$ is a centraliser of I then $C(I)$ is also a closed subgroup of $Gal(K)$. The fact that $H_S^*(C(I),\mathbb{Z}/p) = H^*(C(I),\mathbb{Z}/p)$, which was proven above, implies that $a \in H_{nr}^*(G_i,\mathbb{Z}/p)$ has a property that the restriction of a to $C(I)$ is induced from $C(I)/I$. The converse is also true, but the proof relies on the description of the groups $C(I)$ for different closed cyclic subgroup $I \subset Gal(K)$ (see [**Bog91a**], [**BT02**] for the description of groups I with a non-trivial centraliser and [**Bog92**] for the geometric argument).

10. The spaces V^L/G as universal spaces

We have mentioned above a result which shows that the spaces V^L/G play the role of universal spaces for the stable cohomology of algebraic varieties. In fact, a version of this result also holds for unramified cohomology.

Let $f : X \to V^L/G$ be a map of algebraic varieties. Then the induced map $f^* : H_S^*(G,F) \to H_S^*(X,f^*F)$ maps unramified elements of G into unramified elements of X. The question is whether all the unramified elements $H_{nr}^*(X,\mathbb{Z}/p)$ can be induced from the unramified cohomology elements of finite groups. The answer is yes, for X defined over $\bar{\mathbb{F}}_l$, $l \neq p$ (correcting the result in [**Bog92**]). It is a corollary of the following result describing the centralisers of elements in the Galois group $Gal(K)$, $K = \bar{\mathbb{F}}_l(X)$. Consider the quotient group $Gal^c(K) = (Gal(K)/[[Gal(K),Gal(K)],Gal(K)])_p$ where p stands for the maximal

p-quotient. It is a central extension of abelian group
$Gal^{ab}(K) = (Gal(K)/[Gal(K), Gal(K)])_p$.

Theorem 10.1. *Let a, b be two elements of $Gal^{ab}(K)$ such that the preimages of a, b in $Gal^c(K)$ commute and the group $< a, b >$ topologically generated by a, b has rank 2 over $\mathbb{Z}_p$. Then there is a valuation ν on K such that $a, b \in Gal^{ab}_\nu(K)$ (decomposition group of the valuation) and one of the generators of the group $< a, b >$ belongs to the inertia subgroup I_ν of the valuation.*

The proof of this result may be found in the articles [**Bog91a**],[**BT02**] and a simplified version for $k = \overline{\mathbb{F}}_l(X)$ in [**BT06**].

Corollary 10.2. *Consider an element $a \in H^*(Gal(k), \mathbb{Z}/p)$. Its restriction on any subgroup $C(I) \subset Gal(K)$ is induced from $C(I)/I$ if and only if for any valuation ν of K the restriction of a on Gal_{K_ν} is induced from Gal_{K_ν}/I_ν.*

Indeed, as the previous theorem shows the group $C(I)$ is either abelian or $I \subset I_\nu$ (inertia group for some valuation group ν) with $C(I) \subset Gal_\nu(K)$. Since the latter is equal of the product $Gal_{K_\nu} \times I_\nu$ where I_ν is abelian the result follows immediately follows in this case.

Remark 10.3. In general,the following result holds : let X be an algebraic variety over $\overline{\mathbb{F}}_l, l \neq p$ and $a \in H^*_{nr}(X, \mathbb{Z}/p) \subset H^*(Gal(K), \mathbb{Z}/p)$. Then there are a finite group G , an element $b \in H_{nr}(G, \mathbb{Z}/p)$ and a homomorphism $h_a : Gal(K) \to G$ such that $a = h^*_a(b)$. The proof involves a geometric argument based on the general result above.

Let us discuss the example of unramified cohomology in dimension 2 in more detail. Consider $H^2_{nr}(X, \mathbb{Z}/p)$ where X is an algebraic variety over $k = \overline{\mathbb{F}}_l, k(X) = K$. Any element of $H^2_{nr}(X, \mathbb{Z}/p)$ is induced from $H^2(Gal^{ab}(K), \mathbb{Z}/p)$ by Merkurjev-Suslin theorem (see [**MS82**],[**Bog91a**] [**Bog92**]) and $H^2_{nr}(X, \mathbb{Z}/p) = H^2_{nr}(Gal(K, \mathbb{Z}/p)$ and the latter coincides with $H^2_{nr}(Gal_c(K), \mathbb{Z}/p)$ (see [**Bog91a**]). The group $Gal_c(K)$ is a central extension of abelian group $Gal^{ab}(K)$ and for finite p-groups G of this type the group $H^2_{nr}(G, \mathbb{Z}/p)$ has an explicit description in terms of relations in the group G (see [**Sal84**],[**Bog89**] and Chapter 8 of this article). Similar description holds for the group $H^2_{nr}(Gal_c(K), \mathbb{Z}/p)$.

There are two natural exact sequences of profinite groups:
$0 \to C \to Gal^c(K) \to Gal^{ab}(K) \to 0$
and
$0 \to R \to \Lambda^2(Gal^{ab}(K)) \to C \to 0.$
Here $Gal^{ab}(K)$ is a torsion-free profinite abelian group (similar to $\mathbb{Z}_p^\infty$). The group C is the center of $Gal^c(K)$ and R is the subgroup of non-trivial relations in

$\Lambda^2(Gal^{ab}(K))$. The latter contains a subgroup R_Λ topologically generated by pairs $[x,y] = x^{-1}y^{-1}xy \in R$. The quotient $(R/R_\Lambda$ is a finite dimensional module over $\mathbb{Z}_p$ and there is a natural duality between finite group $(R/R_\Lambda)/p$ and $H^2_{nr}(X,\mathbb{Z}/p)$ (see [**Bog89**] [**Bog87**] [**Bog91a**]). The element $a \in H^2_{nr}(Gal_c(K),\mathbb{Z}/p)$ is induced from some finite abelian quotient of the group $Gal^{ab}(K)$. There exists a finite abelian quotient A of $Gal^{ab}(K), f : Gal^{ab}(K) \to A$ and a finite quotient A_c of $Gal_c(K)$ with a surjective map $f_c : Gal_c(K) \to A_c$ such that:

(a) $A = A_c/[A_c,A_c]$,
(b) the map f induces surjection $f_2 : \Lambda^2(Gal^{ab}(K)) \to \Lambda^2(B)$,
(c) The group $R/R_\Lambda)/p$ imbeds into $\Lambda^2(B)/f(R_\Lambda)$,
(d) The element a is induced from A.

Existence of such quotient follows from the fact that both sequences above are exact as sequences of profinite groups.

Thus, the element a is unramified on the quotient A_c of $Gal^c(K)$ which is obtained as the quotient of $Gal^c(K)$ by a subgroup of finite index generated by R_λ, R^p and $Ker(f)$ due to the criterion from [**Bog87**]. Hence a is induced from an unramified element on A_c.

Thus for any element $a \in H^2_{nr}(X,\mathbb{Z}/p)$ there is a (nonrational) variety $\prod \mathbb{P}^{n_i}/A$ where the abelian group A acts projectively on each $\mathbb{P}^{n_i}$, and an element $b \in H^2_{nr}(\prod \mathbb{P}^{n_i}/A),\mathbb{Z}/p)$ which is induced from $H^2(A,\mathbb{Z}/p)$ and a map $h_a : X \to \prod \mathbb{P}^{n_i}/A$ such that $h_a^*(b) = a$.

Indeed a is induced from an element $b \in H^2_{nr}(A_c,\mathbb{Z}/p)$ and hence from some quotient V^L/G where V is a faithful linear representation of A_c. By taking V to be a sum of representations V_i corresponding to the basis of the characters of the center of A_c we obtain that a is induced from $\prod V_i^L/A_c$.The latter is stably birationally equivalent to the $\prod \mathbb{P}(V_i)/A, A = A_c/[A_c,A_c]$.

In general there is a similar theory of universal spaces (pairs) $(V^L/G,\gamma)$ for unramified elements, and hence for finite birational invariants of algebraic varieties. For any $a \in H^i_{nr}(X,\mathbb{Z}/p), i > 1$ where X is a variety over $\overline{\mathbb{F}}_l, l \neq p$, there is a finite group G, an element $b \in H^i_{nr}(G,\mathbb{Z}/p)$ and a rational map $f_a : X \to V^L/G$ such that $f_a^*(b) = a \in H^i_{nr}(X,\mathbb{Z}/p)$. When the ground field $k \neq \overline{\mathbb{F}}_l$ there is a similar theory for a subgroup of elements in $H^i_{nr}(X,\mathbb{Z}/p), i > 1$ which remain unramified in arbitrary reductions of X (absolutely unramified elements). These two notions are different indeed as follows from the example in [**Bog92**]. It is an abelian surface A over $\overline{\mathbb{Q}}$ which has a multiplicative reduction over some l prime to p. The group $H^2_{nr}(A,\mathbb{Z}/p)$ in this case is nontrvial and the group $H^2_{nr}(Gal(\overline{\mathbb{Q}}(A),\mathbb{Z}/p) \neq H^2_{nr}(A,\mathbb{Z}/p)$.

11. On the Freeness conjecture

As our geometric construction shows, for every finite group G there is a finitely generated group Γ with a surjective homomorphism $h : \Gamma \to G$ such that the kernel of stabilization map $r^* : H^*(G,F) \to H^*_S(G,F)$ coincides with the kernel of the map: $h^* : H^*(G,F) \to H^*(\Gamma,F)$. For example, Γ is a free abelian group if G is a finite abelian group. The proper solution of the stabilization conjecture would give for each G a set of groups Γ, constructed directly from G, which realize the stabilization map.

Partial solution is given by the following two constructions:

Example. Let G be finite group. Then there is surjective map $f : \Gamma' \to G$ with a free abelian group $\mathbb{Z}^n$ as a kernel with the following properties:

(a) Γ' has no torsion elements.
(b) the map $H^2(H,\mathbb{Q}/\mathbb{Z}) \to H^2(f^{-1}(H),\mathbb{Q}/\mathbb{Z})$ is an embedding for any $H \subset G$.

The kernel of the map $f^* : H^*(G,F) \to H^*(\Gamma',F)$ contains the kernel of the stabilization map and coincides with the latter on any abelian subgroup of G. In general, however, it does not coincide with the stabilization map.

There is another construction related to the braid group, and it is not known whether it gives a cohomology stabilization or not.

Example. Let $i : G \subset S_n$ be an imbedding of G into some symmetric group. Consider a natural surjection $s : B_n \to S_n$ and the preimage $\Gamma_G = s^{-1}(i(G)) \subset B_n$. Note that $B_n = \pi_1(\mathbb{C}^n \setminus D) = \pi_1(\mathbb{C}^{n,L}/S_n)$, where $\mathbb{C}^n$ is given a standard permutation action of S_n and D the discriminant divisor in the quotient space which is also isomorphic to $\mathbb{C}^n$. Thus $\Gamma_G = \pi_1(\mathbb{C}^{n,L}/G)$ and hence for the surjection $f : \Gamma_G \to G$ the map $f^* : H^*(G,F) \to H^*(\Gamma_G,F)$ is a partial stabilization map.

The problem of cohomology stabilization is related to the Bloch-Kato conjecture ([**Blo80**],[**BK86**]) on the cohomology of Galois groups. This conjecture for fields $k(X)$ or their inifinite extensions is equivalent to the following two statements (see [**Bog91b**]):

(a) The map

$$h^* : H^*(Gal^{ab}(K),\mathbb{Z}/p) \to H^*(Gal(K),\mathbb{Z}/p)$$

is surjective.

(b) The kernel of h^* coincides with the kernel of

$$h^*_c : H^*(Gal^{ab}(K),\mathbb{Z}/p) \to H^*(Gal_c(K),\mathbb{Z}/p).$$

This conjecture was proved by V. Voevodsky (see[**Voe03b**],[**Voe03a**]) for 2-adic coefficients. His proof is based on his results on the structure of motivic cohomology and integration of the main constructions of homotopy theory into the framework of algebraic geometry. However, there is another approach which relates this result to the structure of Galois groups $Gal(k(X))$ and their Sylow subgroups (see [**Bog92**]). In particular, the following conjecture provides a direct approach to the proof of the Bloch-Kato conjecture and explains why it should be true.

Conjecture 11.1. *Let $k(X)$ be the field of functions over an algebraically closed field k. Then for any prime p the group $[\mathrm{Syl}_p(K), \mathrm{Syl}_p(K)]$ is a cohomologically free group, i.e., $H^i([\mathrm{Syl}_p(K), \mathrm{Syl}_p(K)], M) = 0, i > 1$ for any finite $[\mathrm{Syl}_p(K), \mathrm{Syl}_p(K)]$-module M (see [**Bog95b**]).*

The Freeness conjecture in this form was initially formulated (by the author) in 1986. There are stronger forms of this conjecture but here I am going to dicsuss only this weak version.

There were several reasons to believe this conjecture.

1) It provides with a strong reason for Bloch-Kato conjecture to be true. Namely, a standard argument (see [**BT73**]) implies that it is sufficient to prove the conjecture for $\mathrm{Syl}_p(K)$. Considering the exact sequence of groups:

$$[\mathrm{Syl}_p(K), \mathrm{Syl}_p(K)] \to \mathrm{Syl}_p(K) \to \mathrm{Syl}_p^{ab}(K).$$

It gives a spectral sequence for the cohomology of $\mathrm{Syl}_p(K)$ with:

$$E_2^{p,q} = H^p(\mathrm{Syl}_p^{ab}(K), H^q([\mathrm{Syl}_p(K), \mathrm{Syl}_p(K)], \mathbb{Z}/p).$$

The conjecture above implies that $H^q([\mathrm{Syl}_p(K), \mathrm{Syl}_p(K)], \mathbb{Z}/p) = 0, q > 1$.

Thus all the cohomology of $Gal(K)$ is coming from metabelian finite p-groups and the proof of Bloch-Kato conjecture reduces to the description of differential d_2 in the cohomology spectral sequence.

2) Since the group $[\mathrm{Syl}_p(K), \mathrm{Syl}_p(K)]$ is a topological pro-l-group the Freeness conjecture is equivalent to $H^2([\mathrm{Syl}_p(K), \mathrm{Syl}_p(K)], \mathbb{Z}/p) = 0$ and even to $E_2^{0,2} = 0$.

3) All the elements of $H^*([\mathrm{Syl}_p(K), \mathrm{Syl}_p(K)], \mathbb{Z}/p)$ are unramified and hence any element becomes unramified on a finite abelian extension of K.

I will give a simple proof of the last statement (which was one of the reasons to put forward the above conjecture and whose geometric proof appeared recently in [**CGR06**]).

The differentials d_v link stable cohomology in different dimensions.There is a simple criterion for all cohomology of the Galois group to be unramified:

Proposition 11.2. *Assume that $char F \neq p$ and that for any discrete valuation v on F the group of values $v(F^*)$ is p-divisible. Then $H^i(Gal(F,M)) = H^i_{nr}(Gal(F,M))$, for every finite Galois module M.*

Proof. We have to check that $d_v a = 0$ for any $a \in H^i(Gal(F,M))$ and for any v which corresponds to a divisorial valuation on arbitrary finitely generated subfield over the ground field k. Since v corresponds to a non-trivial divisorial valuation on the subfield of F of the same characteristics, the characteristics of the residue field F_v is prime to p. Thus the decomposition group $Gal_v(F) = Gal(F_v) \times I_v$ where, $I_v = Hom(v(F^*), \mathbb{Z}_p)$. Since p-divisibility of $(v(F^*)$ implies $Hom(v(F^*), \mathbb{Z}_p) = 0$ we conclude that $Gal_v(F) = Gal(F_v)$ and hence $d_v a = 0$ for any valuation v. □

Corollary 11.3. *Let $K : K' = k(X)$ be a finite extension and consider $F = K \times K'^{ab}$ - a composite of K and a maximal abelian pro-p-extension of K'. Then $H^i(Gal(F,M) = H^i_{nr}(Gal(F,M)$ for any finite Galois module M.*

Indeed, for a field K' and a divisorial valuation v on K', the group of values of the extension of v to K'^{ab} is infinitely p-divisible. Since F is an algebraic extension of K' any valuation on F restricts non-trivially on K'. Thus, for any valuation v on F which is divisorial on some extension of K', the set $v(F^*)$ is infinitely p-divisble and the result follows from the proposition above.

Corollary 11.4. *Let $a \in H^i_s(X, \mathbb{Z}/p^n)$ and let $f : X \to Y$ be a finite surjective map. There exists a finite abelian Galois covering $g : Y' \to Y$ such that for a natural map $h : (X \otimes_Y Y') \to X$ the induced element $h^* a$ is contained in $H^i_{nr}((X \times Y'), \mathbb{Z}/p^n)$.*

Remark 11.5. This result and its implications for the Brauer group of K^{ab} open another approach to the proof of the Freeness Conjecture for the commutator subgroup of $\mathrm{Syl}_p(K)$. The question remains wether or not it is possible not only to kill singularities of an element of the Brauer group of X (equal to $H^2(X, \mathbb{Z}/p^n)$) but to kill the element itself by considering abelian extensions of X induced from abelian extensions of an arbitrary cyclic quotient Y of X.

The results above show that the main problem in computation of stable cohomology lies in the computaion of unramified cohomology.

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ANALYSIS ON SYMMETRIC AND LOCALLY SYMMETRIC SPACES (MULTIPLICITIES, COHOMOLOGY AND ZETA FUNCTIONS)

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***Abstract*.** We discuss analysis on symmetric and locally symmetric spaces.

1. Multiplicities of principal series

Let us first fix some notation. Let G be a semisimple Liegroup with finite center and $G \cong KAN$ be the Iwasawa decomposition of G, where K is a maximal compact subgroup, A a maximal $\mathbb{R}$-split torus of G, and N is a nilpotent subgroup. Correspondingly, we have a decomposition of the Lie algebra: $\mathfrak{g} \cong \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$. We denote by $P := MAN$ a minimal parabolic subgroup, where $M = Z_K(A)$ is the centraliser of K in A. For $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$, we define the character $a \mapsto a^\lambda := \exp(\lambda(\log a))$. Furthermore, let $\rho \in \mathfrak{a}^*$ denote $H \mapsto \frac{1}{2} Tr(ad(H)|_{\mathfrak{n}})$. For a finite dimensional representation (σ, V_σ) of M, we define $\sigma_\lambda : P \to GL(V_\sigma)$ by $man \mapsto a^{\rho-\lambda}\sigma(m)$.

Definition 1.1. The induced representation $\pi^{\sigma_\lambda} := Ind_P^G(\sigma_\lambda)$ is called the principal series representation with parameter (σ, λ).

The representation π^{σ_λ} acts on the space of sections of regularity ? of the G-equivariant bundle $V(\sigma,\lambda) := G \times_P V_{\sigma_\lambda}$. The space of sections will be denoted by $V^?_{\pi_{\sigma_\lambda}} := C^?(G/P, V(\sigma,\lambda))$, where ? belongs to $K-fin, \omega, \infty, L^2, -\infty$ or $-\omega$.

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Here, $K-fin$ means K-finite, ω real-analytic, ∞ smooth and L^2 square integrables. To be honest,$C^{K-fin}(G/P,V(\sigma,\lambda))$ is only a $(\mathfrak{g},K)$-module. Denote the $\mathbb{C}$-linear dual of σ by σ^*. To define the hyperfunction and distribution-valued sections, observe that $V(1,-\rho)\to G/P$ is the bundle of densities, and we have a G-equivariant pairing $V(\sigma,\lambda)\otimes V(\sigma^*,-\lambda)\to V(1,-\rho)$. Combining this pairing with integration, we have a pairing of spaces of sections.

Definition 1.2.

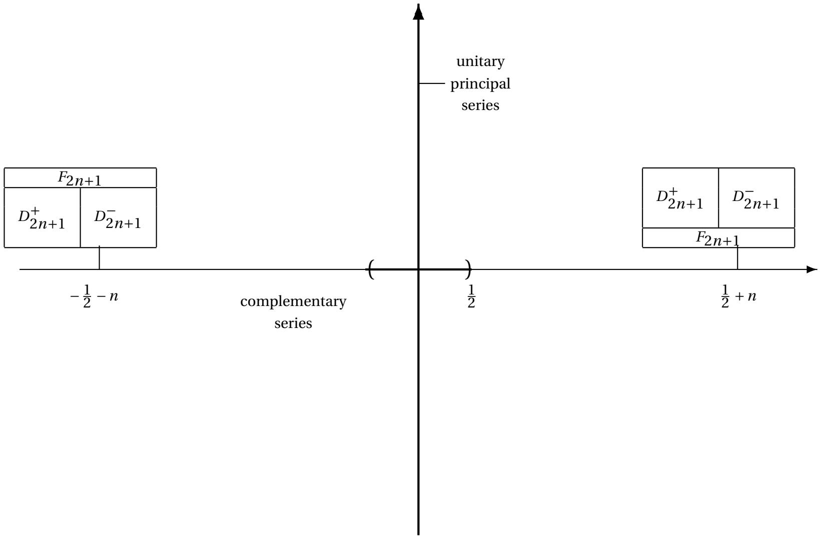
$$\begin{aligned} C^{-\omega}(G/P,V(\sigma,\lambda)) &:= C^{\omega}(G/P,V(\sigma^*,-\lambda))^* \\ C^{-\infty}(G/P,V(\sigma,\lambda)) &:= C^{\infty}(G/P,V(\sigma^*,-\lambda))^*. \end{aligned}$$

$C^{L^2}(G/P,V(\sigma,\lambda))$ is a G-Banach representation. If $\Re(\lambda)=0$, it is unitary in a natural way.

Example 1.3. We consider $G=SL(2,\mathbb{R})$. The Iwasawa components are

$$K=\left\{\begin{pmatrix}\cos\varphi & -\sin\varphi\\ \sin\varphi & \cos\varphi\end{pmatrix}|\varphi\in[0,2\pi)\right\} \qquad A=\left\{\begin{pmatrix}t^{\frac{1}{2}} & 0\\ 0 & t^{-\frac{1}{2}}\end{pmatrix}|\,t\in(0,\infty)\right\}$$
$$N=\left\{\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}|\,x\in\mathbb{R}\right\}$$

In this case, we have $\mathfrak{a}\cong\mathbb{R}H$, where $H=\begin{pmatrix}\frac{1}{2} & 0\\ 0 & -\frac{1}{2}\end{pmatrix}$. Calculating $\rho(H)=\frac{1}{2}$, we obtain that $\rho\cong\frac{1}{2}$. Since $M\cong\left\{\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}\right\}\cong\mathbb{Z}/2\mathbb{Z}$, $\hat{M}=\{1,\theta\}$ consists of two elements. Furthermore, $G/P=K/M\cong S^1$. Choosing $\sigma=1$, the principal series representations are parametrized by $\lambda\in\mathbb{C}$. For generic λ, the representation is irreducible.



$D^{\pm}$ is called the holomorphic/antiholomorphic discrete series, and F_{2n+1} is the $2n+1$-dimensional irreducible representation of $SL(2,\mathbb{R})$. The unitary principal series representations, the complementary series representations, $D^{\pm}_{2n+1}$ and F_1 are unitary. Note that $\pi^{1\pm\frac{1}{2}}$ is a non-trivial extension and therefore not unitary.

Let $\Gamma\subset G$ be a cocompact, torsion-free discrete subgroup, $X:=G/K$, $Y:=\Gamma\backslash X$ and ${}^{\Gamma}V^{-\omega}_{\pi^{\sigma_\lambda}} := \{\varphi\in V^{-\omega}_{\pi^{\sigma_\lambda}} | \pi^{\sigma_\lambda}(\gamma)\varphi=\varphi \quad \forall\ \gamma\in\Gamma\}$, the space of Γ-invariant hyperfunction vectors.

Lemma 1.4 (Frobenius reciprocity).

$$\begin{array}{rcl} {}^{\Gamma}V^{-\omega}_{\pi^{\sigma_\lambda}} & \cong & Hom_G(V^{-\omega}_{\pi^{\sigma^*_{-\lambda}}}, C^\infty(\Gamma\backslash G)) \\ \varphi & \mapsto & (f \overset{i_\varphi}{\mapsto} (g\mapsto \varphi(\pi^{\sigma^*_{-\lambda}}(g)f))) \\ (f\mapsto(\psi(f)(\Gamma 1))) & \leftarrow & \psi \end{array}$$

Since Γ is cocompact, we have a decomposition

$$C^\infty(\Gamma\backslash G)\cong \widehat{\bigoplus}_{(\pi,V_\pi)\in\hat{G}_u} m_\Gamma(\pi)\, V^\infty_\pi,$$

where $\hat{G}_u$ is the unitary dual of G, i.e. the set of equivalence classes of irreducible unitary representations of G. As a consequence, if π^{σ_λ} is unitary and $\lambda\neq 0$, then it is irreducible and therefore $m_\Gamma(\pi^{\sigma_\lambda}) = dim({}^{\Gamma}V^{-\omega}_{\pi^{\sigma_\lambda}})$. Moreover, if π^{σ_λ} is irreducible and not unitary, then ${}^{\Gamma}V^{-\omega}_{\pi^{\sigma_\lambda}}=\{0\}$.

We now return to our example and connect ${}^{\Gamma}V^{-\omega}_{\pi^{\sigma}\lambda}$ with the eigenvalues of the Laplacian on Y. As K-homogenous bundles, $V(\sigma,\lambda) = G \times_P V_{\sigma_\lambda} = K \times_M V_\sigma$, and for $\sigma = 1$, $K \times_M V_1 \cong S^1 \times \mathbb{C}$. Denote by $f \in C^\infty(K \times_M V_1)$ the unique K-invariant section with $f(1) = 1$. For any choice of λ, this corresponds to a real-analytic section f_λ of $V(1,\lambda)$.
Using the reciprocity homomorphism $i : V^{-\omega}_{\pi^1\lambda} \to Hom_G(V^{\omega}_{\pi^1{-\lambda}}, C^\infty(G))$ for trivial Γ, we define the Poisson transformation

$$i.(f_{-\lambda}) : V^{-\omega}_{\pi^1\lambda} \to C^\infty(G).$$

Now, any $\varphi \in V^{-\omega}_{\pi^1\lambda}$ defines $i_\varphi(f_{-\lambda}) \in C^\infty(X)$. The Casimir-operator Ω of $\mathfrak{g}$ acts on π^{1_λ} by a scalar, $\pi^{1_\lambda}(\Omega) = \mu(\lambda) := \frac{1}{4} - \lambda^2$, while $\Omega|_{C^\infty(X)} = \Delta_X$, the Laplace operator. Therefore, $i_\varphi(f_{-\lambda}) \in Ker(\Delta_X - \mu(\lambda))$.

Theorem 1.5 (Helgasson [Hel84]). *For $\lambda \notin \{-\frac{1}{2}, -\frac{3}{2}, -\frac{5}{2}, \dots\}$, the Poisson transformation*

$$i.(f_{-\lambda}) : V^{-\omega}_{\pi^1\lambda} \to Ker(\Delta_X - \mu(\lambda))$$

is an isomorphism.

To calculate $H^*(\Gamma, V^{-\omega}_{\pi^1\lambda})$, we need the following classical theorems.

Theorem 1.6. *Let M be a C^ω-manifold, $E, F \to M$ C^ω-vector bundles and $A : C^\infty(M,E) \to C^\infty(M,F)$ an elliptic operator with C^ω-coefficients. If M is non-compact, then A is surjective.*

Theorem 1.7 ([BO95a], Lemma 2.4, 2.6). *Let a discrete group U act properly on a space M and let $\mathcal{F}$ be a soft or flabby U-equivariant sheaf on M.
Then $H^i(U, \mathcal{F}(M)) = 0$ for all $i \geqslant 1$.*

Combining Theorem 1.5 and Theorem 1.6, we get a short exact sequence for $\lambda \notin \{-\frac{1}{2}, -\frac{3}{2}, -\frac{5}{2}, \dots\}$

$$0 \longrightarrow V^{-\omega}_{\pi^1\lambda} \xrightarrow{i.(f_{-\lambda})} C^\infty(X) \xrightarrow{\Delta_X - \mu(\lambda)} C^\infty(X) \longrightarrow 0.$$

By Theorem 1.7, this is a Γ-acyclic resolution of $V^{-\omega}_{\pi^1\lambda}$, so for $\lambda \notin -\frac{1}{2} - \mathbb{N}_0$, we get

i	$H^i(\Gamma, V^{-\omega}_{\pi^1\lambda})$
0	$ker(\Delta_Y - \mu(\lambda))$
1	$coker(\Delta_Y - \mu(\lambda))$

Since $(\Delta_Y - \mu(\lambda))$ is an elliptic operator, both spaces are finite dimensional. To compute ${}^{\Gamma}V^{-\omega}_{\pi^1\lambda}$ for $\lambda \in -\frac{1}{2} - \mathbb{N}_0$, we need $H^i(\Gamma, V^{-\omega}_{\pi^1\lambda})$ for $i \geq 1, \lambda \in \frac{1}{2} + \mathbb{N}$ as well.
In the case of $\lambda \in \frac{1}{2} + \mathbb{N}$, we have the exact sequence

$$0 \to F_{2n+1} \to V^{-\omega}_{\pi^{1}{}_{n+\frac{1}{2}}} \to D^{-\omega}_{2n+1} \to 0$$

Since in this case $\mu(\lambda) = \frac{1}{4} - \lambda^2$ is negative, $\Delta_Y - \mu(\lambda)$ is a positive operator on a compact manifold and therefore injective,so ${}^{\Gamma}V^{-\omega}_{\pi^1\lambda}$ is trivial. Using the long exact sequence

$$... \to H^i(\Gamma, F_{2n+1}) \to H^i(\Gamma, V^{-\omega}_{\pi^{1}{}_{n+\frac{1}{2}}}) \to H^i(\Gamma, D^{\omega}_{2n+1}) \to ...$$

and the topological result

i	$dim(H^i(\Gamma, F_{2n+1}))$	$dim(H^i(\Gamma, F_1))$
0	0	1
1	(2g-2)(2n+1)	2g
2	0	1

,

where g denotes the genus of Y, we get

i	$dim(H^i(\Gamma, F_{2n+1}))$	$dim(H^i(\Gamma, V^{-\omega}_{\pi^1\lambda}))$	$dim(H^i(\Gamma, D^{-\omega}_{2n+1}))$
0	0	0	(2g-2)(2n+1)
1	(2g-2)(2n+1)	0	0
2	0	0	0

.

For $\lambda = 1/2$, we have that $dim(ker(\Delta_Y)) = dim(coker(\Delta_Y)) = 1$, so we get

i	$dim(H^i(\Gamma, F_1))$	$dim(H^i(\Gamma, V^{-\omega}_{\pi^1 1/2}))$	$dim(H^i(\Gamma, D_1^{-\omega}))$
0	1	1	2g
1	2g	1	2
2	1	0	0

.

Here, the long exact sequence gave us that $dim(H^1(\Gamma, D_1^{-\omega}))$ is either 1 or 2 and since $D_1^{-\omega}$ is the sum of two conjugate isomorphic G submodules, it has to be even.
For $\lambda \in -1/2 - \mathbb{N}_0$, we have a long exact sequence

$$... \to H^i(\Gamma, D^{-\omega}_{2n+1}) \to H^i(\Gamma, V^{-\omega}_{\pi^1\lambda}) \to H^i(\Gamma, F_{2n+1}) \to ...$$

Using the result for positive λ, we obtain immediately for $\lambda \in -\frac{1}{2} - \mathbb{N}$

i	$dim(H^i(\Gamma, D^{-\omega}_{2n+1}))$	$dim(H^i(\Gamma, V^{-\omega}_{\pi^1\lambda}))$	$dim(H^i(\Gamma, F_{2n+1}))$
0	(2n+1)(2g-2)	(2n+1)(2g-2)	0
1	0	(2n+1)(2g-2)	(2n+1)(2g-2)
2	0	0	0

.

In the case of $\lambda = -1/2$, it is

i	$dim(H^i(\Gamma, D^{-\omega}_1))$	$dim(H^i(\Gamma, V^{-\omega}_{\pi^1 -\frac{1}{2}}))$	$dim(H^i(\Gamma, F_1))$
0	2g	2g	1
1	2	2g+1	2g
2	0	1	1

.

Everything except $dim(H^0(\Gamma, V^{-\omega}_{\pi^1 -\frac{1}{2}}))$ is determined by the long exact sequence. For $dim(H^0(\Gamma, V^{-\omega}_{\pi^1 -\frac{1}{2}}))$, we could have $2g$ or $2g+1$, but all invariants are in the submodule $D_1^{-\omega}$ because otherwise we would get an embedding of $V^{\omega}_{\pi^1\frac{1}{2}}$ into $L^2(\Gamma\backslash G)$ which is impossible, since $V^{\omega}_{\pi^1\frac{1}{2}}$ is not unitary.

In all these cases, the cohomology groups are finite-dimensional and $\mathscr{X}(\Gamma, V^{-\omega}_{\pi^1\lambda}) = 0$.

Let us now consider the general case, so let $G_{\mathbb{C}}$ be a connected reductive group over $\mathbb{C}$, $G_{\mathbb{R}}$ a real form, $K \subset G_{\mathbb{R}}$ maximal compact, and $\Gamma \subset G_{\mathbb{R}}$ be cocompact, torsion-free and discrete. Let $Mod(\mathfrak{g}, K)$ the category of $(\mathfrak{g}, K)$ modules and by $\mathscr{HC}(\mathfrak{g}, K) \subset Mod(\mathfrak{g}, K)$ the subcategory of Harish-Chandra modules. Recall that a module is called Harish-Chandra if it is finitely generated over the universal enveloping algebra $\mathscr{U}(\mathfrak{g})$ and admissible, i.e. $\forall \gamma \in \hat{K},\ dim(V(\gamma)) < \infty$, where $V(\gamma)$ is the γ-isotypical component of V. For a G module V denote the K-finite submodule by $V_{K-fin} = \{v \in V | dim(span(Kv)) < \infty\}$. For $V \in \mathscr{HC}(\mathfrak{g}, K)$ we will denote by $\tilde{V} := Hom(V, \mathbb{C})_{K-fin}$ the dual in $\mathscr{HC}(\mathfrak{g}, K)$.

Definition 1.8. For a Harish-Chandra module V, the maximal globalisation of V is $MG(V) := Hom_{(\mathfrak{g},K)}(\tilde{V}, C^\infty(G)_{K-fin})$.

Thus, the maximal globalisation $MG(V)$ is a particular continuous representation of G whose K-finite part is isomorphic to V.
For a Harish-Chandra module V, define the hyperfunction vectors of V to be $V^{-\omega} := (\tilde{V}_B^{\omega})^*$, the topological dual of the analytic vectors in any Banach globalisation of the dual representation.

Theorem 1.9 (Schmid [Sch85]). $MG(V) \cong V^{-\omega}$.

Theorem 1.10 (Bunke/Olbrich [BO97a], 1.4). *For $V \in \mathcal{HC}(\mathfrak{g}, K)$,*

$$H^*(\Gamma, MG(V)) \cong \bigoplus_{(\pi, V_\pi) \in \hat{G}_u} m_\Gamma(\pi) Ext^*_{(\mathfrak{g},K)}(\tilde{V}, V_{\pi,K-fin}).$$

For the proof of this theorem, we need the generalisation of Theorem 1.7

$$H^i(\Gamma, C^\infty(G)_{K-fin}) = 0 \ \forall i \geq 1$$

and the following generalisation of Theorem 1.5.

Theorem 1.11 (Kashiwara-Schmidt [KS94]).

$$Ext^i_{(\mathfrak{g},K)}(\tilde{V}, C^\infty(G)_{K-fin}) = \begin{cases} MG(V), & i = 0 \\ 0, & i > 0 \end{cases}$$

Now, let us chose a projective resolution $P. \to \tilde{V} \to 0$ of $\tilde{V}$ in $Mod(\mathfrak{g}, K)$. By Theorem 1.11, $Hom_{(\mathfrak{g},K)}(P., C^\infty(G)_{K-fin})$ resolves $MG(V)$ Γ-acyclically , therefore $Hom_{\mathfrak{g},K}(P., C^\infty(\Gamma\backslash G)_{K-fin})$ calculates $H^*(\Gamma, MG(V))$. Finally, an analytic argument ([**BO97a**], Lemma 3.1) shows that only a finite part of the decomposition

$$C^\infty(\Gamma\backslash G)_{K-fin} = \widehat{\bigoplus}_{(\pi,V_\pi)\in\hat{G}_u} m_\Gamma(\pi) V^\infty_{\pi,K-fin}$$

contributes to $Hom_{(\mathfrak{g},K)}(P., C^\infty(\Gamma\backslash G)_{K-fin})$.

2. The Selberg Zeta function

In this section we will additionally assume that the real rank of G is equal to one. A group element $g \in G$ is called hyperbolic if there exist $m_g \in M$, $a_g \in A$ such that g is conjugate in G to $m_g a_g$ with $a_g^\rho > 1$. If g is hyperbolic, a_g is unique, and m_g is unique up to conjugation in M. Note that if $\gamma \in \Gamma$, $\gamma \neq 1$, then γ is hyperbolic.
For a hyperbolic $g \in G$, we define

$$Z(g, \sigma, \lambda) := \prod_{k=0}^{\infty} det(1 - \sigma_{\lambda-2\rho}(m_g a_g) \otimes S^k(Ad(m_g a_g)|_{\overline{\mathfrak{n}}})),$$

where $\sigma_{\lambda-2\rho}$ denotes the representation of P as in section 1 and S^k is the k'th symmetric power.
In the example of $G = SL(2,\mathbb{R})$, a hyperbolic g is conjugate to $m_g \begin{pmatrix} t^{1/2} & 0 \\ 0 & t^{-1/2} \end{pmatrix}$, $t > 1$. Therefore, we get

$$Z(g, 1, \lambda) = \prod_{k=0}^{\infty} (1 - t^{-\lambda-k-1/2}).$$

Definition 2.1. The conjugacy class $[\gamma] \in C\Gamma$ is called primitive, if $[\gamma]$ is not of the form $[\gamma] = [\gamma'^n]$ for some $\gamma' \in \Gamma, n > 1$.

Definition 2.2. The Selberg Zeta function is defined for $\Re(\lambda) > \rho$ by the converging product

$$Z(\Gamma,\sigma,\lambda) = \prod_{\substack{[\gamma]\in C\Gamma\\ [\gamma] \text{ primitive}}} Z(\gamma,\sigma,\lambda).$$

Theorem 2.3 (Fried [Fri86]). *$Z(\Gamma,\sigma,\lambda)$ has a meromorphic continuation to $\mathfrak{a}^*_{\mathbb{C}}$.*

The proof uses Ruelles thermodynamic formalism. The disadvantage of this method of proof is that is doesn't give any information about the singularities of the continuation or about a functional equation.
We use the Selberg trace formuala to calculate the singularities of the Zeta function and to determine the functional equation. To this end, we introduce the logarithmic derivative

$$L(\Gamma,\sigma,\lambda) := \frac{Z'(\Gamma,\sigma,\lambda)}{Z(\Gamma,\sigma,\lambda)},$$

considered as a 1-form on $\mathfrak{a}^*_{\mathbb{C}}$. Denote by X^d the dual of the symmetric space X, i.e.

X	H^m	$\mathbb{C}H^m$	$\mathbb{H}H^m$	CaH^2
X^d	S^n	$\mathbb{C}P^m$	$\mathbb{H}P^m$	CaP^2

.

In the case of $n := dimX$ odd, we assume that either σ is irreducible and the isomorphism class of σ is fixed under the action of the Weyl group, or $\sigma = \sigma' \oplus \sigma'^w$ with w the non-trivial element of the Weyl group $W(\mathfrak{g},\mathfrak{a}) \cong \mathbb{Z}/2\mathbb{Z}$ and σ' irreducible.
The positive root system of $(\mathfrak{g},\mathfrak{a})$ is either of the form $\{\alpha\}$ or $\{\frac{\alpha}{2},\alpha\}$. We call α the long root and identify $\mathfrak{a}^*_{\mathbb{C}} \cong \mathbb{C}$ sending α to 1. Define the root vector $H_\alpha \in \mathfrak{a}$ corresponding to a positive root α by the condition that

$$\lambda(H_\alpha) = \frac{\langle\lambda,\alpha\rangle}{\langle\alpha,\alpha\rangle} \quad \forall\lambda\in\mathfrak{a}^*,$$

where $\langle .,.\rangle$ denotes an $Ad(G)$ -invariant inner product on $\mathfrak{g}$. For $\sigma\in\hat{M}$, we define $\varepsilon_\alpha(\sigma)\in\{0,\frac{1}{2}\}$ by the condition $e^{2\pi i\varepsilon_\alpha(\sigma)} = \sigma(exp(2\pi i H_\alpha)$ and $\varepsilon_\sigma\in\{0,\frac{1}{2}\}$ by requiring $\varepsilon_\sigma \equiv |\rho| + \varepsilon_\alpha(\sigma) \bmod \mathbb{Z}$.

Theorem 2.4 (Bunke/Olbrich [BO95b]). *There exists a virtual elliptic operator $A_X(\sigma)$ and a corresponding operator $A_{X^d}(\sigma)$, such that in the case of n even*

$$
\begin{aligned}
&(1) \qquad \frac{L(\Gamma,\sigma,\lambda)}{2\lambda} &=& \; {}'Tr'(\lambda^2 + A_Y(\sigma))^{-1} \\
&(2) & - & \; \frac{(-1)^{\frac{n}{2}} vol(Y)\pi}{vol(X^d)} \frac{P_\sigma(\lambda)}{2\lambda} \begin{cases} tan(\pi\lambda) & if \varepsilon_\sigma = \frac{1}{2} \\ -cot(\pi\lambda) & if \varepsilon_\sigma = 0 \end{cases} \\
&(3) & - & \; \frac{(-1)^{\frac{n}{2}} vol(Y)}{vol(X^d)} {}'Tr'(\lambda^2 - A_{X^d}(\sigma))^{-1}
\end{aligned}
$$

and in the case of n odd,

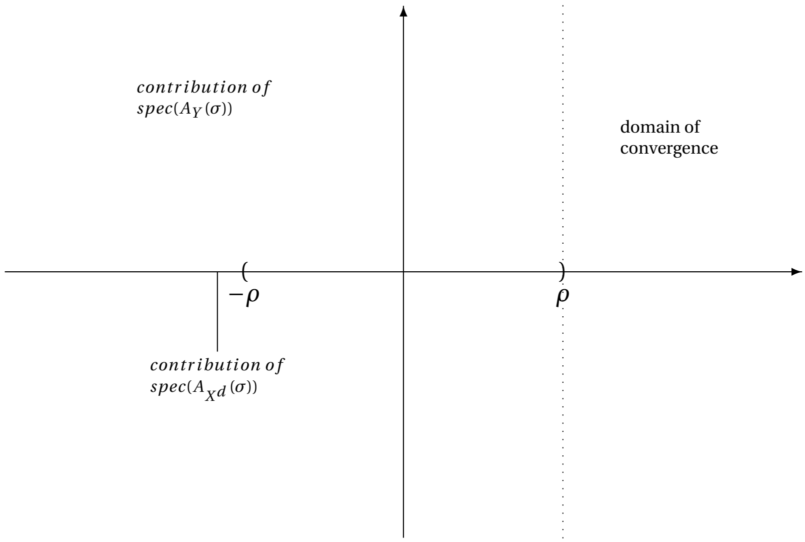
$$
\begin{aligned}
\frac{L(\Gamma,\sigma,\lambda)}{2\lambda} &= {}'Tr'(\lambda^2 + A_Y(\sigma))^{-1} \\
&+ (-1)^{\frac{n+1}{2}} \frac{vol(Y)\pi}{vol(X^d)} \frac{P_\sigma(\lambda)}{2\lambda}
\end{aligned}
$$

In the theorem, P_σ is some polynomial depending on σ. Since $(\lambda^2 + A_Y^2(\sigma))^{-1}$ is not trace-class, we have to use a regularised trace ${}'Tr'$. We refer to [**BO95b**], 3.2. for a discussion of the regularisation.

From this formula, it follows that $L(\Gamma,\sigma,\lambda)$ is meromorphic.
In the case of n even, (2) is odd and (3) is even. The poles of the two terms cancel for $\lambda > 0$, add up for $\lambda < 0$ and $I := 2\lambda((2) + (3))$ is regular at zero ([**BO95b**], 3.2.3). Hence, the set of poles of I is $\varepsilon_\sigma - \mathbb{N}$.
For A an operator on some Hilbert space, ${}'Tr'(\lambda - A)^{-1}$ has first order poles at the eigenvalues of A with $res\ (2\lambda' Tr'(\lambda^2 - A)^{-1})$ equal to the dimension of the corresponding eigenspace, hence

$$res_{-\varepsilon_\sigma - k} I = -\frac{(-1)^{\frac{n}{2}} 2 vol(Y)}{vol(X^d)} dim E_{A_{X^d}(\sigma)}(-\varepsilon_\sigma - k)^2.$$

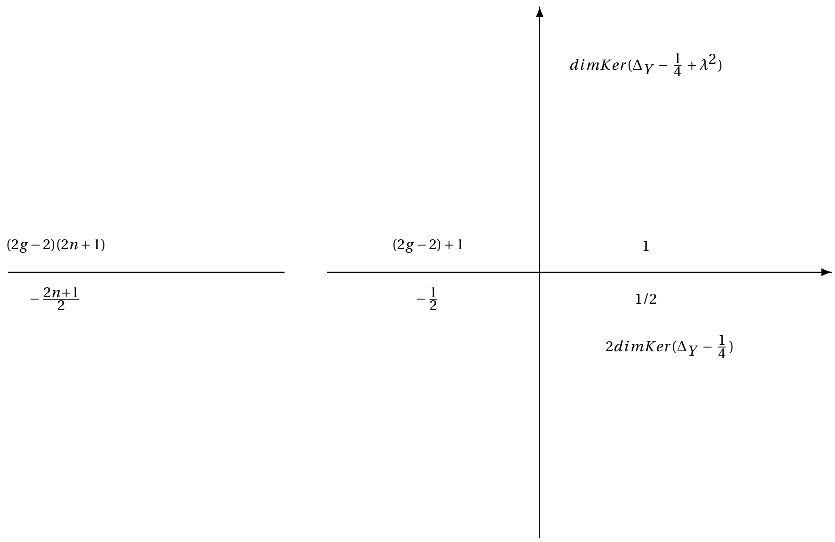
It follows from Hirzebruch proportionality that $\frac{vol(Y)}{vol(X^d)} \in \mathbb{Z}$ ([**BO95b**], Prop. 3.14), so the residues of I are integral.
(1) has first order poles at $\pm is$ with residue $dim E_{A_Y(\sigma)}(s^2)$, where s^2 is a non-zero eigenvalue of $A_Y(\sigma)$ and if zero is an eigenvalue of $A_Y(\sigma)$, there is an additional pole at zero with residue $2 dim ker A_Y(\sigma)$.
In the case of n odd, the only poles are those coming from the first term.
Note that all the residues of L are integral.



We'll now come back to the example of $G = SL(2,\mathbb{R})$, $\sigma = 1$. The above theorem specializes to

$$\frac{L(\Gamma,\sigma,\lambda)}{2\lambda} :=' Tr'(\Delta_Y - \frac{1}{4} + \lambda^2)^{-1} + \frac{2g-2}{2}{}' Tr'(\Delta_{S^2} + \frac{1}{4} - \lambda^2)^{-1} + \frac{2g-2}{2}\,\frac{\pi\tan(\lambda\pi)}{\lambda},$$

where g is the genus of $\Gamma\backslash X$. The eigenvalues of Δ_{S^2} are $\frac{n^2}{4}$, $n \in \mathbb{N}$ with $mult(n^2/4) = 2n+1$. Using this, we get the following picture for $ord(Z(\Gamma,1,\lambda))$.



Theorem 2.5 (Bunke/Olbrich, conjectured by Patterson[BO95a])
For $\lambda \neq 0$ [(1)],

$$ord_\lambda Z(\Gamma,\sigma,\lambda) = -\mathscr{X}'(\Gamma, V^{-\omega}_{\pi^{\sigma_\lambda}}),$$

where $\mathscr{X}'(\Gamma, V^{-\omega}_{\pi^{\sigma_\lambda}}) = \Sigma_{p=0}^{\infty}(-1)^p p\, dim H^p(\Gamma, V^{-\omega}_{\pi^{\sigma_\lambda}})$.

Proof. In the example, the proof is just by inspection. In the general case, use Theorem 1.10 to obtain

$$\begin{aligned}
& H^p(\Gamma, V^{-\omega}_{\pi^{\sigma_\lambda}}) \\
= & \bigoplus_{(\pi,V_\pi)\in\hat{G}_u} m_\Gamma(\pi) Ext^p_{(\mathfrak{g},K)}(V_{\pi^{\sigma^*_{-\lambda}},K-fin}, V_{\pi,K-fin}) \\
= & \bigoplus_{(\pi,V_\pi)\in\hat{G}_u} m_\Gamma(\pi) Ext^p_{(\mathfrak{g},K)}(V_{\pi^*,K-fin}, V_{\pi^{\sigma_\lambda},K-fin}) \\
= & \bigoplus_{(\pi,V_\pi)\in\hat{G}_u} m_\Gamma(\pi) Hom_{MA}(H_p(\mathfrak{n}, V_{\pi^*}) \oplus H_{p-1}(\mathfrak{n}, V_{\pi^*}), V_{\sigma_\lambda}) \\
= & \bigoplus_{(\pi,V_\pi)\in\hat{G}_u} m_\Gamma(\pi) Hom_{MA}([H^{n-1-p}(\mathfrak{n}, V_{\pi^*,K-fin}) \\
& \qquad \oplus H^{n-p}(\mathfrak{n}, V_{\pi^*,K-fin})] \otimes \Lambda^{n-1}\mathfrak{n}, V_{\sigma_\lambda}) \\
= & \bigoplus_{(\pi,V_\pi)\in\hat{G}_u} m_\Gamma(\pi) \left[((H^p(\mathfrak{n}, V_{\pi,K-fin}) \oplus H^{p-1}(\mathfrak{n}, V_{\pi,K-fin})) \otimes V_{\sigma^*_{-\lambda}})^{MA} \right]^*
\end{aligned}$$

Setting

$$\mathscr{X}(\pi,\sigma,\lambda) := \sum_{p=0}^{\infty} (-1)^p dim(H^p(\mathfrak{n}, V_{\pi,K-fin}) \otimes V_{\sigma^*_{-\lambda}})^{MA},$$

the following theorem finishes the proof. □

Theorem 2.6 (Juhl, [Juh], [Juh01]).

$$ord_\mu Z(\Gamma,\sigma,\lambda) = (-1)^{n-1} \sum_{(\pi,V_\pi)\in\hat{G}_u} m_\Gamma(\pi)\mathscr{X}(\pi,\sigma,\mu).$$

3. Generalisations

There are two obvious directions in which the theory could be generalised. First, one could try to generalise the group G. Unfortunately, we do not know of a satisfactory general definition of a Zeta function for groups of higher rank, but see [**MS89**], [**MS91**], [**Dei00**] for special cases.
The other direction is to weaken the conditions on Γ. In [**BO98**] and [**BO97b**], the

(1) For $\lambda = 0$ see [**BO95a**]

theory is generalised to the case of subgroups $\Gamma \subset G$ of finite covolume. Examples of infinite covolume (convex cocompact) are considered in [**BO99**].

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THE EIGENCURVE: A BRIEF SURVEY

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***Abstract*.** We present the notion of a p-adic family of modular eigenforms and survey briefly how it led from its beginnings in Serre's work on p-adic zeta functions to the construction of the Coleman-Mazur eigencurve. We motivate the two constructions of this eigencurve by briefly presenting how the fibre of the eigencurve over a fixed weight-character can be constructed. Finally we survey some of the progress that has been made beyond the case of $\mathrm{GL}_{2,\mathbb{Q}}$.

In his beautiful paper [**Ser73**], Serre presented the notion of a p-adic analytic family of modular eigenforms. Indeed, in that work, Serre used a single such family: the family of p-adic Eisenstein series. He showed how the mere existence of the Eisenstein family, single-handedly, provides the many congruences (and more) that guarantee the p-adic analyticity of the p-adic zeta function.

We let p denote a prime number, and $\mathbb{C}_p$ the completion of an algebraic closure of $\mathbb{Q}_p$. Consider a p-adic closed disc D (defined over $\mathbb{C}_p$) around a positive integer k_0. Let $\mathscr{A}$ denote the ring of p-adic analytic functions over D. For every integer $k \in D$, there is a specialization map $\mathscr{A} \to \mathbb{C}_p$ which is obtained by evaluating a function in $\mathscr{A}$ at the point k. A *p-adic family of modular eigenforms* parameterized by D is a formal q-expansion

$$f(q) = \sum_n a_n q^n \in \mathscr{A}[[q]],$$

such that for all large enough integers $k \in D$, the specialization of $f(q)$ at k, i.e.,

$$f_k(q) = \sum_n a_n(k) q^n \in \mathbb{C}_p[[q]],$$

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is the q-expansion of an eigenform of weight k. Let us be more precise and specify a level: here we assume that all eigenforms are of level $\Gamma_1(Np)$, where N is a positive integer prime to p, and is often referred to as the *tame level*.

This definition is formulated to capture the situation with the Eisenstein series. The constant term of each p-adic Eisenstein series is a special value of the Riemann zeta function(1), and thus, the analyticity of the function defined by the constant terms alone would provide the p-adic zeta function (that is the p-adic analytic interpolation of those special values). On the other hand, the rest of the q-expansion coefficients of the Eisenstein series are of quite a simple nature, and the analyticity of the functions they define can be very simply verified. Serre took advantage of this dichotomy: he showed that the latter analyticity will give us the former one for free, by proving that in a p-adic family the constant terms can be calculated from the rest of the coefficients in an analytic manner.

Serre's work provided the first application of p-adic families, and a dazzling one too. It took, however, more than a decade before further applications of p-adic families were realized, for the simple reason that it was not so clear how to construct p-adic families of modular eigenforms besides the "ur-example" of Eisenstein family. It was Hida who took the first major step in providing families of eigenforms. Hida's results (e.g., [**Hid86b, Hid86a**]), though limited to the case of *ordinary* modular forms, proved instrumental in many number-theoretical applications (e.g., Greenberg-Steven's proof of the Mazur-Tate-Teitelbaum conjecture for elliptic curves). Hida also studied p-adic analytic families of Galois representations attached to ordinary modular eigenforms. Hida's approach inspired the work of Mazur and Wiles [**MW86**] on families of Galois representations, and consequently, Mazur's theory of deformation of Galois representations [**Maz89**] which turned out to be a crucial ingredient of Wiles's proof of Fermat's Last Theorem.

Another decade had to pass before the "ordinary" restriction which appeared in Hida's work could be eased in any substantial way. Already, in Serre's work, it had become clear that one must extend the classical notion of modular forms to include *p-adic* (and later, *overconvergent*) modular forms. These form p-adic Banach (or Frechet) spaces containing the finite-dimensional spaces of classical modular forms, which can be thought of as the p-adic interpolation of those classical spaces. Employing a rigid-analytic vision of overconvergent modular forms which was essentially borrowed from earlier work of Katz [**Kat73**], Coleman [**Col97, Col96**] proved the existence of many families: that almost every overconvergent eigenform of *finite slope* lives in a p-adic family. The slope of an eigenform is the p-adic valuation of its U_p-eigenvalue, and having finite slope is a vast generalization of being ordinary, i.e., having a U_p-eigenvalue which is a p-adic unit. Coleman's important

(1) To be more precise, we shall rather say "the Riemann zeta function devoid of its Euler factor at p."

result [**Col96**] that overconvergent modular forms of small slope are classical, enabled him to prove the existence of an abundance of p-adic families of classical modular forms. Coleman's work was motivated by, and answered, a variety of questions and conjectures that Gouvêa and Mazur had made based on ample numerical evidence (see, for instance, [**GM92**]).

The theory reached an aesthetic culmination in Coleman-Mazur's [**CM98**] organization of Coleman's results (and more) in the form of a geometric object which was labeled the *eigencurve*. It is a rigid-analytic curve whose points correspond to normalized finite-slope p-adic overconvergent modular eigenforms of a fixed tame level N. To avoid misleading the reader, we shall point out that here tame level N signifies a wider class of modular forms than those with level $\Gamma_1(Np)$; for instance it consists of all those of level $\Gamma_1(Np^m)$ for all $m \geqslant 1$. If these modular forms can be put together to form a rigid-analytic variety, it is reasonable to expect their weights to vary p-adically analytically as well. We shall not, therefore, restrict ourselves to integral weights. It turns out that for this intuition to work, one should combine data obtained from the weight and the character of the modular form into a single object called the *weight-character*. The weight-characters will be points on a rigid analytic curve, $\mathscr{W}_{N,p}$, called the *weight space*[(2)] of tame level N. More precisely, the set of $\mathbb{C}_p$-valued points of $\mathscr{W}_{N,p}$ is given by

$$\mathscr{W}_{N,p}(\mathbb{C}_p) = \mathrm{Hom}_{\mathrm{cont}}(\mathbb{Z}_p^* \times (\mathbb{Z}/N\mathbb{Z})^*, \mathbb{C}_p^*),$$

the set of continuous $\mathbb{C}_p^*$-valued characters of $\mathbb{Z}_p^* \times (\mathbb{Z}/N\mathbb{Z})^*$. If a modular form of level $\Gamma_1(Np^m)$ has weight k and $(\mathbb{Z}/Np^m\mathbb{Z})^*$-character ε, then its weight-character is $\kappa := \lambda_k\varepsilon$, where λ_k is the character $z \mapsto z^k$ on $\mathbb{Z}_p^*$, and ε is now thought of as a character of $\mathbb{Z}_p^* \times (\mathbb{Z}/N\mathbb{Z})^*$ by composing with reduction mod p^m. There are however many more possibilities for a weight-character in general. It is easy to show that the weight space, as a rigid analytic variety, is isomorphic to a disjoint union of finitely many p-adic open discs.

The eigencurve, denoted $\mathscr{D}_{N,p}$, admits a projection onto the weight space

$$\mathscr{D}_{N,p} \to \mathscr{W}_{N,p}$$

which to every overconvergent eigenform associates its weight-character.

How does one construct the eigencurve? There are at least two methods: one method uses deformation rings, and another uses Hecke algebras. The rough idea of the first approach is the following. First, using the multiplicity-one theorem, we replace the normalized eigenform f by its system of eigenvalues for the Hecke operators. Then, we encode the Hecke system in the following way: the system of eigenvalues of the operators T_l where l is a prime not dividing Np will be encoded

(2) It would be more appropriate to call it the weight-character space of tame level N.

by the Galois representation attached to f, and the other eigenvalues will live in a finite-dimensional affine space $\mathbb{A}$. Denote by $\mathscr{R}$ the universal deformation space of all tame-level-N modular residual Galois (pseudo-) representations (of which there are finitely many). Then the system of Hecke eigenvalues we are looking for will live inside $\mathscr{R} \times \mathbb{A}$. The rest is to write down enough equations to specify the locus of $\mathscr{D}_{N,p}$, and those equation will be formed using enough of the characteristic power series of compact Hecke operators. A technical point is that one may have to further take the nilreduction of the so-obtained space in case it is not reduced. As a result of this construction, one sees that there is a p-adic family of Galois (pseudo-) representations parameterized by the eigencurve which at a point corresponding to an overconvergent eigenform f, specializes to ρ_f, the Galois representation attached to f.

It is worth noting that in [**Ser62**] Serre, based on Dwork's work, has developed the functional analysis of compact operators on p-adic Banach spaces, though it is Coleman's generalization of this theory in p-adic families that is applied in the above construction. This is a good opportunity to signal our preference of overconvergent modular forms to p-adic modular forms (which were constructed in [**Ser73**]): it is to ensure that U_p (and therefore, every Hecke multiple of it) is a compact operator, and in particular to provide enough defining equations in the above construction.

The Hecke-algebra method is based on the following (roughly-stated) principle: the systems of eigenvalues arising from a commutative algebra of operators T correspond to points of the “space" whose “structure ring” is T (at least under mild conditions). In this naïve form, however, the above principle is not quite useful: in most situations the algebra of operators T is not nearly “good” enough to provide a desirable space. One remedy is to replace T with a projective system of better-behaved algebras of operators which in their totality capture the same systems of eigenvalues. To shed some light on this matter, let me sketch how one can use this idea to construct the fibre $\mathscr{D}_\kappa$ of the Coleman-Mazur eigencurve $\mathscr{D}_{N,p}$ over a weight-character $\kappa \in \mathscr{W}_{N,p}(\mathbb{C}_p)$. The $\mathbb{C}_p$-valued points of $\mathscr{D}_\kappa$ correspond to normalized finite-slope Hecke eigenforms occurring in $M_\kappa^\dagger$, the space of overconvergent modular forms of weight-character κ and tame level N defined over $\mathbb{C}_p$.

Let $P_\kappa(x)$ denote the characteristic power series of the compact operator U_p acting on $M_\kappa^\dagger$. Let $\mathscr{H}_\kappa$ denote the Hecke algebra of $M_\kappa^\dagger$. Let $\mathscr{Z}_\kappa$ denote the fibre of the spectral curve over κ, i.e., the zero locus of P_κ in the rigid analytic affine line. Its points correspond to the inverses of non-zero eigenvalues of U_p acting on $M_\kappa^\dagger$. Let F denote a polynomial whose roots form a finite subset (with multiplicities) of these inverse eigenvalues. We have a factorization $P_\kappa(x) = F(x)G(x)$, and by p-adic Riesz theory, as developed by Serre in [**Ser62**], we can find a corresponding decomposition $M_\kappa^\dagger = M_F \oplus M_G$ such that $F(U_p)$ is nilpotent (and one can show in this case it is indeed zero) on M_F and invertible on M_G. Let T_F denote the algebra of operators of M_F induced by $\mathscr{H}_\kappa$. Then, it is easy to see that T_F is the ring of functions on an affinoid rigid analytic space $\mathscr{D}_{\kappa,F}$. This space is going to be the part of $\mathscr{D}_\kappa$ accounting for eigenforms for which the inverse of the U_p-eigenvalue is a root of $F(x)$. It turns out that varying F, we can glue the various $\mathscr{D}_{\kappa,F}$'s along $\mathscr{Z}_\kappa$ and construct $\mathscr{D}_\kappa$. In essence, the Hecke-algebra construction of $\mathscr{D}_{N,p}$ is an implementation of the above procedure in p-adic families over the weight space, though it is much more technically involved. Among other things, it requires Coleman's generalization of Serre's Fredholm theory and Riesz theory used in the above simplified construction.

The geometric picture provided by the eigencurve is so fascinating that it might make us forget, for a moment perhaps, how little we know about the eigencurve as a geometric object. There are many basic questions that are still unresolved; for instance, we still don't know whether the eigencurve has a finite or infinite number of connected components. Or whether it is smooth or proper over the weight space. Progress is being made, however. Recent work of Buzzard and Kilford [**BK05**] shows us that the 2-adic eigencurve indeed looks quite simple at the boundary of the weight space(3): it is the disjoint union of infinitely many p-adic annuli. Using this, Buzzard and Calegari [**BC**] have shown that the 2-adic eigencurve is proper over the weight space.

Another direction in which research in this area is taking place is the investigation of a similar theory for various reductive groups, other than $\mathrm{GL}_{2,\mathbb{Q}}$. Buzzard has axiomatized and generalized Coleman-Mazur's Hecke-algebra construction of the eigencurve in [**Buz04**] (see also Chenevier's work [**Che04**]). With this machinery at hand, at least two important steps have to be taken in the case of a general reductive group G. First, one needs to formulate a definition of overconvergent automorphic forms and Hecke algebras for the reductive group in question, and also a definition of p-adic analytic families of these objects. Equipped with an appropriate definition, one can apply Buzzard's general machinery to produce an *eigenvariety*, a rigid analytic variety whose points would correspond to a certain class of overconvergent automorphic forms for G. The second step, is to prove a "classicality criterion"; one which decides when an overconvergent automorphic form is classical, and hence enables us to identify (at least to some extent) the classical locus on the eigenvariety. This will allow us to use p-adic methods to obtain results about classical automorphic forms. In the case of $\mathrm{GL}_{2,\mathbb{Q}}$, for instance, these two steps were taken, respectively, in [**Col97, Col96**].

(3) Far from the set of classical weight-characters that is!

Let me mention, very briefly, some of the progress that has been made for groups other than $\mathrm{GL}_{2,\mathbb{Q}}$. Based on an idea of Stevens [**Ste**], Buzzard [**Buz**] developed the theory of overconvergent automorphic forms for the group of units of a quaternion algebra over $\mathbb{Q}$ which is ramified at infinity, completing the two steps alluded to earlier in this case. Later, following [**Buz**] closely, Buzzard [**Buz04**] and Yamagami [**Yam**] (with some technical differences) studied the case of a quaternion algebra over a totally real field which is ramified at all infinite primes (i.e., is *definite*). Chenevier [**Che04**] also used this idea to study the case of some unitary groups which are forms of $\mathrm{GL}_{n,\mathbb{Q}}$ and which are compact modulo center. These constructions are combinatorial in nature (the relevant "Shimura variety" is zero-dimensional), and some of the results such as the classicality criterion follow with considerably less effort than in Coleman's approach which is geometric in nature. The picture in these cases, however, is more intrinsic since unlike in Coleman's work, it doesn't wholly depend on the existence of a specific p-adic family (the Eisenstein family). In fact this dependence has been a main obstacle in creating an equally complete picture in other geometric situations. In [**Kas99, Kas04**] we investigated the above-mentioned first step in two cases: the group of units of a quaternion algebra over $\mathbb{Q}$ which is split at infinity, and the unitary group arising from a twist of a quaternion algebra over a totally real field $F \neq \mathbb{Q}$ which is split at one infinite place. In both cases the relevant Shimura curve is PEL [(4)] and one-dimensional, and we use Coleman's geometric approach. In [**Kas**] we also prove a result which specializes to an analogue of Coleman's classicality criterion for both cases. The proof, however, does not follow Coleman's, and is based on a more conceptual method presented in [**Kas06**]. In fact the results of [**Kas**] allow one to implement our *two* steps for Shimura curves which are not necessarily PEL; for instance one can study the case of the group of units of a quaternion algebra over a totally real field which is split at one place at infinity. In the case of $\mathrm{GL}_{2,F}$, where F is a totally real field, Kisin and Lai [**KL05**] have undertaken the study of the first step of the above strategy. The classicality criterion is however still missing in that context (though they provide a consolation classicality result for some applications). We are hopeful that the method in [**Kas06**] will be able to address this question at least after acquiring a better understanding of the canonical subgroups of Hilbert-Blumenthal abelian schemes.

As hinted at before, a common shortcoming of the (geometric) methods described in [**Kas99, Kas04, KL05**] is their incomplete treatment of the weights. This is because in these cases, one does not have an analogue of the Eisenstein family which is "full" with respect to the weight. A partial remedy comes with the lifting

(4) In other words, a moduli space of abelian varieties with polarization, endomorphism by a certain order, and level structure.

of a Hasse invariant over the special fibre of the Shimura variety to characteristic zero to create a family in a one-dimensional direction of the weight space. This underlines the need for a more intrinsic definition of overconvergent automorphic forms, even in the classical case of $\mathrm{GL}_{2,\mathbb{Q}}$. In fact even more is desirable: to have a p-adic overconvergent analogue of the classical formalism of automorphic forms and automorphic representations of general reductive groups. An important and promising approach is Emerton's more conceptual construction of the eigencurve [**Eme06**] using the theory of locally p-adic analytic representations of p-adic reductive groups developed by Schneider and Teitelbaum [**ST02b, ST01, ST02a, ST03**]. The approach is quite general, but it is less concrete[(5)] than Coleman-Mazur's and technical difficulties can arise for many groups. We Would like to mention the very recent approach of Iovita and Stevens [**IS**] where overconvergent modular forms of a general weight (and families thereof) are interpreted as sections of certain sheaves over rigid analytic regions in modular curves. There is also a general approach to the construction of the eigenvarieties by E. Urban (yet unpublished).

With an appropriate general theory at our disposal, it is tempting to conjecture that the various Langlands functoriality principles would p-adically interpolate to rigid analytic maps between eigenvarieties of the corresponding groups. For instance, Chenevier [**Che05**] has shown the existence of a p-adic Jacquet-Langlands correspondence between overconvergent automorphic forms for $\mathrm{GL}_{2,\mathbb{Q}}$, and those for a definite quaternion algebra over $\mathbb{Q}$. The proof proceeds by taking a p-adic "closure" of the classical Jacquet-Langlands correspondence[(6)].

Investigation of the Langlands philosophy in p-adic families is emerging as one of the mainstream directions of research in number theory for the years to come.

Remark 0.1 (Added after revision). Progress has been made since these notes were written. I will suffice it to mention that in light of recent work by Ash, Stevens, and collaborators on families of automorphic forms for GL_n, we now know that the eigenvariety machine introduced in [**Buz**] will not suffice for the construction of eigenvarieties in general as automorphic forms do not necessarily vary in families of the same dimension as the weight space.

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(5) Progress in [**BC, BK05**] depends on the rather concrete nature of Coleman-Mazur's construction.

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FLAG VARIETIES AND SCHUBERT CALCULUS

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***Abstract*.** We discuss recent developments in Schubert calculus.

1. Introduction

In 1879, H. Schubert laid some foundations for enumerative geometry.

Question 1.1. What is the number of lines in 3-space incident to four given lines in general position?

Schubert's "Principle of Conservation of Number": In a degeneration in which the number of solutions remains finite, this number remains constant, provided that multiplicities are properly taken into account.

Notation 1.2. Schubert uses the following notation:

– g = lines incident to a given line
– g_e = lines contained in a given plane ("e" for "Ebene")
– g_p = lines through a given point ("p" for "Punkt")
– g_s = lines through a given point in a given plane ("s" for "Strahlenbüschel")

Example 1.3. In this notation, Question 1.1 asks for the number g^4. By the above principle, two of the lines in general position can be moved such that they intersect. Then for a line to be incident to both requires it to lie in the plane that

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they span, or to pass through the point where they intersect; hence $g^2 = g_e + g_p$. It is easy to see that $g_e^2 = g_p^2 = 1$ and $g_e \cdot g_p = 0$. Hence $g^4 = (g_e + g_p)^2 = 2$.

In modern language, Question 1.1 is a calculation in the cohomology ring of the *Grassmannian variety* (or manifold)

$$\begin{aligned} G(2,4) &= \{\text{lines in 3-space } (=\mathbb{P}^3_{\mathbb{C}})\} \\ &= \{\text{2-dimensional linear spaces in } \mathbb{C}^4\}. \end{aligned}$$

The modern notation for Schubert calculus is for example

$$\int_{G(2,4)} \sigma_1^4 = ? \qquad \sigma_1^2 = \sigma_{11} + \sigma_2 \qquad \ldots \qquad \int_{G(2,4)} \sigma_1^4 = 2.$$

2. The Grassmannian $G(k,n)$

$$\begin{aligned} G(k,n) &= \{\Sigma \subset \mathbb{C}^n \mid \dim \Sigma = k\} \\ &= \{M \in \mathrm{Mat}_{k\times n} \mid \mathrm{Rank}\, M = k\}/(\text{row operations}) \\ &= P_k \backslash GL(n) \end{aligned}$$

is the Grassmannian manifold of linear subspaces of dimension k in $\mathbb{C}^n$ which can also be written as a $k \times n$ matrix of k basis vectors in $\mathbb{C}^n$ (up to a change of basis) or as a quotient of GL(n) where P_k is the parabolic subgroup of $n \times n$ matrices with a 0-block of size $k \times (n-k)$ in the upper right corner:

$$\begin{pmatrix} a_{1,1} & \ldots & a_{1,k} & 0 & \ldots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{k,1} & \ldots & a_{k,k} & 0 & \ldots & 0 \\ a_{k+1,1} & \ldots & a_{k+1,k} & a_{k+1,k+1} & \ldots & a_{k+1,n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n,1} & \ldots & a_{n,k} & a_{n,k+1} & \ldots & a_{n,n} \end{pmatrix}$$

Fact 2.1. $G(k,n)$ is a nonsingular projective variety of (complex) dimension $k \cdot (n-k)$. It embeds in $\mathbb{P}^{\binom{n}{k}-1}$ via the *Plücker embedding*

$$\begin{aligned} G(k,n) &\to \mathbb{P}^{\binom{n}{k}-1} = \{[\cdots : z_{i_1,\ldots,i_k} : \ldots]\} \\ \mathrm{Mat}_{k\times n} \ni M &\mapsto [\cdots : \det(M_{i_1,\ldots,i_k}) : \ldots] \end{aligned}$$

where the coordinates are indexed by $1 \le i_1 < \cdots < i_k \le n$ and $M_{i_1,\ldots,i_k}$ is the minor of columns $i_1, \ldots, i_k$.

$G(k,n)$ is covered by the open sets with $z_{i_1,\dots,i_k} \neq 0$ for any $1 \le i_1 < \cdots < i_k \le n$ denoted by $U_{i_1,\dots,i_k}$. Bringing the corresponding minor into the form

$$\begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 1 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 1 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \end{pmatrix},$$

we see that

$$U_{i_1,\dots,i_k} = \begin{pmatrix} & 0 & & 0 & & & & 0 & & 1 & \\ & 0 & & 0 & & & & 1 & & 0 & \\ * & \vdots & * & \vdots & * & \dots & * & \vdots & * & \vdots & * \\ & 0 & & 1 & & & & 0 & & 0 & \\ & 1 & & 0 & & & & 0 & & 0 & \end{pmatrix} \cong \mathbb{C}^{k(n-k)},$$

which explains that the dimension of $G(k,n)$ is $k(n-k)$.

Example 2.2. For $G(2,4)$, one open set is

$$U_{1,2} \ni \begin{pmatrix} 0 & 1 & a & b \\ 1 & 0 & c & d \end{pmatrix} \mapsto [z_{1,2} : z_{2,3} : z_{2,4} : z_{1,3} : z_{1,4} : z_{3,4}] = [-1 : -a : -b : c : d : ad-bc].$$

The image in $\mathbb{P}^5$ satisfies the relation $z_{1,2}z_{3,4} - z_{1,3}z_{2,4} + z_{1,4}z_{2,3} = 0$, and $G(2,4)$ is the quadric defined by this.

In general, $G(k,n) \in \mathbb{P}^{\binom{n}{k}-1}$ is cut out by quadratic equations, but it is not a complete intersection.

Every matrix is row-equivalent to a unique matrix in row-echelon form, and its rank is the number of pivots. In our situation, every $M \in \mathrm{Mat}_{k\times n}$ of rank k can be transformed by row transformations to a unique matrix of the form

$$\begin{pmatrix} * & \dots & * & 0 & * & \dots & * & 0 & * & \dots & * & 1 & 0 & \dots & 0 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots & & & & & & & & \vdots \\ \vdots & & \vdots & 0 & * & \dots & * & 1 & 0 & \dots & \dots & \dots & \dots & \dots & 0 \\ * & \dots & * & 1 & 0 & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & 0 \end{pmatrix} \cong \mathbb{C}^{i_1+\cdots+i_k-\frac{k(k+1)}{2}}$$

Definition 2.3. For any $1 \le i_1 < \cdots < i_k \le n$, the *Schubert cell* $X^0_{i_1,\dots,i_k}$ is defined as

$$\{M \in \mathrm{Mat}_{k\times n} \mid \text{in echelon form, the pivots are in columns } i_1,\dots,i_k\} / \mathrm{GL}(k).$$

Fact 2.4.

$$G(k,n) = \bigsqcup_{1 \le i_1 < \cdots < i_k \le n} X^0_{i_1,\dots,i_k}.$$

Definition 2.5. The *Schubert varieties* are defined as $X_{i_1,\dots,i_k} = \overline{X^0_{i_1,\dots,i_k}}$.

Fact 2.6. Schubert varieties are algebraic subvarieties of $G(k,n)$, each defined by vanishing of some set of the coordinates $z_{i_1,\dots,i_k}$.

There is a cellular structure on $G(k,n)$, given by the union of Schubert cells of real dimension not larger than j for $j = 0,1,2,\dots,2k(n-k)$:

$$\{\text{point}\} = X^0 = X^1 \subset X^2 = X^3 \subset \cdots \subset X^{2k(n-k)} = G(k,n)$$

For cellular structures in general, the condition $H_i(X^j, X^{j-1}) = 0$ must be satisfied if $i \neq j$. In our situation, as $X^0 = X^1, X^2 = X^3, \dots$, we have

$$H_*(G(k,n),\mathbb{Z}) = H_*(\mathbb{Z} \leftarrow 0 \leftarrow \mathbb{Z} \leftarrow 0 \leftarrow \mathbb{Z}^{m_2} \leftarrow \dots \leftarrow 0 \leftarrow \mathbb{Z}^{m_j} \leftarrow 0 \leftarrow \dots)$$

where

$$m_j = \#\{1 \le i_1 < \cdots < i_k \le n \mid i_1 + \cdots + i_k - \frac{k(k+1)}{2} = j\}.$$

Therefore

$$H_{2j}(G(k,n),\mathbb{Z}) = \mathbb{Z}^{m_j}, \qquad H_{2j+1}(G(k,n),\mathbb{Z}) = 0.$$

Cohomology is obtained from this by Poincare duality.

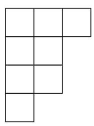
Theorem 2.7. *$G(k,n)$ has cohomology only in even degrees, and $H^{2j}(G(k,n),\mathbb{Z}) = \mathbb{Z}^{m_j}$ where m_j is the number of partitions of the integer j into at most k parts with each part less than or equal to $n-k$.*

Definition 2.8 (The language of partitions). Consider a partition, usually denoted by λ, for example

$$8 = 3 + 2 + 2 + 1.$$

The order is irrelevant, each term is called a *part* and must be positive. By convention, the parts are usually given in decreasing order. We allow extra, irrelevant parts which are equal to 0.

The corresponding diagram is:



The weight of λ is denoted by $|\lambda|$ and is 8 in this case.

Denote the partition

$$k \cdot (n-k) = (n-k) + \cdots + (n-k)$$

consisting of k equal parts $n-k$ by $(n-k)^k$, and for two partitions $\lambda = a_1 + \cdots + a_k$ and $\lambda' = a'_1 + \cdots + a'_k$, we write $\lambda \subset \lambda'$ if the diagram of λ is contained in the diagram of λ', i.e. if $a_i \leq a'_i$ for all $i \in \{1, \ldots, k\}$.

In this language, the number m_j as above is given by the number of partitions of j contained in a rectangle of size $k \times (n-k)$.

$$\begin{aligned} m_j &= \#\{a_1 \geqslant \cdots \geqslant a_k \geqslant 0 \mid a_1 \leq n-k, \sum_i a_i = j\} \\ &= \#\{\text{partitions } \lambda \mid |\lambda| = j, \lambda \subset (n-k)^k\} \end{aligned}$$

The Schubert cells can also be defined as

$$X^0_{i_1,\ldots,i_k} = \{\Sigma \subset \mathbb{C}^n \mid \dim(\Sigma \cap \mathbb{C}^j) = \#(\{i_1, \ldots, i_k\} \cap \{1, \ldots, j\}) \text{ for } j = 1, \ldots, n\}$$

Fact 2.9. We have the following decomposition of a Schubert variety into Schubert cells:

$$\begin{aligned} X_{i_1,\ldots,i_k} &= \bigcup_{i'_1 \leq i_1, \ldots, i'_k \leq i_k} X^0_{i'_1,\ldots,i'_k} \\ &= \{\Sigma \subset \mathbb{C}^n \mid \dim(\Sigma \cap \mathbb{C}^j) \geq \#(\{i_1, \ldots, i_k\} \cap \{1, \ldots, j\}) \text{ for } j = 1, \ldots, n\} \end{aligned}$$

Using the language of partitions, we write

$$X_\lambda = X_{n-k+1-\lambda_1, n-k+2-\lambda_2, \ldots, n-\lambda_k}$$

where $\lambda = (\lambda_1 \geqslant \cdots \geqslant \lambda_k \geqslant 0)$. Then

$$X_\lambda = \{\Sigma \subset \mathbb{C}^n \mid \dim(\Sigma \cap \mathbb{C}^{n-k+i-\lambda_i}) \geq i \text{ for } i = 1, \ldots, k\}.$$

The corresponding *Schubert classes* are defined as $\sigma_\lambda = [X_\lambda] \in H^{2|\lambda|}(G(k,n), \mathbb{Z})$. We have

$$H^*(G(k,n), \mathbb{Z}) = \bigoplus_{\lambda \subset (n-k)^k} \mathbb{Z} \cdot \sigma_\lambda.$$

Example 2.10. Explicitly, for $G(2,4)$, we have the following Schubert classes:

– $\sigma_\varnothing = \begin{pmatrix} * & * & 0 & 1 \\ * & * & 1 & 0 \end{pmatrix}$ corresponding to all $\Sigma \subset \mathbb{C}^4$ (lines in $\mathbb{P}^3$)

– $\sigma_1 = \begin{pmatrix} * & 0 & * & 1 \\ * & 1 & 0 & 0 \end{pmatrix}$ corresponding to Σ meeting $\mathbb{C}^2$ non-trivially (lines incident to a given line)

– $\sigma_{11} = \begin{pmatrix} * & 0 & 1 & 0 \\ * & 1 & 0 & 0 \end{pmatrix}$ corresponding to all $\Sigma \subset \mathbb{C}^3$ (lines in a plane)

– $\sigma_2 = \begin{pmatrix} 0 & * & * & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$ corresponding to Σ containing $\mathbb{C}^1$ (lines through a given point)

– $\sigma_{21} = \begin{pmatrix} 0 & * & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$ corresponding to $\mathbb{C}^1 \subset \Sigma \subset \mathbb{C}^3$ (lines in a plane through a point)

– $\sigma_{22} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$ corresponding to $\Sigma = \mathbb{C}^2$ (a given line)

Schubert varieties can be defined with respect to any *complete flag*

$$F_\bullet = F_1 \subset F_2 \subset \cdots \subset F_n = \mathbb{C}^n.$$

Let $E_\bullet$ be the standard complete flag $E_1 = \mathbb{C}^1 \subset E_2 = \mathbb{C}^2 \subset \cdots \subset E_n = \mathbb{C}^n$.

Defining $X_\lambda(F_\bullet)$ in an obvious way, we have $X_\lambda(E_\bullet) = X_\lambda$ with X_λ defined as above. Any $X_\lambda(F_\bullet)$ is the translate of X_λ by a suitable element of $\mathrm{GL}(n)$.

Consider an intersection

$$X_{\lambda^1}(F_\bullet^1) \cap \cdots \cap X_{\lambda^j}(F_\bullet^j) \supset X_{\lambda^1}^0(F_\bullet^1) \cap \cdots \cap X_{\lambda^j}^0(F_\bullet^j).$$

In good situations, this is a dense open inclusion, and the variety on the right is non-singular and either empty or of dimension $k(n-k) - |\lambda^1| - \cdots - |\lambda^j|$.

By Kleiman's Bertini theorem (see [**Har77**], Theorem III.10.8, or the original paper [**Kle74**]), for a general tuple $(F_\bullet^1, \ldots, F_\bullet^j)$, we are in a good situation. When $|\lambda^1| + \cdots + |\lambda^j| = k(n-k)$, this tells us that the intersection is a finite set of points, each a point of transverse intersection, and hence

$$c_{\lambda^1,\ldots,\lambda^j} := \int_{G(k,n)} \sigma_{\lambda^1} \cup \cdots \cup \sigma_{\lambda^j} = \#(X_{\lambda^1}(F_\bullet^1) \cap \cdots \cap X_{\lambda^j}(F_\bullet^j)).$$

Here, $\int_{G(k,n)}$ denotes evaluation of the degree of a zero-dimensional cycle class:

$$\sigma_{(n-k)^k} \mapsto 1$$
$$\sigma_\lambda \mapsto 0 \text{ if } |\lambda| < k(n-k).$$

For $j = 2$, a result of Richardson (see [**Ric92**]) states that the intersection of two Schubert varieties in general position is irreducible.

As an exercise, one can prove that any general pair of flags can be mapped by a suitable element of $\mathrm{GL}(n)$ to a specific pair consisting of the standard flag $E_\bullet$ where $E_i = \mathrm{span}(e_1, \ldots, e_i)$ and the opposite flag $F_\bullet$ with $F_i = \mathrm{span}(e_{n+1-i}, \ldots, e_n)$.

Proposition 2.11. *$X_\lambda(E_\bullet) \cap X_\mu(F_\bullet)$ is empty if and only if $\lambda_i + \mu_{k+1-i} > n-k$ for some i. Otherwise, a dense open subset consists of elements*

$$\begin{pmatrix}
0 & \cdots\cdots\cdots\cdots\cdots\cdots & 0 & * & \cdots & * & 1 & 0 & \cdots & 0 \\
0 & \cdots\cdots\cdots\cdots & 0 & * & \cdots & * & 1 & 0 & \cdots\cdots & 0 \\
\vdots & & & & & & & & & \vdots \\
0 & \cdots\cdots & 0 & * & \cdots & * & 1 & 0 & \cdots\cdots\cdots & 0 \\
0 & \cdots & 0 & * & \cdots & * & 1 & 0 & \cdots\cdots\cdots\cdots\cdots\cdots & 0
\end{pmatrix}$$

Here, the first $*$ in row i is in column $k+1-i+\mu_i$, and the 1 is in column $n+1-i-\lambda_{k+1-i}$ (i.e. the 1 are in the same positions as the pivots in σ_λ, and the positions of the first $*$'s are the pivots of σ_μ turned around). The leftmost $*$ in each row stands for a non-zero entry while the others can be arbitrary.

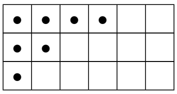
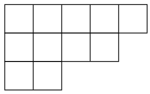
Proof. Suppose $\lambda_i + \mu_{k+1-i} > n-k$ for some i, and suppose there is a $\Sigma \in X_\lambda(E_\bullet) \cap X_\mu(F_\bullet)$. Then $\dim(\Sigma \cap E_{n-k+i-\lambda_i}) \geqslant i$, and $\dim(\Sigma \cap F_{n+1-i-\mu_{k+1-i}}) \geqslant k+1-i$. This implies $\dim(\Sigma \cap E_{n-k+i-\lambda_i} \cap F_{n+1-i-\mu_{k+1-i}}) \geqslant 1$, which gives us a contradiction since $E_{n-k+i-\lambda_i} \cap F_{n+1-i-\mu_{k+1-i}} = 0$.

For the second part, the matrix above describes a subset of the right dimension, and by Richardson's theorem, the intersection is irreducible. Therefore, this subset must be dense. □

Next, consider the special case $|\lambda| + |\mu| = k(n-k)$. Then

$$\int_{G(k,n)} \sigma_\lambda \cup \sigma_\mu = \delta_{\mu,\lambda^\vee}$$

where $\lambda^\vee = (n-k-\lambda_k, \ldots, n-k-\lambda_1)$

$\lambda =$  $\lambda^\vee =$ 

(Here, λ consists of the boxes marked by $\bullet$ inside the $k \times (n-k)$ box, and $\lambda^\vee$ is the complement of λ inside this box turned around.)

Therefore, in general,

$$\sigma_\lambda \cup \sigma_\mu = \sum_\nu c^\nu_{\lambda\mu} \sigma_\nu \text{ where } c^\nu_{\lambda\mu} = c_{\lambda\mu\nu^\vee}.$$

Hence, triple intersection numbers determine the multiplication in $H^*(G(k,n))$.

3. Combinatorial rules: Schubert calculus

Generally, *Schubert calculus* refers to Schubert's methods, involving symbolic manipulations, for the solution of enumerative problems. More specifically, it refers to methods, both classical and modern, for computing in $H^*(G(k,n))$ and similar rings.

The three main methods of Schubert calculus are:

1. degeneration techniques (Schubert, Pieri, ...)
2. triple intersections (Hodge–Pedoe, ...)
3. algebraic methods

Definition 3.1. The Schubert class corresponding to the partition (i) for some i is called a *special Schubert class* and is denoted by σ_i.

Products of an arbitrary and a special Schubert class can be calculated easily:

Proposition 3.2 (Pieri's formula). *We have*

$$\sigma_\lambda \cup \sigma_i = \sum_\mu \sigma_\mu,$$

where μ must fulfill the conditions that $\mu \supset \lambda$, $|\mu| = |\lambda| + i$, and μ/λ has at most one box in every column.

Here, μ/λ denotes the set of boxes in μ which are not in λ.

Example 3.3. In $H^*(G(2,4))$, we calculate $\sigma_1^2 = \sigma_1 \cup \sigma_1 = \sigma_{11} + \sigma_2$ where

$$\sigma_1 = \square \qquad \sigma_{11} = \begin{array}{c}\square\\[-4pt]\square\end{array} \qquad \sigma_2 = \square\!\square$$

Similarly, we compute (note that μ in Pieri's formula above must fit in the $k \times (n-k)$ box)

$$\sigma_1^3 = \sigma_{11} \cup \sigma_1 + \sigma_2 \cup \sigma_1 = \sigma_{21} + \sigma_{21} = 2\sigma_{21}$$

and

$$\sigma_1^4 = 2\sigma_{21} \cup \sigma_1 = 2\sigma_{22}.$$

Therefore,

$$\int_{G(2,4)} \sigma_1^4 = 2.$$

Proposition 3.4 (Giambelli's formula). *We can write an arbitrary Schubert class σ_λ, where $\lambda = (\lambda_1, \ldots, \lambda_\ell)$, in terms of the special Schubert classes $\sigma_1, \ldots, \sigma_{n-k}$:*

$$\sigma_\lambda = \det \begin{pmatrix} \sigma_{\lambda_1} & \sigma_{\lambda_1+1} & \sigma_{\lambda_1+2} & \cdots & \cdots \\ \sigma_{\lambda_2-1} & \sigma_{\lambda_2} & \sigma_{\lambda_2+1} & \cdots & \cdots \\ \sigma_{\lambda_3-2} & \sigma_{\lambda_3-1} & \sigma_{\lambda_3} & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \sigma_{\lambda_\ell} \end{pmatrix}$$

Here, the determinant is evaluated using the $\cup$-product, and we use the convention that $\sigma_0 = 1$ and $\sigma_i = 0$ if $i < 0$ or $i > n-k$.

Also we have a ring presentation $H^*(G(k,n),\mathbb{Z}) = \mathbb{Z}[\sigma_1,\dots,\sigma_{n-k}]/(R_{k+1},\dots,R_n)$ where

$$R_i = \det\begin{pmatrix} \sigma_1 & \sigma_2 & \sigma_3 & \dots\dots \\ 1 & \sigma_1 & \sigma_2 & \dots\dots \\ 0 & 1 & \sigma_1 & \dots\dots \\ \dots\dots & \dots\dots & \dots\dots & \dots\dots \\ 0 & \dots & 0 \quad 1 & \sigma_1 \end{pmatrix}$$

is the determinant of an $i \times i$-matrix.

4. Flag varieties

Having considered only Grassmannian varieties so far, we now look at complete flag varieties

$$F(n) = \{\Sigma_1 \subset \Sigma_2 \subset \cdots \subset \mathbb{C}^n\} = B\setminus \mathrm{GL}(n)$$

where B is the Borel group of lower triangular matrices.

We can also consider partial flag varieties

$$F(a_1,\dots,a_j;n) = \{\Sigma_1 \subset \cdots \subset \Sigma_j \subset \mathbb{C}^n \mid \dim\Sigma_i = a_i \text{ for } i = 1,\dots,j\} = P\setminus \mathrm{GL}(n)$$

where P is a lower parabolic matrix corresponding to $a_1,\dots,a_j$.

Example 4.1. For $j = 2$ and $F(3,5;7)$, the Schubert cell / variety corresponding to $\sigma_{136,27} \in H^*(F(3,5;7))$ is described by the matrix

$$\left(\begin{array}{ccccccc} * & 0 & * & * & 0 & * & 1 \\ * & 0 & * & * & 1 & 0 & 0 \\ * & 1 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & * & * & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

where the first three rows span Σ_1, and all five rows span Σ_2.

In general,

$$H^*(F) = \bigoplus_{s\in S} \mathbb{Z}\cdot\sigma_s$$

for a suitable set S. In case of the complete flag variety $F(n)$, we have $S = S_n$, the permutation group.

In the case of a Grassmannian, i.e., $j = 1$, $k = a_1$, we have $S = \{1 \le i_1 < \cdots < i_k \le n\}$. The permutation corresponding to $\sigma_{i_1,\dots,i_k}$ can be described in the following way: We define $1 \le j_1 < \cdots < j_{n-k} \le n$ so that $\{i_1,\dots,i_k\} \cup \{j_1,\dots,j_{n-k}\} = \{1,\dots,n\}$. Then the corresponding Grassmannian permutation is

$$\begin{pmatrix} 1 & \dots & k & k+1 & \dots & n \\ i_1 & \dots & i_k & j_1 & \dots & j_{n-k} \end{pmatrix} \in S_n$$

In a 2-step flag, for example $\sigma_{136,27}$ corresponds to

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 3 & 6 & 2 & 7 & 4 & 5 \end{pmatrix} \in S_7$$

consisting of three blocks of increasing numbers.

Remark 4.2. Certain minimum-length coset representatives of the Weyl group S_n (in this case) index the Schubert varieties – this is the situation for general homogeneous spaces $P\backslash G$.

Returning to the case $G(k,n)$, let $d \le \min(k, n-k)$ and consider the diagram

$$\begin{tikzcd} F(k-d,k,k+d;n) \arrow[r, "\psi"] \arrow[d, "\pi"'] & F(k-d,k+d;n) \\ G(k,n) & \end{tikzcd}$$

and set $\sigma_\lambda^{(d)} = \psi_* \pi^* \sigma_\lambda \in H^*(F(k-d,k+d;n))$. Looking at the permutation corresponding to σ_λ, this means sorting the $2d$ entries on positions $k-d+1,\dots,k+d$ to get three blocks of increasing numbers.

Theorem 4.3. *For $\lambda,\mu,\nu \subset (n-k)^k$, $|\lambda|+|\mu|+|\nu| = k(n-k)+dn$, we have*

$$\begin{aligned} c^{(d)}_{\lambda,\mu,\nu} &:= \int_{F(k-d,k+d;n)} \sigma_\lambda^{(d)} \cup \sigma_\mu^{(d)} \cup \sigma_\nu^{(d)} \\ &= \#\{\text{degree } d \text{ rational curves on } G(k,n) \text{ incident to } X_\lambda(E_\bullet), X_\mu(F_\bullet), X_\nu(G_\bullet)\} \end{aligned}$$

for general triples $(E_\bullet, F_\bullet, G_\bullet)$ of flags.

Remark 4.4. The setting for this result is the following: $H^* G(k,n)$ admits a "quantum" deformation

$$QH^* G(k,n) = \bigoplus_{\lambda \subset (n-k)^k} \mathbb{Z}[q] \cdot \sigma_\lambda$$

(as $\mathbb{Z}[q]$-modules), with

$$\sigma_\lambda * \sigma_\mu = \sum_{\nu, d \ge 0} c^{(d)}_{\lambda,\mu,\nu^\vee} q^d \sigma_\nu.$$

It is an amazing fact that this multiplication is associative! It comes from the general theory of *quantum cohomology*, which associates $QH^* X$ to $H^* X$ for general complex projective manifolds.

The multiplication in $QH^* X$ encodes enumerative information about rational curves in X.

Question 4.5. Getting back to $H^* G(k,n)$, we are interested in the constants $c_{\lambda,\mu,\nu^\vee}$ in the general product

$$\sigma_\lambda \cup \sigma_\mu = \sum_\nu c_{\lambda,\mu,\nu^\vee} \sigma_\nu$$

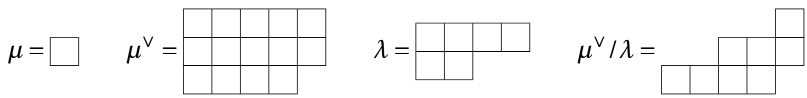
and similarly in the values of $c^{(d)}_{\lambda,\mu,\nu^\vee}$ for quantum extensions.

More generally, what are the structure constants in the product of Schubert classes for partial flag varieties?

The classical Littlewood–Richardson rule gives a combinatorial interpretation for $c_{\lambda,\mu,\nu}$, which has been extended recently to partial flag varieties.

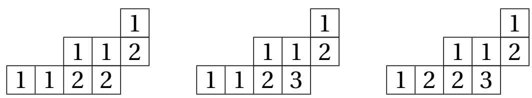
The classical rule for $G(k,n)$ is: $c_{\lambda,\mu,\nu}$ is equal to the number of *semistandard tableaux* of content ν on the skew diagram $\mu^\vee/\lambda$ (i.e. λ with $\mu^\vee$ removed as introduced in Lemma 3.2; the result is 0 if $\lambda \not\subset \mu^\vee$) satisfying a further combinatorial condition which is best illustrated by the following example:

Consider $G(3,8)$, $\lambda = 42$, $\mu = 1$. We want to calculate $\sigma_{42} \cup \sigma_1$. We have $\mu^\vee = 554$.

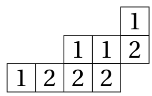


The resulting $\mu^\vee/\lambda$ can be filled with numbers which must satisfy the following conditions: in rows, the numbers must increase weakly from left to right; in columns, the numbers must increase strictly from top to bottom; and as the further "lattice word property", reading the numbers row by row from right to left must give a sequence such that for any initial subwords, $\#1's \geqslant \#2's \geqslant \ldots$ must hold.

These conditions can be fulfilled in three different ways



while the diagram



gives the sequence 12112221 which does not satisfy the lattice word property since the initial subword of length 7 contains more $2's$ than $1's$.

These diagrams with numbers are translated into the partition $\nu = (\#1's, \#2's, \ldots)$ of weight $|\mu^\vee| - |\lambda|$, and each of them contributes 1 to $c_{\lambda,\mu,\nu}$. The three examples give $\nu_1 = (53)$, $\nu_2 = (521)$, $\nu_3 = (431)$, with respective dual partitions (52), (43), (421). Therefore,

$$\sigma_{42} \cup \sigma_1 = \sigma_{52} + \sigma_{43} + \sigma_{421}.$$

Most proofs of the Littlewood–Richardson rule are combinatorial. Recently, there have been two geometric proofs by R. Vakil ([**Vak06**]) and I. Coskun ([**Cos04**]). These approaches are based on the technique of degeneration.

Example 4.6. To calculate $\sigma_1^2 \in H^* G(2,4)$, consider

$$\left\{\begin{pmatrix} 0 & 0 & * & * \\ * & * & 0 & 0 \end{pmatrix}\right\}$$

whose closure is 2-dimensional in $G(2,4)$. This describes the space of all Σ meeting $\operatorname{span}(e_3, e_4)$ and $\operatorname{span}(e_1, e_2)$ non-trivially. We degenerate the latter to $\operatorname{span}(e_2, e_3)$. In the limit, any Σ not meeting the new intersection must be contained in the new span, i.e. must lie in the following translate of σ_{11} or of σ_2:

$$\sigma_{11} = \left\{\begin{pmatrix} 0 & 0 & * & * \\ 0 & * & * & 0 \end{pmatrix}\right\} = \left\{\begin{pmatrix} 0 & * & * & * \\ 0 & 0 & * & * \end{pmatrix}\right\} \qquad \sigma_2 = \left\{\begin{pmatrix} * & * & * & * \\ 0 & 0 & * & 0 \end{pmatrix}\right\}$$

One can apply this repeatedly and get an algorithm, i.e., a combinatorial formula, for $\sigma_\lambda \cup \sigma_\mu$.

Example 4.7. $\sigma_1 \cup \sigma_{42} \in H^*(G(3,8))$:

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & * & * & * & * & 0 \\ * & * & * & * & * & 0 & 0 & 0 \end{pmatrix}$$

Degenerating $\operatorname{span}(e_1, \ldots, e_5)$ to $\operatorname{span}(e_2, \ldots, e_6)$ gives in the case that Σ intersects the intersection $\operatorname{span}(e_4, e_5, e_6)$:

$$\begin{pmatrix} * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & * & * & * & 0 & 0 \end{pmatrix}$$

and in the case that Σ is contained in the new spans:

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & * & * & * & * & 0 \\ 0 & * & * & * & * & * & 0 & 0 \end{pmatrix}$$

Coskun has extended this to a Littlewood–Richardson rule for two-step flag varieties ([**Cos04**]) and general flag varieties (research announcement). This gives a complete solution for the problem of multiplying Schubert classes in $\mathrm{GL}(n)$-homogeneous spaces.

Example 4.8. $\sigma_{2,15} \cup \sigma_{3,16} \in H^* F(1,3;8)$:

$$\left(\begin{array}{cccccccc} 0 & 0 & * & * & * & * & * & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & * & * & * \\ * & * & * & * & 0 & 0 & 0 & 0 \end{array}\right) = \begin{pmatrix} 0 & * & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & * & * & * \\ 0 & 0 & * & * & * & * & * & 0 \\ * & * & * & * & 0 & 0 & 0 & 0 \end{pmatrix}$$

This can be transformed into

$$\begin{pmatrix} 0 & 0 & * \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & * & * & * \\ * & * & * & * & * & * & * & 0 \\ 0 & 0 & * & * & * & 0 & 0 & 0 \end{pmatrix}$$

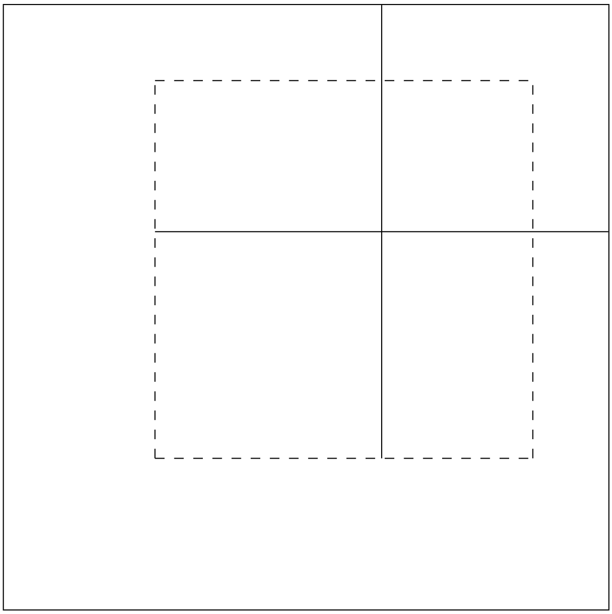
and

$$\begin{pmatrix} 0 & * & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & * & * & * \\ 0 & 0 & * & * & * & * & * & 0 \\ 0 & * & * & * & * & 0 & 0 & 0 \end{pmatrix}$$

and, as a new type of move which did not occur in the previous example:

$$\begin{pmatrix} 0 & * & * \end{pmatrix} \cdot \begin{pmatrix} * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & * & * & 0 \\ 0 & 0 & * & * & * & 0 & 0 & 0 \end{pmatrix}$$

Coskun also introduced the new notation of "Mondrian tableaux". For the last variety, it is given by the following picture:



5. Bibliographic remarks

General references on Schubert varieties are for example an article by Kleiman and Laksov [**KL72**] and Fulton's book [**Ful97**].

On enumerative geometry, we mention the original book by Schubert [**Sch79**] and Kleiman's modern treatment [**Kle76**].

Modern work in this area can be found in articles by Buch, Kresch, and Tamvakis [**BKT03**], Vakil [**Vak06**], and Coskun [**Cos04**].

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ZETA FUNCTION OF GRAPHS

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***Abstract*.** In this note we describe three algorithms for calculating the number of closed, reduced paths of a graph and discuss some applications.

1. Basic definitions

First we start with some basic definitions and facts about graphs.

Let $X = (V, E)$ be a graph with $V = VX$ as the set of vertices and $E = EX$ as the set edges. Assume that $|VX| = r_0 < \infty$, $|EX| = 2r_1 < \infty$ and that the graph X is directed in the sense that if $e \in EX$ then the inverse edge $\bar{e}$ also is in EX.

Definition 1.1 (Adjacency Matrix). The adjacency matrix of the graph X is the matrix A indexed by the pairs of vertices $x, y \in V$, such that $A = (a_{xy})$, where

$$a_{xy} = \text{ number of edges joining } x \text{ to } y.$$

1.1. Regular and simple graphs. A graph is called *simple* if there is at most one edge joining (adjacent) vertices and if there are no loops; hence, X is simple if and only if $a_{xy} \in \{0, 1\}$ for all $x, y \in V$ and $a_{xx} = 0$ for all $x \in V$.

If every vertex x of X possesses exactely k edges departing (or arriving) from x then X is called *k-regular*.

In this talk all graphs will be simple and regular.

July 2006.

1.2. Spectrum of a graph. For a finite graph X on r_0 vertices, the adjacency matrix A is a symmetric matrix, hence it has r_0 real eigenvalues. Counting multiplicities we may list the eigenvalues in decreasing order:

$$\lambda_0 \geq \lambda_1 \geq \ldots \geq \lambda_{r_0-1}.$$

The *spectrum* of X is the set of eigenvalues of A. If X is k-regular we have:

- $\lambda_0 = k$;
- λ_0 has multiplicity $1 \Longleftrightarrow X$ is connected;
- $-k \leq \lambda_i \leq k$ for $i = 0, 1, \ldots, r_0 - 1$.

Set $\lambda(X) := \max_{\lambda_i \neq \pm k} \{|\lambda_i|\}$.

1.3. Ramanujan graphs. For families of k-regular graphs with increasing number of vertices we have the following result:

Theorem 1.2 (Alon - Boppana [Alo86]). *Let $\{X_m\}_{m\geq 1}$ be a family of connected, k-regular, finite graphs, with $|V_m| \to \infty$ as $m \to \infty$. Then*

$$\liminf_{m\to\infty} \lambda(X_m) \geq 2\sqrt{k-1}.$$

This bound gives us a motivation for the following definition:

Definition 1.3 (Ramanujan graph). A finite, connected, k-regular graph X is a *Ramanujan graph* if, $\lambda(X) \leq 2\sqrt{k-1}$.

1.4. Some Operators and Definitions. For any field K we can associate to our graph X the K-vectorspaces:

- $C_0(X)$: K-vectorspace with base VX, thus $\dim_K C_0(X) = r_0$.
- $C_1(X)$: K-vectorspace with base EX, thus $\dim_K C_1(X) = 2r_1$.

We define the following operators:

$$\begin{array}{lrcl}
\text{(Vertex-Adjacency)} & A: C_0(X) & \longrightarrow & C_0(X) \\
 & x & \longmapsto & \sum_{\partial_0(e)=x} \partial_1(e) \\
\text{(Edge-Adjacency)} & T: C_1(X) & \longrightarrow & C_1(X) \\
 & e & \longmapsto & \sum_{(e,e_1)_{\text{red}}} e_1 \\
\text{(Reversal of edges)} & J: C_1(X) & \longrightarrow & C_1(X) \\
 & e & \longmapsto & J(e) = \bar{e}
\end{array}$$

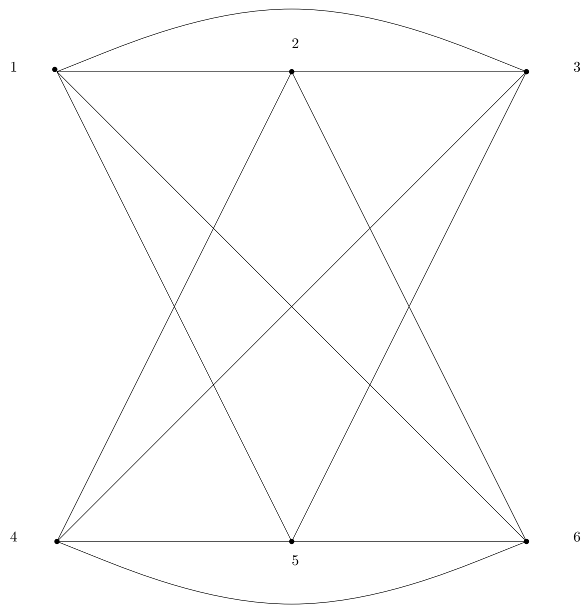
Finally:

$$\begin{array}{lrcl} & \partial_0, \partial_1 : C_1(X) & \longrightarrow & C_0(X) \\ \text{(Startpoint)} & e & \longmapsto & \partial_0(e) \\ \text{(Endpoint)} & e & \longmapsto & \partial_1(e) \end{array}$$

Next we will define some basic definitions about paths and cycles.

- A path form x to y: Tuple of edges $\gamma = (e_1, \dots, e_n)$ s.t. $\partial_0(e_1) = x$, $\partial_1(e_n) = y$ and $\partial_1(e_{i-1}) = \partial_0(e_i)$ for $1 < i \leqslant n$. Length of a path $l(\gamma) = n$.
- Backtrack: $\overline{e_{i+1}} = e_i$.
- Tail: $\overline{e_n} = e_1$.
- Cycle: $\partial_0(e_1) = \partial_1(e_n)$.
- $R_n^x := \#\{$Cycles around x without backtrack (reduced) of length $n\}$.
- $C_n^x := \#\{$Cycles around x without backtrack and tail of lenght $n\}$.
- Primitive cycle: Not of the shape $(e_1, \dots, e_k, e_1, \dots, e_k, \dots, e_1, \dots, e_k)$.
- Cycles are eqivalent iff: $(e_1, \dots, e_n) \sim (e_i, \dots, e_n, e_1, \dots, e_{i-1})$.
- $\mathscr{P}$: Eqivalenceclasses of primitive cycles.

1.5. Example: 4-regular finite upper half-plane graph $X_3(1,-1)$ with 6 Vertices – Computation with magma.



Adjacency matrix:

```
0 1 1 0 1 1
1 0 1 1 0 1
1 1 0 1 1 0
0 1 1 0 1 1
1 0 1 1 0 1
1 1 0 1 1 0
```

Spectrum:

```
Eig := [-2,-2,0,0,0,4];
```

We have $2 \leqslant 2\sqrt{3} \Longrightarrow$ Ramanujan graph.

Matrix T of $X_3(1,-1)$

```
[0  0  0  0  0  0  0  0  1  0  0  0  0  0  0  0  1  0  0  0  1  0  0  0]
[0  0  0  0  1  0  0  0  0  0  0  0  0  0  0  0  1  0  0  0  1  0  0  0]
[0  0  0  0  1  0  0  0  1  0  0  0  0  0  0  0  0  0  0  0  1  0  0  0]
[0  0  0  0  1  0  0  0  1  0  0  0  0  0  0  0  1  0  0  0  0  0  0  0]
[0  0  0  0  0  0  0  0  0  1  0  0  1  0  0  0  0  0  0  0  0  1  0  0]
[1  0  0  0  0  0  0  0  0  0  0  0  1  0  0  0  0  0  0  0  0  1  0  0]
[1  0  0  0  0  0  0  0  0  1  0  0  0  0  0  0  0  0  0  0  0  1  0  0]
[1  0  0  0  0  0  0  0  0  1  0  0  1  0  0  0  0  0  0  0  0  0  0  0]
[0  0  0  0  0  1  0  0  0  0  0  0  0  1  0  0  0  1  0  0  0  0  0  0]
[0  1  0  0  0  0  0  0  0  0  0  0  0  1  0  0  0  1  0  0  0  0  0  0]
[0  1  0  0  0  1  0  0  0  0  0  0  0  0  0  0  0  1  0  0  0  0  0  0]
[0  1  0  0  0  1  0  0  0  0  0  0  0  1  0  0  0  0  0  0  0  0  0  0]
[0  0  0  0  0  0  0  0  0  0  1  0  0  0  0  0  0  0  1  0  0  0  1  0]
[0  0  0  0  0  0  1  0  0  0  0  0  0  0  0  0  0  0  1  0  0  0  1  0]
[0  0  0  0  0  0  1  0  0  0  1  0  0  0  0  0  0  0  0  0  0  0  1  0]
[0  0  0  0  0  0  1  0  0  0  1  0  0  0  0  0  0  0  1  0  0  0  0  0]
[0  0  0  0  0  0  0  0  0  0  0  1  0  0  1  0  0  0  0  0  0  0  0  1]
[0  0  1  0  0  0  0  0  0  0  0  0  0  0  1  0  0  0  0  0  0  0  0  1]
[0  0  1  0  0  0  0  0  0  0  0  1  0  0  0  0  0  0  0  0  0  0  0  1]
[0  0  1  0  0  0  0  0  0  0  0  1  0  0  1  0  0  0  0  0  0  0  0  0]
[0  0  0  0  0  0  0  1  0  0  0  0  0  0  0  1  0  0  0  1  0  0  0  0]
[0  0  0  1  0  0  0  0  0  0  0  0  0  0  0  1  0  0  0  1  0  0  0  0]
[0  0  0  1  0  0  0  1  0  0  0  0  0  0  0  0  0  0  0  1  0  0  0  0]
[0  0  0  1  0  0  0  1  0  0  0  0  0  0  0  1  0  0  0  0  0  0  0  0]
```

Magma code for calculation of T out of A:

```
 //Construction of edge adjacency out of vertex adjacency

//First: Construct a list of the orientated egdges

function ad2T(A)
    n := NumberOfRows(A);
    L :=[];
    for i:=1 to n do
        for j:= 1 to n do
            if i ne j then
                if A[i,j] ne 0 then
                    Append(~L,[i,j]);
                    L1 :=[IntegerRing()|];
                    for k:=1 to n do
                        if k eq i then
                            continue;
```

```
                        end if;
                        if A[j,k] ne 0 then
                            Append(~L1,k);
                        end if;
                    end for;
                    Append(~L,L1);
                end if;
            end if;
        end for;
    end for;
    return L;
end function;
```

```
//Second: Construct the edge adjacency matrix out of the above list.

function L2T(L)
    n := Round(#L/2);
    print n;
    A := ZeroMatrix(IntegerRing(),n,n);
    for t:=1 to n do
        for s:=1 to #L[2*t] do
            A[Round((Index(L,[L[2*t-1][2],L[2*t][s]])+1)/2),t]:=1;
        end for;
    end for;
    return A;
end function;
```

2. Explicit Construction of Families of Ramanujan Graphs

All known explicit constructions are using number theory. These constructions exist only if k is of the form $k = p^n + 1$, where p is a prime number and $n \geq 1$.

Open question. Does there exist an explicit construction of a family of Ramanujan graphs for all $k \geq 3$? The first interesting case will be a family of 7-regular Ramanujan graphs.

We are considering here the following families of Ramanujan graphs:

- LPS graphs.
- Finite upper half-plane graphs.

Both families are Cayley graphs.

Cayley graphs. Let G be a group (finite or infinte) and let S be a nonempty, finite subset of G. We assume that S is symmetric; that is, $S = S^{-1}$.

Definition 2.1. The Cayley graph $\mathscr{C}(G,S)$ is the graph with vertex set $V = G$ and edge set

$$E := \{(x, y) \mid x, y \in G; \exists s \in S \text{ s.t. } y = xs\}.$$

We define $k := |S|$.

Properties of Cayley graphs. Let $\mathscr{C}(G,S)$ be a Caley graph.

- $\mathscr{C}(G,S)$ is a simple, k-regular, vertex-transitive graph.
- $\mathscr{C}(G,S)$ has no loop if and only if $1 \notin S$.
- $\mathscr{C}(G,S)$ is connected if and only if S generates G.
- If there exsts a homomorphism ψ from G to the multiplicative group $\{-1,1\}$, such that $\psi(S) = -1$, then $\mathscr{C}(G,S)$ is bipartite. The converse holds provided $\mathscr{C}(G,S)$ is connected.

3. LPS family of Ramanujan graphs

Let $p \equiv 1 \bmod 4$ be a prime number. Jacobi's theorem implies that the set of integer solutions

$$\begin{aligned} S := \{(x_0,x_1,x_2,x_3) \in \mathbb{Z}^4 \mid\ & x_0^2+x_1^2+x_2^2+x_3^2 = p \\ & \text{and } x_0 \geq 0, x_0 \text{ is odd} \\ & \text{and } x_i \text{ is even for } i = 1,2,3\} \end{aligned}$$

has exactly $p+1$ elements.

Let $q \equiv 1 \bmod 4$ be a another prime number. We associate to S a set of generators in the groups $\mathrm{PSL}_2(\mathbb{F}_q)$ and $\mathrm{PGL}_2(\mathbb{F}_q)$ respectively. Let $i \in \mathbb{F}_q$ be an element s.t. $i^2 = -1$.

- $\left(\frac{p}{q}\right) = 1$ i.e., the equation $x^2 = p$ has a solution δ in $\mathbb{F}_q$.

$$\begin{aligned} \tilde{S} := & \left\{ \frac{1}{\delta} \begin{pmatrix} x_0 + i x_1 & x_2 + i x_3 \\ -x_2 + i x_3 & x_0 - i x_1 \end{pmatrix} \text{ s.t. } (x_0,x_1,x_2,x_3) \in S \right\} \\ & \subseteq \mathrm{PSL}_2(\mathbb{F}_q) \end{aligned}$$

- $\left(\frac{p}{q}\right) = -1$ i.e, the equation $x^2 = p$ has no solution in $\mathbb{F}_q$.

$$\begin{aligned} \tilde{S} := & \left\{ \begin{pmatrix} x_0 + i x_1 & x_2 + i x_3 \\ -x_2 + i x_3 & x_0 - i x_1 \end{pmatrix} \text{ s.t. } (x_0,x_1,x_2,x_3) \in S \right\} \\ & \subseteq \mathrm{PGL}_2(\mathbb{F}_q) \end{aligned}$$

The graph $X^{(p,q)}$. We define the Cayley graph $X^{(p,q)}$ as follows:

$$X^{(p,q)} := \begin{cases} \mathscr{C}(\mathrm{PSL}_2(\mathbb{F}_q), \tilde{S}) & \text{if } \left(\frac{p}{q}\right) = 1, \\ \mathscr{C}(\mathrm{PGL}_2(\mathbb{F}_q), \tilde{S}) & \text{if } \left(\frac{p}{q}\right) = -1. \end{cases}$$

The latter will be bipartite.

The Theorem of Lubotzky-Philips-Sarnak. For all primes p, q as above we have the following theorem:

Theorem 3.1 (Lubotzky-Philips-Sarnak [Lub94]). *$X^{p,q}$ is a $(p+1)$-regular Ramanujan graph.*

Sketch of the proof.

- The Bruhat-Tits tree $\mathscr{T}_p$ of $\mathrm{PGL}_2(\mathbb{Q}_p)$ is a universal covering of the $(p+1)$ regular graph $X^{p,q}$ and

$$X^{p,q} \cong \Gamma(q)\backslash\mathscr{T}_p$$

where

$$\Gamma(q) := \ker(\mathrm{PSL}_2(\mathbb{Z}[\frac{1}{p}]) \longrightarrow \mathrm{PSL}_2(\mathbb{Z}[\frac{1}{p}]/q\mathbb{Z}[\frac{1}{p}]))$$

is a congruence subgroup of $\mathrm{PSL}_2(\mathbb{Q}_p)$.

- Interpretation of the adjacency matrix of $X^{p,q}$ as a Hecke operator defined over cusp forms of weight 2 of the congruence subgroup $\Gamma(q)$.
- Now the proof will be complete from Deligne's theorem about Ramanujan-Petersson conjecture:

  All eigenvalues of the Hecke operator defined over the cusp forms of weight g of the congruence subgroup $\Gamma(q)$ satisfy: $|\lambda| \leq 2p^{\frac{g-1}{2}}$.

In fact the Ramanujan-Petersson conjecture is a result of Weil's conjecture which was proved by Deligne in 1973. Deligne's proof uses a pure algebraic geometric method, up to date there is no other proof.

4. The family of finite upper half-plane graphs - Terras family

Let $\mathbb{F}_q$ be a finite field of order q and $\delta \in \mathbb{F}_q$ a nonsquare.
The finite upper half-plane $\mathbb{H}_q$ is by definition:

$$\mathbb{H}_q := \{z = x + \sqrt{\delta}y \mid x, y \in \mathbb{F}_q,\ y \neq 0\}.$$

Let be $a \in \mathbb{F}_q$ with $a \neq 0, 4\delta$. Define a graph $X_q(a,\delta)$ in the following way:
The set of vertices V are the points of the upper half-plane $\mathbb{H}_q$.
Two points $z_1, z_2 \in \mathbb{H}_q$ are adjacent if $d(z_1, z_2) = a$ where

$$d(z_1, z_2) := \frac{N(z_1 - z_2)}{\Im(z_1)\Im(z_2)},\ N(x + \sqrt{\delta}y) := x^2 - \delta y^2,\ \Im(x + \sqrt{\delta}y) := y.$$

The upper half-plane graph $X_q(a,\delta)$ is Ramanujan, [Cel94], [Kat93]. The graph $X_q(a,\delta)$ is $(q+1)$-regular. In fact it is a Caley graph:

$$X_q(a,\delta) = \mathscr{C}(\mathbb{A}_2(q), S_q(a,\delta))$$

where

$$\mathbb{A}_2(q) := \left\{ \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} \mid x, y \in \mathbb{F}_q\,, y \neq 0 \right\}$$

and

$$S_q(a,\delta) := \left\{ \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} \in \mathbb{A}_2(q) \ : \ x^2 = ay + \delta(y-1)^2 \right\}.$$

For $a \neq 0, 4\delta$ the equation $x^2 = ay + \delta(y-1)^2$ has exactly $q+1$ solution in $\mathbb{F}_q$. A. Terras conjectured that they are Ramanujan graphs, and this was proved by N. Katz.

5. Riemann zeta funktion and graphs

Theorem 5.1 (A. Sarveniazi, [Sar06]). *Let $\mathscr{F} = \{X_p\}_{p\in\mathscr{P}}$ be a certain family Ramanujan where $\mathscr{P}$ is the set of all prime numbers. For a complex number s with $\Re(s) > 2$, the Riemann zeta function $\zeta(s)$ satisfies:*

$$\zeta(s) = \prod_{p\in\mathscr{P}} (1+p^{1-s})^{\frac{2}{p-1}} \exp\left(\sum_{p\in\mathscr{P}} \frac{1}{-\chi_p} \sum_{n=1}^{\infty} \frac{B^{\mathscr{F}}_{p,n}(s)}{n} p^{\frac{-ns}{2}} \right)$$

where

$$B^{\mathscr{F}}_{p,n}(s) := C_n(X_p) - \frac{\mathrm{Tr}(A^n_{X_p})}{(1+p^{1-s})^n}\ .$$

$C_n(X_p)$ is the number of closed and tail-less paths without backtracking in X_p and $\chi_p := r_{0,p} - r_{1,p}$.

Goal.

- Find the most efficient way to calculate Riemann zeta function and its zero's from the formula (5.1) using a certain family of Ramanujan graphs.
- Find the most efficient algorithm to calculate $C_n(X_p)$ for possibly large numbers n and prime numbers p.
- Find the best family of Ramanujan graphs with the most efficient algorithm in order to calculate $C_n(X_p)$.
- Determine the distribution of C_n's for each family.
- Compare the growth of C_n's between different families.

Calculation-Algorithm I of C_n.

1. Calculate the adjacency matrix A;
2. Calculate from the matrix A the edge adjacency matrix T;
3. Calculate the eigenvalues of T using *Octave*;
4. Calculate the values of C_n as a sum of powers of these eigenvalues.

Matrix T associated to $X_3(1,-1)$.

```
[0 0 0 0 0 0 0 0 1 0 0 0 0 0 0 0 1 0 0 0 1 0 0 0]
[0 0 0 0 1 0 0 0 0 0 0 0 0 0 0 0 1 0 0 0 1 0 0 0]
[0 0 0 0 1 0 0 0 1 0 0 0 0 0 0 0 0 0 0 0 1 0 0 0]
[0 0 0 0 1 0 0 0 1 0 0 0 0 0 0 0 1 0 0 0 0 0 0 0]
[0 0 0 0 0 0 0 0 0 1 0 0 1 0 0 0 0 0 0 0 0 1 0 0]
[1 0 0 0 0 0 0 0 0 0 0 0 1 0 0 0 0 0 0 0 0 1 0 0]
[1 0 0 0 0 0 0 0 0 1 0 0 0 0 0 0 0 0 0 0 0 1 0 0]
[1 0 0 0 0 0 0 0 0 1 0 0 1 0 0 0 0 0 0 0 0 0 0 0]
[0 0 0 0 0 1 0 0 0 0 0 0 0 1 0 0 0 1 0 0 0 0 0 0]
[0 1 0 0 0 0 0 0 0 0 0 0 0 1 0 0 0 1 0 0 0 0 0 0]
[0 1 0 0 0 1 0 0 0 0 0 0 0 0 0 0 0 1 0 0 0 0 0 0]
[0 1 0 0 0 1 0 0 0 0 0 0 0 1 0 0 0 0 0 0 0 0 0 0]
[0 0 0 0 0 0 0 0 0 0 1 0 0 0 0 0 0 0 1 0 0 0 1 0]
[0 0 0 0 0 0 1 0 0 0 0 0 0 0 0 0 0 0 1 0 0 0 1 0]
[0 0 0 0 0 0 1 0 0 0 1 0 0 0 0 0 0 0 0 0 0 0 1 0]
[0 0 0 0 0 0 1 0 0 0 1 0 0 0 0 0 0 0 1 0 0 0 0 0]
[0 0 0 0 0 0 0 0 0 0 0 1 0 0 1 0 0 0 0 0 0 0 0 1]
[0 0 1 0 0 0 0 0 0 0 0 0 0 0 1 0 0 0 0 0 0 0 0 1]
[0 0 1 0 0 0 0 0 0 0 0 1 0 0 0 0 0 0 0 0 0 0 0 1]
[0 0 1 0 0 0 0 0 0 0 0 1 0 0 1 0 0 0 0 0 0 0 0 0]
[0 0 0 0 0 0 0 1 0 0 0 0 0 0 0 1 0 0 0 1 0 0 0 0]
[0 0 0 1 0 0 0 0 0 0 0 0 0 0 0 1 0 0 0 1 0 0 0 0]
[0 0 0 1 0 0 0 1 0 0 0 0 0 0 0 0 0 0 0 1 0 0 0 0]
[0 0 0 1 0 0 0 1 0 0 0 0 0 0 0 1 0 0 0 0 0 0 0 0]
```

Cycles (around x) for $X_3(1,-1)$ - $C_n/6 = C_n^x$ for $n = 1,\ldots,40$.

```
0, 0, 8, 20, 40, 112, 336, 1188, 3344, 9440, 29656,
89524, 264888, 795984, 2391968, 7177028, 21529888,
64549312, 193692840, 581227860, 1743368264,
5229933104, 15690734320, 47071910884, 141214574640,
423643727520, 1270931460728, 3812803380980
11438400876760, 34315165468432, 102945569829696,
308836760219268, 926510045017664, 2779530195862400,
8338590914608456, 25015772682195604, 75047317963623528,
225141951917163504, 675425857623271568,
2026277582020942628
```

Calculation-Algorithm II of C_n.

1. Calculate the adjacency matrix A;
2. Calculate from the matrix A the matrix A_n defined as follows: For $x, y \in X$ we define

$$A_n(x,y) := \#\ \text{Paths without backtracks from } x \text{ to } y \text{ of length.}$$

It is an exercise to proof:

$$A_1 = A$$
$$A_2 = A_1^2 - kE_{r_0}$$
$$A_{n+1} = A_1 A_n - (k-1)A_{n-1} \text{ if } n > 2.$$

3. Let y be a vertex connected with x. We define:
   - $R_n^{xy} :=$ #{Red. paths from x to y of length n} $= (A_n)_{xy}$.
   - $K_n^y :=$ #{Red. cycles around y of length n s.t. $e_1 \neq e_{yx}$ and $e_n \neq e_{xy}$}.
   - $L_n^{xy} :=$ #{ Red. paths form x to y of lenght n s.t. $e_1 \neq e_{xy}$}.
4. We conclude the following, recursive formulas:
   - $C_n^x = R_n^x - kK_{n-2}^y$.
   - $K_n^y = R_n^y - 2L_{n-1}^{xy} + K_{n-2}^x$.
   - $L_n^{xy} = R_n^{xy} - R_{n-1}^x + L_{n-2}^{xy}$.

   Initial values:
      a) $L_1^{xy} = 0$, $L_2^{xy} = R_2^{xy}$.
      b) $K_1^y = 0$, $K_2^y = 0$.
      c) $C_1^x = 0$, $C_2^x = 0$.
5. Calculate C_n recursively.

Zeta function of graphs.

Definition 5.2 (Zeta function).

$$Z(X,u) := \prod_{\gamma \in \mathscr{P}} \frac{1}{1-u^{l(\gamma)}}.$$

If we define $q := k-1$ and $u := q^{-s}$:

$$Z(X,s) = \prod_{\gamma \in \mathscr{P}} \frac{1}{1-q^{-sl(\gamma)}}.$$

Identities for the Zeta function.

Lemma 5.3.

$$Z(X,u) = \exp\left(\sum_{n=1}^{\infty} C_n \frac{u^n}{n}\right)$$
$$Z(X,u) = \frac{1}{\det(I_{r_0} - uT)},$$

where $\mathrm{Tr}(T^n) = C_n$ *for* $n > 0$.

Corollary 5.4. *The Zeta function is a rational function without zeros. The poles are the eigenvalues of T.*

Ihara-Bass formula [Iha66], [Has89], [Bas92].

Theorem 5.5. *Let be $\chi := r_0 - r_1$, $q := k-1$ and let I_{r_0} be the identity matrix. Then:*

$$Z(X,u) = \frac{(1-u^2)^\chi}{\det(I_{r_0} - uA + u^2 q I_{r_0})}.$$

From the lemma above we conlcude:

Corollary 5.6. $Z(X,u) = \frac{1}{\det(I_{2r_1} - uT)} = \frac{(1-u^2)^\chi}{\det((1+u^2q)I_{r_0} - uA)}$.

Ihara-Bass formula for $X_3(1,-1)$.

```
Determinant(1-uT) =
729*u^24 - 3888*u^22 - 432*u^21 + 7938*u^20 + 2160*u^19
- 6912*u^18 - 4032*u^17 + 639*u^16 + 3008*u^15 + 2976*u^14
+ 96*u^13 - 1412*u^12 - 1248*u^11 - 384*u^10 + 320*u^9
+ 327*u^8 + 192*u^7 + 16*u^6 - 48*u^5 - 30*u^4 - 16*u^3 + 1
```

```
Determinant(1-u*A+3*u^2)*(1-u^2)^6 =
729*u^24 - 3888*u^22 - 432*u^21 + 7938*u^20 + 2160*u^19
- 6912*u^18 - 4032*u^17 + 639*u^16 + 3008*u^15 + 2976*u^14
+ 96*u^13 - 1412*u^12 - 1248*u^11 - 384*u^10 + 320*u^9 +
327*u^8 + 192*u^7 + 16*u^6 - 48*u^5 - 30*u^4 - 16*u^3 + 1
```

Factorisation of $\det(1-uT)$ **and of** $\det(1-uA+pu^2)$**.**

```
 > f:=Determinant(1 - uT);
> f;
729*u^24 - 3888*u^22 - 432*u^21 + 7938*u^20 + 2160*u^19 - 6912*u^18 - 4032*u^17 + 639*u
     ^16 + 3008*u^15 + 2976*u^14 + 96*u^13 - 1412*u^12 -
    1248*u^11 - 384*u^10 + 320*u^9 + 327*u^8 + 192*u^7 + 16*u^6 - 48*u^5 - 30*u^4 - 16*u
          ^3 + 1
> Factorisation(f);
[
    <u - 1, 7>,
    <u + 1, 6>,
    <3*u - 1, 1>,
    <3*u^2 + 1, 3>,
    <3*u^2 + 2*u + 1, 2>
]
```

```
 > f:=Determinant(1 - u*A+3 *u^2);
> f;
729*u^12 + 486*u^10 - 432*u^9 - 81*u^8 - 432*u^7 - 108*u^6 - 144*u^5 - 9*u^4 - 16*u^3 +
     6*u^2 + 1
> Factorisation(f);
[
    <u - 1, 1>,
    <3*u - 1, 1>,
    <3*u^2 + 1, 3>,
    <3*u^2 + 2*u + 1, 2>
]
```

Calculation-Algorithm III for C_n using Ihara-Bass formula.

1. Calculate the adjacency matrix A.
2. Calculate the eigenvalues of A using *Matlab* or *Octave*.
3. From Ihara Bass formula, we have:

$$\begin{aligned}\mathrm{Eig}(T) = {} & [1/2(\lambda + i\sqrt{4p-\lambda^2} \mid \lambda \in \mathrm{Eig}(A)] \\ & \cup [1/2(\lambda - i\sqrt{4p-\lambda^2} \mid \lambda \in \mathrm{Eig}(A)] \\ & \cup [\underbrace{1,\dots,1}_{-\chi \text{ pieces}}] \\ & \cup [\underbrace{-1,\dots,-1}_{-\chi \text{ pieces}}].\end{aligned}$$

4. Calculate the values of the C_n's as a sum of powers of these eigenvalues.

6. Images of the Terras family

Thanks to Tim Oliver Kaiser ([**Kai06**]) we can show some images of the Terras family. They are available as an avi-movie at:

`http://www.uni-math.gwdg.de/wiedmann/movie.avi`

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LOOP GROUPS AND STRING TOPOLOGY

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***Abstract*.** In this note, a concise treatment of the representation theory of loop groups of compact Lie groups is given, following mainly the book "Loop groups" of Pressley and Segal. Starting with the basic definitions, we end up with the discussion of central extensions of loop groups, and with the classification of positive energy representations. We always compare with the parallel theory for compact Lie groups, and try to sketch concisely the basic constructions and ingredients of proofs.

1. Introduction

Let G be a compact Lie group. Then the space LG of all maps from the circle S^1 to G becomes a group by pointwise multiplication. Actually, there are different variants of LG, depending on the classes of maps one considers, and the topology to be put on the mapping space. In these lectures, we will always look at the space of smooth (i.e., C^∞) maps, with the topology of uniform convergence of all derivatives.

These groups certainly are not algebraic groups in the usual sense of the word. Nevertheless, they share many properties of algebraic groups (concerning e.g., their representation theory). There are actually analogous objects which are very algebraic (compare e.g. [**Fal03**]), and it turns out that those have properties remarkably close to those of the smooth loop groups.

The lectures are organized as follows.

(1) Lecture 1: Review of compact Lie groups and their representations, basics of loop groups of compact Lie groups.

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(2) Lecture 2: Finer properties of loop groups
(3) Lecture 3: the representations of loop groups (of positive energy)

The lectures and these notes are mainly based on the excellent monograph "Loop groups" by Pressley and Segal [**PS86**].

2. Basics about compact Lie groups

2.1 Definition. A Lie group G is a smooth manifold G with a group structure, such that the map $G \times G \to G; (g,h) \mapsto gh^{-1}$ is smooth.

The group acts on itself by left multiplication: $l_g(h) = gh$. A vector field $X \in \Gamma(TG)$ is called *left invariant*, if $(l_g)_* X = X$ for each $g \in G$. The space of all left invariant vector fields is called the *Lie algebra* $Lie(G)$. If we consider vector fields as derivations, then the commutator of two left invariant vector fields again is a left invariant vector field. This defines the Lie bracket

$$[\cdot,\cdot]\colon Lie(G) \times Lie(G) \to Lie(G); [X,Y] = XY - YX.$$

By left invariance, each left invariant vector field is determined uniquely by its value at $1 \in G$, therefore we get the identification $T_1 G \cong Lie(G)$; we will frequently use both variants.

To each left invariant vector field X we associate its flow $\Psi_X\colon G \times \mathbb{R} \to G$ (a priori, it might only be defined on an open subset of $G \times \{0\}$). We define the *exponential map*

$$\exp\colon Lie(G) \to G; X \mapsto \Psi_X(1,1).$$

This is defined on an open subset of $0 \in G$. The differential

$$d_0 \exp\colon Lie(G) \to T_1(G) = Lie(G)$$

is the identity, therefore on a suitably small open neighborhood of 0, exp is a diffeomorphism onto its image.

A maximal torus T of the compact Lie group G is a Lie subgroup $T \subset G$ which is isomorphic to a torus T^n (i.e. a product of circles) and which has maximal rank among all such. It's a theorem that for a connected compact Lie group G and a given maximal torus $T \subset G$, an arbitrary connected abelian Lie subgroup $A \subset G$ is conjugate to a subgroup of T.

2.2 Example. The group $U(n) := \{A \in M(n,\mathbb{C}) \mid AA^* = 1\}$ is a Lie group, a Lie submanifold of the group $Gl(n,\mathbb{C})$ of all invertible matrices.

In this case, $T_1 U(n) = \{A \in M(n,\mathbb{C}) \mid A + A^* = 0\}$. The commutator of $T_1 U(n) = L(U(n))$ is the usual commutator of matrices: $[A,B] = AB - BA$ for $A, B \in T_1 U(n)$.

The exponential map for the Lie group $U(n)$ is the usual exponential map of matrices, given by the power series:

$$\exp\colon T_1U(n) \to U(n); A \mapsto \exp(A) = \sum_{k=0}^{\infty} A^k/k!.$$

The functional equation shows that the image indeed belongs to $U(n)$.

Similarly, $Lie(SU(n)) = \{A \in M(n,\mathbb{C}) \mid A + A^* = 0, \mathrm{tr}(A) = 0\}$.

2.3 Theorem. *If G is a connected compact Lie group, then* $\exp\colon Lie(G) \to G$ *is surjective.*

2.4 Definition. Every compact connected Lie group G can be realized as a Lie-subgroup of $SU(n)$ for big enough n. It follows that its Lie algebra $Lie(G)$ is a sub-Lie algebra of $Lie(U(n)) = \{A \in M(n,\mathbb{C}) \mid A^* = -A\}$. Therefore, the complexification $Lie(G) \otimes_{\mathbb{R}} \mathbb{C}$ is a sub Lie algebra of

$$Lie(U(n)) \otimes_{\mathbb{R}} \mathbb{C} = \{A + iB \mid A, B \in M(n,\mathbb{C}), A^* = -A, (iB)^* = -(iB)\} = M(n,\mathbb{C}).$$

Note that the bracket (given by the commutator) is complex linear on these complex vector spaces.

The corresponding sub Lie group $G_{\mathbb{C}}$ of $Gl(n,\mathbb{C})$ (the simply connected Lie group with Lie algebra $M(n,\mathbb{C})$) with Lie algebra $Lie(G) \otimes_{\mathbb{R}} \mathbb{C}$ is called the complexification of G.

The group $G_{\mathbb{C}}$ is a *complex Lie group*, i.e., the manifold $G_{\mathbb{C}}$ has a natural structure of a complex manifold (charts with holomorphic transition maps), and the composition $G_{\mathbb{C}} \times G_{\mathbb{C}} \to G_{\mathbb{C}}$ is holomorphic.

2.5 Exercise. Check the assertions made in Definition 2.4, in particular about the complex structure.

2.6 Example. The complexified Lie algebra

$$Lie(SU(n)) \otimes_{\mathbb{R}} \mathbb{C} = \{A \in M(n,\mathbb{C}) \mid \mathrm{tr}(A) = 0\},$$

and the complexification of $SU(n)$ is $Sl(n,\mathbb{C})$.

2.7 Definition. Let G be a compact Lie group. It acts on itself by conjugation: $G \times G \to G; (g,h) \mapsto ghg^{-1}$.

For fixed $g \in G$, we can take the differential of the corresponding map $h \mapsto ghg^{-1}$ at $h = 1$. This defines the adjoint representation $ad\colon G \to Gl(Lie(G))$.

We now decompose $Lie(G)$ into irreducible sub-representations for this action, $Lie(G) = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k$. Each of these are Lie subalgebras, and we have $[\mathfrak{g}_i, \mathfrak{g}_j] = 0$ if $i \neq j$.

G is called semi-simple if none of the summands is one-dimensional, and *simple* if there is only one summand, which additionally is required not to be one-dimensional.

2.8 Remark. The simply connected simple compact Lie groups have been classified, they consist of $SU(n)$, $SO(n)$, the symplectic groups Sp_n and five exceptional groups (called G_2, F_4, E_6, E_7, E_8).

2.9 Definition. Let G be a compact Lie group with a maximal torus T. G acts (induced from conjugation) on $Lie(G)$ via the adjoint representation, which induces a representation on $Lie(G) \otimes_{\mathbb{R}} \mathbb{C} =: \mathfrak{g}_{\mathbb{C}}$. This Lie algebra contains the Lie algebra $\mathfrak{t}_{\mathbb{C}}$ of the maximal torus, on which T acts trivially (T is abelian).

Since T is a *maximal* torus, it acts non-trivially on every non-zero vector of the complement.

As every finite dimensional representation of a torus, the complement decomposes into a direct sum $\bigoplus \mathfrak{g}_\alpha$, where on each $\mathfrak{g}_\alpha$, $t \in T$ acts via multiplication with $\alpha(t) \in S^1$, where $\alpha\colon T \to S^1$ is a homomorphism, called the *weight* of the summand $\mathfrak{g}_\alpha$.

We can translate the homomorphisms $\alpha\colon T \to S^1$ into their derivative at the identity, thus getting a linear map $\alpha'\colon \mathfrak{t} \to \mathbb{R}$, i.e. an element of the dual space $\mathfrak{t}^*$, they are related by $\alpha(\exp(x)) = e^{i\alpha'(x)}$.

This way, we think of the group of characters $\hat{T} = \mathrm{Hom}(T, S^1)$ as a lattice in t^*, called the lattice of weights. It contains the set of *roots*, i.e., the non-zero weights occurring in the adjoint representation of G.

2.10 Remark. As $\mathfrak{g}_{\mathbb{C}}$ is the complexification of a real representation, if α is a root of G, so is $-\alpha$, with $\mathfrak{g}_{-\alpha} = \overline{\mathfrak{g}_\alpha}$.

It is a theorem that the subspaces $\mathfrak{g}_\alpha$ are always 1-dimensional.

2.11 Example. For $U(n)$, $Lie(U(n))_{\mathbb{C}} = M(n, \mathbb{C})$. A maximal torus is given by the diagonal matrices (with diagonal entries in S^1). We can then index the roots by pairs (i, j) with $1 \le i \ne j \le n$. We have $\alpha_{ij}(\mathrm{diag}(z_1, \dots, z_n)) = z_i z_j^{-1} \in S^1$. The corresponding subspace $\mathfrak{g}_{ij}$ consists of matrices which are zero except for the (i, j)-entry. As an element of $\mathfrak{t}^*$ it is given by the linear map $\mathbb{R}^n \to \mathbb{R}; (x_1, \dots, x_n) \mapsto x_i - x_j$.

2.12 Definition. According to Remark 2.10, the spaces $\mathfrak{g}_\alpha$ are 1-dimensional. Choose a vector $0 \ne e_\alpha \in \mathfrak{g}_\alpha$, such that $e_{-\alpha} = \overline{e_\alpha}$. Then $h_\alpha := -i[e_\alpha, e_{-\alpha}] \in \mathfrak{t}$ is non-zero. We can normalize the vector e_α in such a way that $[h_\alpha, e_\alpha] = 2ie_\alpha$. Then h_α is canonically determined by α, it is called the *coroot* associated to α.

We call G a *simply laced* Lie group, if there is a G-invariant inner product on $Lie(G)$ such that $\langle h_\alpha, h_\alpha \rangle = 2$ for all roots α.

This inner product gives rise to an isomorphism $\mathfrak{t} \cong \mathfrak{t}^*$; this isomorphism maps h_α to α.

2.13 Example. For $SU(2)$, $T = \{\mathrm{diag}(z, z^{-1}) \mid z \in S^1\}$, there is only one pair of roots $\alpha, -\alpha$, obtained by restriction of the roots α_{12} and α_{21} of $U(2)$ to the (smaller) maximal torus of $SU(2)$. We get $\alpha(\mathrm{diag}(z, z^{-1})) = z^2$, $e_\alpha = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$, $h_\alpha = \left(\begin{smallmatrix} i & 0 \\ 0 & -i \end{smallmatrix}\right)$.

All the groups $U(n)$ and $SU(n)$ are simply laced, as well as $SO(2n)$. The canonical inner product of $Lie(U(n))_{\mathbb{C}} = M(n, \mathbb{C})$ is given by $\langle A, B\rangle = -tr(A^* B)/2$.

2.14 Proposition. *Given any semi-simple Lie group, the Lie algebra is spanned by the vectors $e_\alpha, e_{-\alpha}, h_\alpha$ as α varies over the set of (positive) roots.*

For each such triple one can define a Lie algebra homomorphism $Lie(SU(2)) \to Lie(G)$ which maps the standard generators of $Lie(SU(2))$ to the chosen generators of $Lie(G)$.

We get the technical useful result that for a semi-simple compact Lie group G the Lie algebra $Lie(G)$ is generated by the images of finitely many Lie algebra homomorphisms from $Lie(SU(2))$ to $L(G)$.

2.15 Definition. Let G be a compact Lie group with maximal torus T, and let $N(T) := \{g \in G \mid gTg^{-1} \subset T\}$ be the normalizer of T in G. The *Weyl group* $W(T) := N(T)/T$ acts on T by conjugation, and via the adjoint representation also on $\mathfrak{t}$ and then on $\mathfrak{t}^*$. It is a finite group.

This action preserves the lattice of weights $\hat{T} \subset \mathfrak{t}^*$, and also the set of roots.

Given any root α of G, the Weyl group contains an element s_α of order 2. It acts on $\mathfrak{t}$ by reflection on the hyperplane $H_\alpha := \{X \mid \alpha(X) = 0\}$. More precisely we have $s_\alpha(X) = X - \alpha(X) h_\alpha$, where $h_\alpha \in \mathfrak{t}$ is the coroot associated to the root α.

The reflections s_α together generate the Weyl group $W(G)$.

The Lie algebra $\mathfrak{t}$ is decomposed into the union of the root hyperplanes H_α and into their complement, called the set of *regular* elements. The complement decomposes into finitely many connected components, which are called the *Weyl chambers*. One chooses one of these and calls it the *positive Weyl chamber*. The Weyl group acts freely and transitively on the Weyl chambers.

A root of G is called positive or negative, if it assumes positive or negative values on the positive chamber. A positive root α is called *simple*, if its hyperplane H_α is a wall of the positive chamber.

2.16 Example. For the group $U(n)$, the Weyl group is the symmetric group S_n, it acts by permutation of the diagonal entries on the maximal torus. Lifts of the elements of $W(U(n))$ to $N(T) \subset U(n)$ are given by the permutation matrices.

For $SU(n)$, the Weyl group is the alternating group S_n, again acting by permutation of the diagonal entries of the maximal torus.

Since these groups are simply laced, we can depict their roots, coroots etc. in a simply Euclidean picture of $\mathfrak{t} \cong \mathfrak{t}^*$ (look at the case $G = SU(3)$ where $Lie(T)$ is 2-dimensional).

In general, the positive roots of $U(n)$ or $SU(n)$ can be chosen to be the roots α_{ij} with $i < j$.

2.17 Theorem. *Let G be a compact semi-simple Lie group. There is a one-to-one correspondence between the irreducible representations of G and the set of dominant weights, where a weight α is called dominant if $\alpha(h_\beta) \geq 0$ for each positive root β of G.*

We don't prove this theorem here; we just point out that the representation associated to a dominant weight is given in as the space of holomorphic sections of a line bundle over a homogeneous space of $G_\mathbb{C}$ associated to the weight.

2.18 Remark. One of the goals of these lectures is to explain how this results extends to loop groups.

3. Basics about loop groups

3.1 Definition. An infinite dimensional smooth manifold (modeled on a locally convex complete topologically vector space X) is a topological space M together with a collection of charts $x_i \colon U_i \to V_i$, with open subset U_i of M and V_i of X and a homeomorphism x_i, such that the change of coordinate maps $x_j \circ x_i^{-1} \colon V_i \to V_j$ are smooth maps between the topological vector space X.

Differentiability of a map $f \colon X \to X$ is defined in terms of convergence of difference quotients, if it exists, the differential is then a map

$$Df \colon X \times X \to X; (v, w) \mapsto \lim_{t \to 0} \frac{f(u + tv) - f(u)}{t}$$

(where the second variable encodes the direction of differentiation), iterating this, we define higher derivatives and the concept of smooth, i.e., C^∞ maps.

3.2 Definition. Let G be a compact Lie group with Lie algebra $Lie(G) = T_1 G$ (a finite dimensional vector space). For us, the model topological vector space X will be $X = C^\infty(S^1, Lie(G))$. Its topology is defined by the collection of semi-norms $q_i(f) := \sup_{x \in S^1} \{|\partial^i \varphi / \partial t^k(x)|\}$ for any norm on $Lie(G)$.

This way, X is a complete separable (i.e., with a countable dense subset) metrizable topological vector space. A sequence of smooth functions $\varphi_k \colon S^1 \to Lie(G)$ converges to $\varphi \colon S^1 \to Lie(G)$ if and only if the functions and all their derivatives converge uniformly.

3.3 Lemma. *There is a canonical structure of a smooth infinite dimensional manifold on LG. A chart around the constant loop* 1 *is given by the exponential map* $C^\infty(S^1, U) \to LG; \chi \mapsto \exp\circ\chi$ *where* U *is a sufficiently small open neighborhood of* $0 \in T_1(G)$. *Charts around any other point* $f \in LG$ *are obtained by translation with* f, *using the group structure on* LG.

We define the topology on LG to be the finest topology (as many open sets as possible) such that all the above maps are continuous.

It is not hard to see that the transition functions are then actually smooth, and that the group operations (defined pointwise) are smooth maps.

3.4 Exercise. Work out the details of Lemma 3.3, and show that it extends to the case where S^1 is replaced by any compact smooth manifold N.

3.5 Remark. There are many variants of the manifold LG of loops in G. Quite useful are versions which are based on maps of a given Sobolev degree, one of the advantages being that the manifolds are then locally Hilbert spaces.

3.6 Exercise. Show that for $G = SU(2)$ the group LG is connected, but the exponential map is not surjective.

3.7 Exercise. There are other interesting infinite dimensional Lie groups. One which is of some interest for loop groups is $\mathrm{Diffeo}(S^1)$, the group of all diffeomorphisms of S^1 (and also its identity component of orientation preserving diffeomorphism).

Show that this is indeed a Lie group, with Lie algebra (and local model for the smooth structure) $Vect(S^1)$, the space of all smooth vector fields. The canonical map $\exp\colon Vect(S^1) \to \mathrm{Diffeo}(S^1)$ maps a vector field to the (time 1) flow generated by it.

Show that there are no neighborhoods $U \subset Vect(S^1)$ of 0 and $V \subset \mathrm{Diffeo}(S^1)$ of id_{S^1} such that $\exp|\colon U \to V$ is injective or surjective.

3.8 Lemma. *Consider the subgroup of based loops*

$$\Omega(G) = \{f\colon S^1 \to G \in LG \mid f(1) = 1\} \subset LG,$$

and the subgroup of constant loops $G \subset LG$.

The multiplication map $G \times \Omega(G) \to LG$ *is a diffeomorphism.*

Proof. The inverse map is given by $LG \to G \times \Omega(G); f \mapsto (f(1), f(1)^{-1}f)$. Clearly the two maps are continuous and inverse to each other. The differentiable structure of $\Omega(G)$ makes the maps smooth. □

3.9 Corollary. *It follows that the homotopy groups of LG are easy to compute (in terms of those of G); the maps of Lemma 3.8 give an isomorphism $\pi_k(LG) \cong \pi_k(G) \oplus \pi_{k-1}(G)$.*

In particular, LG is connected if and only if G is connected and simply connected, else the connected components of G are parameterized by $\pi_0(G) \times \pi_1(G)$.

3.10 Definition. Embed the compact Lie group G into $U(n)$. Using Fourier decomposition, we can now write the elements of LG in the form $f(z) = \sum A_k z^k$ with $A_k \in M(n, \mathbb{C})$.

We define now a number of subgroups of LG.

(1) $L_{pol}G$ consists of those loops with only finitely many non-zero Fourier coefficients. It is the union (over $N \in \mathbb{N}$) of the subsets of functions of degree $\leq N$. The latter ones are compact subsets of $M(n, \mathbb{C})^N$, and we give $L_{pol}G$ the direct limit topology.

(2) $L_{rat}G$ consists of those loops which are rational functions $f(z)$ (without poles on $\{|z| = 1\}$).

(3) $L_{an}G$ are those loops where the series $\sum A_k z^k$ converges for some annulus $r \leq |z| \leq 1/r$ with $0 < r < 1$. For fixed $0 < r < 1$ this is Banach Lie group (of holomorphic functions on the corresponding annulus), with the topology of uniform convergence. We give $L_{an}G$ the direct limit topology.

3.11 Exercise. In general, $L_{pol}G$ is not dense in G. Show this for the case $G = S^1$.

3.12 Proposition. *If G is semi-simple, then $L_{pol}G$ is dense in LG.*

Proof. Set $H := \overline{L_{pol}G}$, and

$$V := \{X \in C^\infty(S^1, Lie(G) \mid \exp(tX) \in H \forall t \in \mathbb{R}\}) \subset C^\infty(S^1, Lie(G)).$$

Then V is a vector space, as

$$\exp(X+Y) = \lim_{k\to\infty} \big(\exp(X/k)\exp(Y/k)\big)^k$$

converges in the C^∞-topology if X and Y are close enough to 0. Since exp is continuous, V is a closed subspace. It remains to check that V is dense in $L\mathfrak{g}$.

Because of Proposition 2.14 it suffices to check the statement for $SU(2)$. Here, we first note that the elements $X_n\colon z \mapsto \begin{pmatrix} 0 & z^n \\ -z^{-n} & 0 \end{pmatrix}$ and $Y_n\colon z \mapsto \begin{pmatrix} 0 & iz^n \\ iz^{-n} & 0 \end{pmatrix}$ belong to V, $X_n^2 = Y_n^2 = -1$, so that $\exp(tX_n) = \sum_k (tX_n)^k/k!$ actually is a family of polynomial loops. By linearity and the fact that V is closed, every loop of the form $z \mapsto \begin{pmatrix} 0 & f(z)+ig(z) \\ -f(z)+ig(z) & 0 \end{pmatrix}$ belongs to V, for arbitrary smooth real valued functions $f, g\colon S^1 \to \mathbb{R}$ (use their Fourier decomposition). Since V is finally invariant under conjugation by polynomial loops in $LSU(2)$, it is all of $\mathfrak{su}(2)$. □

3.1. Abelian subgroups of LG. We have already used the fact that each compact Lie group G as a maximal torus T, which is unique upto conjugation.

3.13 Proposition. *Let G be a connected compact Lie group with maximal torus T. Let $A \subset LG$ be a (maximal) abelian subgroup. Then $A(z) := \{f(z) \in G \mid f \in A\} \subset G$ is a (maximal) abelian subgroup of G.*

In particular, we get a the maximal abelian subgroup A_λ for each smooth map $\lambda \colon S^1 \to \{T \subset G \mid T \text{ maximal torus}\} \cong G/N$, where N is the normalizer of T in G, with $A_\lambda := \{f \in LG \mid f(z) \in \lambda(z)\}$.

The conjugacy class of the space A_λ depends only on the homotopy class of λ. Since $\pi_1(G/N) = W = N/T$, $[S^1, G/N]$ is the set of conjugacy classes of elements of the Weyl group W.

An element $w \in W$ acts on T by conjugation, and the corresponding A_λ is isomorphic to $\{\gamma \colon \mathbb{R} \to T \mid \gamma(t+2\pi) = w^{-1}\gamma(t)w$ for all $t \in \mathbb{R}$.

Proof. It is clear that all evaluation maps of A have abelian image. If all the image sets are maximal abelian, then A is maximal abelian.

Since all maximal tori are conjugate, the action of G on the set of all maximal tori is transitive, with stabilizer (by definition) the normalizer N, so that this space is isomorphic to G/N.

Next, $W = N/T$ acts freely on G/T, with quotient G/N. Since G/T is simply connected (a fact true for every connected Lie group), we conclude from covering theory that $\pi_1(G/T) \cong W$. But then the homotopy classes of non base-point preserving maps are bijective to the conjugacy classes of the fundamental group.

Given a (homotopy class of maps) $\lambda \colon S^1 \to G/T$, represented by $w \in W = N/T$ (with a lift $w' \in N$ and with $x \in Lie(G)$ such that $\exp(2\pi x) = w'$), we can choose λ with $\lambda(z) = \exp(zx)T\exp(-zx)$ (recall that the bijection between G/N and the set of maximal tori is given by conjugation of T).

We then get a bijection from A_λ to the twisted loop group of the assertion by sending $f \in A_\lambda$ to $\tilde{f} \colon \mathbb{R} \to T : t \mapsto \exp(-tx) f(t) \exp(tx)$. □

3.14 Example. The most obvious maximal abelian of a compact Lie group G with maximal torus T is LT, which itself contains in particular T (as subgroup of constant loops).

3.15 Exercise. Find other maximal abelian subgroups of LG.

3.16 Example. If $G = U(n)$ with maximal torus T, its Weyl group W is isomorphic to the symmetric group S_n. Given a cycle $w \in W = S_n$, the corresponding maximal abelian subgroup $A_w \subset LU(n)$ is isomorphic to LS^1.

More generally, if w is a product of k cycles (possibly of length 1), then A_w is a product of k copies of LS^1.

Proof. We prove the statement if w consists of one cycle (of length n). Then, in the description of Proposition 3.13, A_w consists of functions $\mathbb{R} \to T$ which are periodic of period $2\pi l$. Moreover, the different components are all determined by the first one, and differ only by a translation in the argument of 2π or a multiple of 2π. □

3.17 Proposition. *If G is semi-simple and compact, then LG_0, the component of the identity of LG, is perfect.*

4. Root system and Weyl group of loop groups

4.1 Definition. Let G a compact Lie group with maximal torus T. Consider its complexified Lie algebra $Lie(LG)_{\mathbb{C}} = L\mathfrak{g}_{\mathbb{C}}$.

It carries the action of S^1 by reparametrization of loops: $(z_0 \cdot X)(z) := X(z_0 z)$.

We get a corresponding decomposition of $L\mathfrak{g}_{\mathbb{C}} = \bigoplus_{k\in\mathbb{Z}} \mathfrak{g}_{\mathbb{C}} z^k$ into Fourier components (the sums has of course to be completed appropriately).

The action of S^1 used in the above decomposition still commutes with the adjoint representation of the subgroup T of constant loops, so the summands can further be decomposed according to the weights of the action of T, to give a decomposition

$$L\mathfrak{g}_{\mathbb{C}} = \oplus_{(k,\alpha)\in\mathbb{Z}\times\hat{T}} \mathfrak{g}_{(k,\alpha)} z^k.$$

The index set $\mathbb{Z} \times \hat{T}$ is the Pontryagin dual of $S^1 \times T$ (i.e. the set of all homomorphisms $S^1 \times T \to S^1$). Those homomorphisms which occur (now also with possibly $\alpha = 0$) are called the *roots* of LG.

The (infinite) set of roots of LG is permuted by the so called *affine Weyl group*

$$W_{aff} = N(T \times S^1)/(T \times S^1),$$

considered inside the semidirect product $LG \rtimes S^1$, where we use the action of S^1 by reparametrization (rotation) of loops on LG to construct the semidirect product. This follows because we decompose $L\mathfrak{g}_{\mathbb{C}}$ as a representation of $LG \rtimes S^1$ with respect to the subgroup $T \times S^1$.

4.2 Proposition. *W_{aff} is a semidirect product* $\mathrm{Hom}(S^1, T) \rtimes W$*, where W is the Weyl group of G, with its usual action on the target T.*

Proof. Clearly $\mathrm{Hom}(S^1, T)$ is a subgroup of LG which conjugates the constant loops with values in T into itself, and the action of the constant loops with values in the normalizer N does the same (and factors through W). It is also evident that all of S^1 belongs to the normalizer of $T \times S^1$.

On the other hand, if $R_{z_0} \in S^1 \in LG \rtimes S^1$ acts by rotation by $z_0 \in S^1$, then for $f \in LG$ we get $f^{-1} R_{z-0} f = f^{-1}(\cdot) f(\cdot z_0) R_{z_0}$.

This belongs to $T \times S^1$ if and only if $z \mapsto f(z)^{-1} f(zz_0)$ is a constant element of T for each z_0, i.e., if and only if $z \mapsto f(1)^{-1} f(z)$ is a group homomorphism $z \to T$. Additionally, f conjugates T to itself if and only if $f(1) \in N$. Therefore, $N(T \times S^1)$ is the product of N, $\mathrm{Hom}(S^1, T)$ and S^1 inside $LG \rtimes S^1$, and the quotient by $T \times S^1$ is as claimed. □

4.3 Definition. We think of the weights and roots of LG not as linear forms on $\mathbb{R} \times \mathfrak{t}$ (derivatives of elements in $\mathrm{Hom}(S^1 \times T, S^1)$), but rather as *affine linear* functions on $\mathfrak{t}$, where we identify $\mathfrak{t}$ with the hyperplane $1 \times \mathfrak{t}$ in $\mathbb{R} \times \mathfrak{t}$; this explains the notation "affine roots".

Moreover, the group W_{aff} acts linearly on $\mathbb{R} \times \mathfrak{t}$ and preserves $1 \times \mathfrak{t}$, where $\lambda \in \mathrm{Hom}(S^1, T)$ acts by translation by $\lambda'(1) \in \mathfrak{t}$.

An affine root (k, α) is (for $\alpha \neq 0$) determined (upto sign) by the affine hyperplane $H_{k,\alpha} := \{x \in \mathfrak{t} \mid \alpha(x) = -k\} \subset \mathfrak{t}$ which is the set where it vanishes.

The collection of these hyperplanes is called the diagram of LG. It contains as the subset consisting of the $H_{0,\alpha}$ the diagram of G.

Recall that the connected components of the complement of all $H_{0,\alpha}$ were called the chambers of G, and we choose one which we call the positive chamber. The components of the complement of the diagram of LG are called *alcoves*. Each chamber contains a unique alcove which touches the origin, and this way the positive chamber defines a positive alcove, the set $\{x \in \ \mid 0 < \alpha(x) < 1$ for all positive roots $\alpha\}$. An affine root is called positive or negative, if it has positive or negative values at the positive alcove. The positive affine roots corresponding to the walls of the positive alcove are called the simple affine roots.

4.4 Example. The diagram for $SU(3)$ is the tessellation of the plane by equilateral triangles.

In general, if G is a simple group, then each chamber is a simplicial cone, bounded by the l hyperplanes $H_{0,\alpha_1}, \dots, H_{0,\alpha_l}$ (where $\alpha_1, \dots, \alpha_l$ are the simple roots of G). There is a highest root α_{l+1} of G, which dominates all other roots (on the positive chamber). The positive alcove is then an l-dimensional simplex cut out of the positive chamber by the wall $H_{1,-\alpha_{l+1}}$, and we get $l+1$ simple affine roots of LG, $(0, \alpha_1), \dots, (0, \alpha_l), (1, -\alpha_{l+1})$.

In general, if G is semi-simple with q simple factors, the positive alcove is a product of q simplices, bounded by the walls of the simple roots of G, and walls $H_{1,-\alpha_i}$, $i = l+1, \dots, l+q$ being the highest weights of the irreducible summands of the adjoint representation.

4.5 Proposition. *Let G be a connected and simply connected compact Lie group. Then W_{aff} is generated by reflections in the hyperplanes (the reflections in the walls of the positive alcove suffice), and it acts freely and transitively on the set of alcoves.*

Proof. From the theory of compact Lie groups, we know that W is generated by the reflections at the $H_{0,\alpha}$, and that $\mathrm{Hom}(S^1,\mathfrak{t})$ is generated by the coroots h_α.

Recall that the reflection s_α was given by $s_\alpha(x) = x - \alpha(x)h_\alpha$. Recall that we normalized such that $\alpha(h_\alpha) = 2$, therefore $-kh_\alpha/2 \in H_{k,\alpha}$.

Now the reflection $s_{k,\alpha}$ in the hyperplane $H_{k,\alpha}$ is given by

$$s_{k,\alpha}(x) = x + kh_\alpha/2 - \alpha(x + kh_\alpha/2)h_\alpha - kh_\alpha/2 = s_\alpha(x) - kh_\alpha.$$

Since $s_\alpha \in W$ and h_α in $\mathrm{Hom}(S^1, T)$ (identified with the value of its derivative at 1), $s_{k,\alpha} \in W_{aff} = \mathrm{Hom}(S^1, T) \rtimes W$.

To show that these reflections generate W_{aff}, it suffices to show that they generate the translation by h_α. But this is given by $s_{a,-\alpha}s_{0,\alpha}$.

We now show that W_{aff} acts transitively on the set of alcoves. For an arbitrary alcove A, we have to find $\gamma \in W_{aff}$ such that γA is the positive alcove C_0. Now the orbit $W_{aff}a$ of a point $p \in A$ is a subset S of $\mathfrak{t}$ without accumulation points. Choose a point $c \in C_0$ and one of the points $b \in S$ with minimal distance to c. If b would not belong to C_0, then b and c are separated by a wall of C_0, in which we can reflect b to obtain another point of S, necessarily closer to c than b.

Since W acts freely on the set of chambers, W_{aff} acts freely on the set of alcoves (since each element of W preserves the distance to the origin, and each translation moves the positive alcove away from the origin, only elements of W could stabilize the positive alcove). □

5. Central extensions of LG

We want to study the representations of LG for a compact Lie group G. It turns out, however, that most of the relevant representations are no honest representations but only projective representations., i.e., $U_f U_g = c(f,g)U_{fg}$ for every $f,g \in LG$, where U_f is the operator by which f acts, and $c(f,g) \in S^1$ is a scalar valued function (a cocycle).

More precisely, the actions we consider are actions of central extensions of LG (in some sense defined by this cocycle). We don't want to go into the details of the construction and classification of these central extensions, but only state the main results.

Let G be a compact connected Lie group.

(1) LG has many central extensions $1 \to S^1 \to \tilde{L}G \to LG \to 1$.

(2) The corresponding Lie algebras $\tilde{L}\mathfrak{g}$ are classified as follows: for every symmetric invariant bilinear form $\langle\cdot,\cdot\rangle\colon \mathfrak{g}\times\mathfrak{g} \to \mathbb{R}$ we get a form

$$\omega\colon L\mathfrak{g}\times L\mathfrak{g} \to \mathbb{R}; (X,Y) \mapsto \frac{1}{2\pi}\int_0^{2\pi} \langle X(z), Y'(z)\rangle\, dz.$$

Then $\tilde{L}\mathfrak{g} = \mathbb{R} \oplus L\mathfrak{g}$ with bracket

$$[(a, X), (b, Y)] = (\omega(X, Y), [X, Y]).$$

(3) The extension $0 \to \mathbb{R} \to \tilde{L}\mathfrak{g} \to L\mathfrak{g} \to 0$ with bracket given by the inner product $\langle , \rangle$ on $\mathfrak{g}$ corresponds to an extension of Lie groups $1 \to S^1 \to \tilde{L}G \to LG \to 1$ if and only if $\langle h_\alpha, h_\alpha \rangle \in 2\mathbb{Z}$ for every coroot h_α of G.

(4) If this integrality condition is satisfied, the extension $\tilde{L}G$ is uniquely determined. Moreover, there is a unique action of $\mathrm{Diffeo}^+(S^1)$ on $\tilde{L}G$ which covers the action on LG. In particular, there is a induced canonical lift of the action of S^1 on LG by rotation of the argument to $\tilde{L}G$.

(5) The integrality condition is satisfied if and only if $\omega/2\pi$, considered as an invariant differential form on LG (which is closed by the invariance of $\langle , \rangle$ and therefore of ω), lifts to an integral cohomology class. It then represents the first Chern class of the principle S^1-bundle $S^1 \to \tilde{L}G \to LG$ over LG. It follows that the topological structure completely determines the group extension.

(6) If G is simple and simply connected there is a *universal* central extension $1 \to S^1 \to \tilde{L}G \to LG \to 1$ (universal means that there is a unique map of extensions to any other central extension of LG).

If G is simple, all invariant bilinear forms on $\mathfrak{g}$ are proportional, and the universal extension corresponds to the smallest non-trivial one which satisfies the integrality condition. We call this also the basic inner product and the basic central extension.

(7) For $SU(n)$ and the other simply laced groups, the basic inner product is the canonical inner product such that $\langle h_\alpha, h_\alpha \rangle = 2$ for every coroot α.

(8) There is a precise formula for the adjoint and coadjoint action of suitable elements, compare Lemma 6.10.

6. Representations of loop groups

6.1 Definition. A representation of a loop group LG (or more generally any topological group) is for us given by a locally convex topological vector space V (over $\mathbb{C}$) with an action

$$G \times V \to V; (g, v) \to gv$$

which is continuous and linear in the second variable.

Two representations V_1, V_2 are called *essentially equivalent*, if they contain dense G-invariant subspaces $V_1' \subset V_1$, $V_2' \subset V_2$ with a continuous G-equivariant bijection $V_1' \to V_2'$

The representation V is called *smooth*, if there is a dense subspace of vectors $v \in V$ such that the map $G \to V; g \mapsto gv$ is smooth —such vectors are called *smooth vectors* of the representation.

A representation V is called irreducible if it has no closed invariant subspaces.

6.2 Example. The actions of S^1 on $C^\infty(S^1)$, $C(S^1)$ and $L^2(S^1)$ by rotation of the argument are all equivalent and smooth.

However, rotation of the argument does not define an action in our sense of S^1 on $L^\infty(S^1)$ because the corresponding map $S^1 \times L^\infty(S^1) \to L^\infty(S^1)$ is not continuous.

6.3 Remark. Given a representation V of LG and $z_0 \in S^1$ (or more generally any diffeomorphism of S^1, z_0 gives rise to a diffeomorphism by translation), then we define a new representation $\varphi^* V$ by composition with the induced automorphism of LG.

We are most interested in representations which are symmetric, meaning that $\varphi^* V \cong V$. Actually, we require a somewhat stronger condition in the following Definition 6.4.

6.4 Definition. When we consider representation V of LG, we really want to consider actions of $LG \rtimes S^1$, i.e. we want an action of S^1 on V which intertwines the action of LG; for $z_0 \in S^1$ we want operators R_{z_0} on V such that $R_{z_0} f R_{z_0}^{-1} v = f(\cdot + z_0) v$ for all $v \in V$, $f \in LG$.

Moreover, we will study *projective* representations, i.e., representations such that $f_1 \cdot (f_2 \cdot v) = c(f_1, f_2)(f_1 f_2) \cdot v$ with $c(f_1, f_2) \in \mathbb{C} \setminus \{0\}$.

More precisely, these are actions of a central extension $1 \to \mathbb{C}^\times \to \tilde{L}G \to LG \to 1$.

Since S^1 acts on V, we get a decomposition $V = \overline{\bigoplus_{k\in\mathbb{Z}} V(k)}$, where $z \in S^1$ acts on $V(k)$ by multiplication with z^{-k}.

We say V is a representation of *positive energy*, if $V(k) = 0$ for $k < 0$.

6.5 Example. The adjoint representation of LG on $L\mathfrak{g}$, or the canonical representation of $LSU(n)$ on the Hilbert space $L^2(S^1, \mathbb{C}^n)$ are *not* of positive energy, nor of negative energy. However, they are not irreducible.

6.6 Remark. One can always modify the action of S^1 on a representation V of LG by multiplication with a character of S^1, so that “positive energy” and “energy bounded below” are more or less the same.

The complex conjugate of a representation of negative energy is a representation of positive energy.

6.7 Proposition. *An irreducible unitary representation V of $\tilde{L}G \rtimes S^1$ which is of positive energy (and this makes only sense with the action of the extra circle) is also irreducible as a representation of $\tilde{L}G$.*

Proof. Let T be the projection onto a proper $\tilde{L}G$-invariant summand of V. This operator commutes with the action of $\tilde{L}G$. Let R_z be the operator through which $z \in S^1$ acts on V. We define the bounded operators $T_q := \int_{S^1} z^q R_z T R_z^{-1}\, dz$. They all commute with $\tilde{L}G$, and T_q maps $V(k)$ to $V(k+q)$.

Let m be the lowest energy of V. Then $T_q(V(m)) = 0$ for all $q < 0$. Since V is irreducible, V is generated as a representation of $LG \rtimes S^1$ by $V(m)$. Since $V(m)$ is S^1-invariant, $T_q(V) = 0$ for $q < 0$. Since $T_{-q} = T_q^*$, we even have $T_q = 0$ for all $q \neq 0$. Now, the T_q are the Fourier coefficients of the loop $z \mapsto R_z T R_z^{-1}$. It follows that this loop is constant, i.e., that T commutes also with the action of S^1. But since V was irreducible, this implies that $T = 0$.

□

The representation of positive energy of a loop group behave very much like the representation of a compact Lie group. This is reflected in the following theorem.

6.8 Theorem. *Let G be a compact Lie group. Let V be a smooth representation of positive energy of $\tilde{L}G \rtimes S^1$. Then, upto essential equivalence:*

(1) *If V is non-trivial then it does not factor through an honest representation of LG, i.e., it is truly projective.*
(2) *V is a discrete direct sum of irreducible representations (of positive energy).*
(3) *V is unitary.*
(4) *The representation extends to a representation of $\tilde{L}G \rtimes \mathrm{Diffeo}^+(S^1)$, where $\mathrm{Diffeo}^+(S^1)$ denotes the orientation preserving diffeomorphisms and contains S^1 (acting by translation).*
(5) *V extends to a holomorphic projective representation of $LG_{\mathbb{C}}$.*

Granted this theorem, it is of particular importance to classify the irreducible representations of positive energy.

6.9 Definition. Let $T_0 \times T \times S^1 \subset \tilde{L}G \rtimes S^1$ be a "maximal torus", with $T_0 = S^1$ the kernel of the central extension, T a maximal torus of G and S^1 the rotation group. We can then refine the energy decomposition of any representation V to a decomposition

$$V = \overline{\bigoplus_{n,\alpha,h \in \mathbb{Z}\times\hat{T}\times\mathbb{Z}} V_{n,\alpha,h}}$$

according to the characters of $T_0 \times T \times S^1$. The characters which occur are called the *weights* of V. n is called the *energy* and h the *level*.

6.10 Lemma. *The action of $\xi \in \mathrm{Hom}(S^1, T)$ on a weight (n,α,h) is given by*

$$\xi(n,\alpha,h) = (n + \alpha(\xi) + h|\xi|^2/2, \alpha + h\xi^*, h)$$

where we identify ξ with $\xi'(1) \in \mathfrak{t}$.

Moreover, the norm is obtained from the inner product on $L\mathfrak{g}$ which corresponds to the central extension $\tilde{L}G$, and $\xi^* \in \hat{T}$ is the image of ξ under the map $\mathfrak{t} \to \mathfrak{t}^*$ defined by this inner product.

6.11 Remark. Note that T_0 commutes with every element of $\tilde{L}G \rtimes S^1$. Consequently, each representation decomposes into subrepresentations with fixed level, and an irreducible representation has only one level h. The level is a measure for the "projectivity" of the representation; it factors through an honest representation of LG if and only if the level is 0.

The weights of a representation are permuted by the normalizer of $T_0 \times T \times S^1$, hence by the affine Weyl group $W_{aff} = \mathrm{Hom}(S^1, T) \rtimes W$ (where W is the Weyl group of G).

6.12 Definition. Given a root (n, α) of LG, we define the *coroot* $(-n|h_\alpha|^2/2, h_\alpha) \in \mathbb{R} \oplus \mathfrak{t} \subset \mathbb{R} \oplus \mathfrak{t} \oplus \mathbb{R}$, where we use again the inner product for the central extension $\tilde{L}G$.

Our irreducible representation have a number of additional important properties.

6.13 Theorem. *Let V be a smooth irreducible representation of LG of positive energy (i.e., we really take a representation of $\tilde{L}G \rtimes S^1$).*

(1) *Then V is of finite type, i.e., for each energy n the subspace $V(n)$ is finite dimensional. In particular, each weight space $E_{h,\lambda,n}$ is finite dimensional.*

(2) *V has a unique lowest weight (h, λ, n) with $E_{h,\lambda,n} \neq 0$.* Lowest weight *means by definition, that for each positive root (α, m) the character $(h, \lambda - \alpha, n - m)$ does not occur as a weight in V.*

This lowest weight is antidominant, *i.e., for each positive coroot $(-m|h_\alpha|^2/2, h_\alpha)$ we have $\langle (h, \lambda, n), (-m|h_\alpha|^2/2), h_\alpha, -0 \rangle = \lambda(h_\alpha) - hm|h_\alpha|^2/2 \leq 0$.*

Since we have in particular to consider the positive roots $(\alpha, 0)$ and $(-\alpha, 1)$ (for each positive root α of G), this is equivalent to

$$-h|h_\alpha|^2/2 \leq \lambda(h_\alpha) \leq 0 \tag{6.14}$$

for each positive root α of G.

(3) *There is a bijection between isomorphism classes of irreducible representations of $\tilde{L}G \rtimes S^1$ as above and antidominant weights.*

6.15 Corollary. *If the level $h = 0$, only $\lambda = 0$ satisfies Inequality* (6.14). *In other words, among the representations considered here, only the trivial representation is an honest representation of LG, all others are projective.*

For a given level h, there are only finite possible antidominant weights (with $n =$ 0*), because the h_α generate* $\mathfrak{t}$.

6.16 Example. If G is simple and we look at an antidominant weight $(h, \lambda, 0)$, then $-\lambda$ is a dominant weight in the usual sense of G, i.e., it is contained in the corresponding simplicial cone in $\mathfrak{t}^*$, but with the extra condition that it is contained in the simplex cut off by $\{\mu \mid \mu(\alpha_0) = h\}$, where α_0 is the highest weight of G.

We get in particular the so called *fundamental weights*

(1) $(1,0,0)$
(2) $(\langle\omega_i,\alpha_0\rangle,-\omega_i,0)$, with ω_i the fundamental weights of G are determined by $\omega_i(h_{\alpha_j})=\delta_{ij}$.

The antidominant weights are exactly the linear combinations of the fundamental weights with coefficients in $\mathbb{N}\cup\{0\}$.

Given an irreducible representation V of $\tilde{L}G\rtimes S^1$ of lowest weight $(h,\lambda,0)$, we want to determine which other weights occur in V (or rather, we want to find restrictions for those weights).

First observation: the whole orbit under W_{aff} occurs. This produces, for the $\eta\in\operatorname{Hom}(S^1,T)$, the weights $(h,\lambda+h\eta^*,\lambda(\eta)+h\left|\eta\right|^2/2)$,

6.17 Example. If $G=SU(2)$, we have the isomorphism $(\hat{T}\subset\mathfrak{t})\cong(\mathbb{Z}\subset\mathbb{R})$, where $\left(\begin{smallmatrix}2\pi it & 0\\ 0 & -2\pi it\end{smallmatrix}\right)\in\mathfrak{t}$ is mapped to $t\in\mathbb{R}$, $\mathfrak{t}^*$ is identified with $\mathfrak{t}$ using the standard inner product and this way the character $\operatorname{diag}(z,z^{-1})\mapsto z^\mu$ in $\hat{T}$ is mapped to $\mu\in\mathbb{Z}$.

Under this identification, the lowest weight α_0 is identified with $1\in\mathbb{Z}$.

There are exactly two fundamental weights. The W_{aff}-orbit of the weight $(1,0,0)$ is (with this identification of $\hat{T}$ with $\mathbb{Z}$) $\{(1,2k,m)\mid(2k)^2=2m\}$, the set of all weights of the corresponding irreducible representations is $\{(1,2k,m)\mid(2k)^2\le 2m\}$. Similarly, the orbit of $(1,-1,0)$ is $\{(1,2k+1,m)\mid(2k+1)^2=2m+1\}$, the set of all weights is $\{(1,2k+1,m)\mid(2k+1)^2\le 2m+1\}$.

For arbitrary G, the orbit of the lowest weight (h,μ,m), $\left|(h,\mu,m)\right|^2=\left|\mu\right|^2+2mh$, is contained in the parabola $\{(h,\mu',m')\mid\left|\mu'\right|^2=\left|(h,\mu,m)\right|^2+2m'h\}$.

All other weights are contained in the interior of this parabola (i.e., those points with "=" replaced by "≤").

The last statement follows because, by translation with an element of W_{aff} we can assume that (h,μ',m') is antidominant. Then

$$\left|(h,\mu',m')\right|^2-\left|(h,\mu,m)\right|^2=\langle(h,\mu',m')+(h,\mu,m),(h,\mu',m')-(h,\mu,m)\rangle\le 0,$$

because the first entry is antidominant and the second one is positive, (h,μ,m) being a lowest weight.

We use the fact (not proved here) that the extension inner product on $\mathfrak{t}$ extends to an inner product on $\mathbb{R}\oplus\mathfrak{t}\oplus\mathbb{R}$ which implements the pairing between roots and coroots.

7. Proofs and Homogeneous spaces of LG

We now indicate the proofs of the statements of Section 6. The basic idea is that we can mimic the Borel-Weil theorem for compact Lie groups. It can be stated as follows:

7.1 Theorem. *The homogeneous space G/T has a complex structure, because it is isomorphic to $G_{\mathbb{C}}/B^+$, where $G_{\mathbb{C}}$ is the complexification of G and B^+ is the Borel subgroup. In case $G = U(n)$, $G_{\mathbb{C}} = Gl(n,\mathbb{C})$ and $B^+ \subset Gl(n,\mathbb{C})$ is the subgroup of upper triangular matrices; the homogeneous space is the flag variety.*

To each weight $\lambda\colon T \to S^1$ there is a uniquely associated holomorphic line bundle L_λ over $G_{\mathbb{C}}/B^+$ with action of $G_{\mathbb{C}}$.

L_λ has non-trivial holomorphic sections if and only if λ is an antidominant weight. In this case, the space of holomorphic sections is an irreducible representation with lowest weight λ.

For loop groups, the relevant homogeneous space is

$$Y := LG/T = LG_{\mathbb{C}}/B^+G_{\mathbb{C}}, \qquad B^+G_{\mathbb{C}} = \{\sum_{k\geq 0} \lambda_k z^k \mid \lambda_0 \in B^+\}.$$

Recall that for $GL(n,\mathbb{C})$, the *Borel subgroup* B^+ is the subgroup of upper triangular matrices.

Note that the second description defines on Y the structure of a complex manifold.

We have also $Y = \tilde{L}G/\tilde{T} = \tilde{L}G_{\mathbb{C}}/\tilde{B}^+G_{\mathbb{C}}$.

7.2 Lemma. *Each character $\lambda\colon \tilde{T} \to S^1$ has a unique extension*

$$\tilde{B}^+G_{\mathbb{C}} = \tilde{T}_{\mathbb{C}} \cdot \tilde{N}^+G_{\mathbb{C}} \to \mathbb{C}^x$$

with $N^+G_{\mathbb{C}} := \{\sum_{k\geq 0}\lambda_k z^k \mid \lambda_0 \in N^+G_{\mathbb{C}}\}$, where N^+ is the nilpotent subgroup of $G_{\mathbb{C}}$ whose Lie algebra is generated by the positive root vectors, for $Gl(n,\mathbb{C})$ it is the group of upper triangular matrices with 1s on the diagonal. This way defines a holomorphic line bundle

$$L_\lambda := \tilde{L}G_{\mathbb{C}} \times_{\tilde{B}^+G_{\mathbb{C}}} \mathbb{C} \text{ over } Y.$$

Write Γ_λ for the space of holomorphic sections of $L\lambda$. This is a representation of $\tilde{L}G \rtimes S^1$.

7.3 Lemma. *The space $Y = LG/T$ contains the affine Weyl group $W_{aff} = (\mathrm{Hom}(S^1,T)\cdot N(T))/T$, where $N(T)$ is the normalizer of T in $G \subset LG$.*

The space Y is stratified by the orbits of W_{aff} under the action of the group $N^-LG_{\mathbb{C}} := \{\sum_{k\leq 0}\lambda_k z^k \mid \lambda_0 \in N^-G_{\mathbb{C}}\}$. Here $N^-G_{\mathbb{C}}$ is the nilpotent Lie subgroup whose Lie algebra is spanned by the negative root vectors of $\mathfrak{g}_{\mathbb{C}}$. For $Gl(n,\mathbb{C})$ this is the group of lower triangular matrices with 1s on the diagonal.

7.4 Theorem. *Assume that $\lambda \in \mathrm{Hom}(\tilde{T}, S^1)$ is a weight such that the space Γ_λ of holomorphic sections of L_λ is non-trivial. Then*

(1) Γ_λ *is a representation of positive energy,*
(2) Γ_λ *is of finite type, i.e. each fixed energy subspace $\Gamma_\lambda(n)$ is finite dimensional,*
(3) λ *is the lowest weight of Γ_λ and is antidominant,*
(4) Γ_λ *is irreducible.*

Proof. We use the stratification of Y to reduce to the top stratum. It turns out that the top stratum is $N^- LG_{\mathbb{C}}$ and that L_γ trivializes here, so Γ_λ restricts to the space of holomorphic functions on $N^- LG_{\mathbb{C}}$. Because holomorphic sections are determined by their values on the top stratum, this restriction map is injective. We can further, with the exponential map (which is surjective in this case), pull back to holomorphic functions on $N^- L\mathfrak{g}_{\mathbb{C}}$, and then look at the Taylor coefficients at 0.

This way we finally map invectively into $\prod_{p\geq 0} S^p(N^- L\mathfrak{g}_{\mathbb{C}})^*$, where $S^p(V)^*$ is the space of p-multilinear maps $V\times\cdots\times V \to \mathbb{C}$. This map is indeed $\tilde{T}\times S^1$-equivariant, if we multiply the obvious action on the target with λ.

It now turns out that $N^- L\mathfrak{g}_{\mathbb{C}}$ has (essentially by definition) negative energy, and therefore the duals $S^p(N^- L\mathfrak{g}_{\mathbb{C}})$ all have positive energy; and the weights are exactly the positive roots (before multiplication with λ). Consequently λ is of lowest weight. If for some positive root (α, n), we had $\lambda(\alpha, n) = m > 0$, then reflection in W_{aff} corresponding to α would map λ to $\lambda - m\alpha$, which on the other hand can not be a root of Γ_λ if λ is a root. Consequently, λ is antidominant.

Explicit calculations also show that the image of the “restriction map” is contained in a subspace of finite type.

To prove that Γ_λ is irreducible, we look at the subspace of lowest energy. This is a representation of $G_{\mathbb{C}}$. Pick a lowest weight vector for this representation, it is then invariant under the nilpotent subgroup N^-. Since it is of lowest energy, it is even invariant under $N^- LG_{\mathbb{C}}$. On the other hand, since the top stratum of Y is $N^- LG_{\mathbb{C}}$, the value of an invariant section at one point completely determines it, so that the space of such sections is 1-dimensional. The group $B^- LG_{\mathbb{C}}$ acts on this space by multiplication with the holomorphic homomorphism $\lambda\colon B^- LG_{\mathbb{C}} \to \mathbb{C}^\times$, so that λ really occurs as lowest weight.

We now show that this vector is actually a cyclic vector, i.e., generates Γ_λ under the action of $\tilde{L}G \rtimes S^1$. Else choose a vector of lowest energy not in this subrepresentation, of lowest weight for the corresponding action of the compact group G, and we get with the argument as above a second $N^- LG_{\mathbb{C}}$ invariant section. □

7.5 Lemma. $\Gamma_\lambda \neq 0$ *if and only If λ is antidominant.*

Proof. If α is a positive root of G (therefore $(\alpha, 0)$ is a positive root of $LG \rtimes S^1$) we get a corresponding inclusion $i_\alpha\colon Sl_2(\mathbb{C}) \to \tilde{L}G_{\mathbb{C}}$ whose restriction to $\mathbb{C}^\times \subset Sl_2(\mathbb{C})$ is the exponential of h_α. Since α is positive, $B^+ Sl_2(\mathbb{C})$ is mapped to $B^+ \tilde{L}G_{\mathbb{C}}$. Therefore, we get an induced map $P^1(\mathbb{C}) = Sl_2(\mathbb{C})/B^+ \to Y$. The pullback of L_λ under this map is the line bundle associated to $\lambda \circ h_\alpha$. If Γ_λ is non-trivial, we can therefore pull back to obtain a non-trivial holomorphic section of this bundle over $P^1(\mathbb{C})$. These exist only if $\lambda \circ h_\alpha = \lambda(h_\alpha) \leq 0$.

This prove half the conditions for antidominance. We omit the prove of the other half, where we have to consider the positive root $(-\alpha, 1)$.

If, on the other hand, λ is antidominant, one constructs a holomorphic section along the stratification of Y. □

7.6 Theorem. *An arbitrary smooth representation of $\tilde{L}G \rtimes S^1$ of positive energy splits (upto essential equivalence) as a direct sum of representations of the form Γ_λ.*

In particular, the Γ_λ are exactly the irreducible representations.

*Proof***.** If E is a representation of positive energy, so is $\overline{E}^*$. Pick in the G-representation $\overline{E}^*(0)$ a vector ε of lowest weight λ. For each smooth vector $v \in E$, the map

$$s_v\colon \tilde{L}G_{\mathbb{C}} \to \mathbb{C}; f \mapsto \varepsilon(f^{-1}v)$$

turns out to define a holomorphic section of L_λ. This give a non-trivial map $E \to \Gamma_\lambda$. If E is irreducible it therefore is essentially equivalent to Γ_λ.

Using $\overline{\Gamma}_\lambda^*$ and similar constructions, we can split off factor Γ_λ successively from an arbitrary representation E. □

The proofs of the statements about the structure of these homogenous spaces uses the study of related Grassmannians. These we define and study for $LU(n)$; results for arbitrary compact Lie groups follow by embedding into $U(n)$ and reduction to the established case.

Let $H = H_+ \oplus H_-$ be a (polarized) Hilbert space. The important example for us is $H = L^2(S^1, \mathbb{C}^n)$, with H_+ generated by functions z^k for $k \geq 0$ and H_- generated by z^k with $k < 0$ (negative or positive Fourier coefficients vanish).

On this Hilbert space, the complex loop group $LGl(n, \mathbb{C})$ acts by pointwise multiplication.

We define the Grassmannian

$$Gr(H) := \{W \subset H \mid \mathrm{pr}_+\colon W \to H_+ \text{ is Fredholm}, \mathrm{pr}_-\colon W \to H_- \text{ is Hilbert-Schmidt}\}.$$

This is a Hilbert manifold.

We define the restricted linear group $Gl_{res}(H) := \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mid b, c \text{ Fredholm}\}$. (This implies that a, d are Hilbert-Schmidt). Set $U_{res}(H) := U(H) \cap Gl_{res}(H)$. These groups act on $Gr(H)$.

7.7 Lemma. *The image of $LGl(n, \mathbb{C})$ in $Gl(H)$ is contained in $Gl_{res}(H)$.*

Proof. Write $\gamma = \sum_k \gamma_k e^{ik\theta} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in LGl(n,\mathbb{C})$. Then for each $n > 0$, $b(e^{-in\theta}) = \sum_{k\geq n} \gamma_k e^{i(k-n)\theta}$, and for $n \geq 0$, $c(e^{in\theta}) = \sum_{k<-n} \gamma_k e^{i(k+n)\theta}$. Therefore

$$\|b\|_{HS} = \sum_{n\geq 0}\sum_{k\geq n} \left|\gamma_k\right|^2 = \sum_{k\geq 0} (1+k)\left|\gamma_k\right|^2 < \infty$$

since smooth functions have rapidly decreasing Fourier expansion. Similarly for c. □

$Gr(H)$ has a cell decomposition/stratification in analogy to the finite dimensional Grassmannians (with infinitely many cells of all kinds of relative dimensions). Details are omitted here, but this is an important tool in many proofs.

$Gr(H)$ contains the subspace

$$Gr_\infty(H) := \{W \in Gr(H) \mid \mathrm{im}(pr_-) \cup \mathrm{im}(\mathrm{pr}_+) \subset C^\infty(S^1,\mathbb{C}^n)\}.$$

Inside this one we consider the subspace $Gr_\infty^{(n)} := \{W \in Gr_\infty(H) \mid zW \subset W\}$.

It turns out that $Gr_\infty(H)^{(n)} = LGl(n,\mathbb{C})/L^+Gl(n,\mathbb{C}) = LU(n)/U(n)$.

The main point of the definition of $Gr(H)$ is that we can define a useful (and fine) virtual dimension for its elements; measuring the dimension of $W \cap z^m H_-$ for every m. This is used to stratify these Grassmannians and related homogeneous spaces, and these stratifications were used in the study of the holomorphic sections of line bundles over these spaces.

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THE SPINOR L-FUNCTION

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***Abstract*.** In this paper we survey some recent results regarding automorphic forms on the similitude symplectic group of order four. We will also explain recent progress on analytic properties of L-functions associated to such automorphic forms.

1. Introduction

The purpose of this note is to put together in a suggestive way some of the recent results in the theory of automorphic forms on the similitude symplectic group of order four. Our emphasis will be on the spinor L-function as the title indicates. This is by no means to suggest that the standard L-function is not equally as interesting. A quick search in `mathscinet` reveals however that the standard L-function is better understood, and it may be time to devote some energy to the spinor L-function.

The paper is organized as follows. In the first section we review the basic structure of the group under study. We also define Siegel modular forms and explain how one might associate adelic automorphic forms to them. In the next section, we describe the theory of the spinor L-function. Here we will define two integral representations for the L-function; one due to Novodvorsky and one due to Piatetski-Shapiro. The latter integral works for all automorphic representations, whereas the former works only for generic representations. We include a resume of the results of the author on the determination of the local non-archimedean Euler factors of the Novodvorsky's integral. We will also review Moriyama's result on the archimedean

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representations. Piatetski-Shapiro's integral representation involve the Bessel functionals. For this reason, we have devoted the third section to the study of Bessel functionals and their existence. As an application of the spinor L-function, we have considered CAP representations in the fourth section where a theorem of Piatetski-Shapiro is discussed. In the last paragraph we have listed a number of recent developments which have a bearing on our understanding of the spinor L-function, e.g., Asgari-Shahidi's transfer of generic cusp forms on GSp(4) to GL(4) which among other things implies the holomorphy of the spinor L-function of such forms.

There are many interesting and important topics that are not mentioned in these notes; in particular, no connection to the arithmetic of special values of L-functions is discussed. This is the topic that has fueled this author's interest in the subject. These notes are based on the talks given by the author at the Summer School on Algebraic Groups at the Mathematisches Institut at Göttingen during June and July of 2005. The author wishes to thank the hospitality of the Mathematisches Institut. He also wishes to thank Yuri Tschinkel for making the visit possible. The author was partially supported by the NSA.

2. Classical Siegel forms and automorphic representations

2.1. Preliminaries on GSp(4)**.** In this paper, the group GSp(4) over an arbitrary field K is the group of all matrices $g \in \mathrm{GL}_4(K)$ that satisfy the following equation for some scalar $\nu(g) \in K$:

$$^t g J g = \nu(g) J,$$

where $J = \begin{pmatrix} & & 1 & \\ & & & 1 \\ -1 & & & \\ & -1 & & \end{pmatrix}$. It is a standard fact that $G = \mathrm{GSp}(4)$ is a reductive group. The map $(F^\times)^3 \longrightarrow G$, given by

$$(a, b, \lambda) \mapsto \mathrm{diag}(a, b, \lambda a^{-1}, \lambda b^{-1})$$

gives a parameterization of a maximal torus T in G. The Weyl group is a dihedral group of order eight. We have three standard parabolic subgroups: The Borel subgroup B, The Siegel subgroup P, and the Klingen subgroup Q with the following Levi decompositions:

$$B = \left\{ \begin{pmatrix} a & & & \\ & b & & \\ & & a^{-1}\lambda & \\ & & & b^{-1}\lambda \end{pmatrix} \begin{pmatrix} 1 & x & & \\ & 1 & & \\ & & 1 & \\ & & -x & 1 \end{pmatrix} \begin{pmatrix} 1 & & s & r \\ & 1 & r & t \\ & & 1 & \\ & & & 1 \end{pmatrix} \right\},$$

$$P = \left\{ \begin{pmatrix} g & \\ & \alpha\, {}^t g^{-1} \end{pmatrix} \begin{pmatrix} 1 & & s & r \\ & 1 & r & t \\ & & 1 & \\ & & & 1 \end{pmatrix} ; g \in \mathrm{GL}(2) \right\},$$

and finally Q is the maximal parabolic subgroup with non-abelian unipotent radical associated to the long simple root. If ψ is an additive character of the field K, we define a character θ of the unipotent radical $N(B)$ of the Borel subgroup by the following:

$$\theta(\begin{pmatrix} 1 & x & & \\ & 1 & & \\ & & 1 & \\ & & -x & 1 \end{pmatrix} \begin{pmatrix} 1 & & s & r \\ & 1 & r & t \\ & & 1 & \\ & & & 1 \end{pmatrix}) = \psi(x+t).$$

When K is a local field, we always take ψ to be the unramified Tate character. Our notation for representation theory of the symplectic group is from [**TB00**].

We define various subgroups of the group $G = \mathrm{Sp}(4)$ over the real numbers. We have

$$G(\mathbb{R}) = \{g \in \mathrm{GL}_4(\mathbb{R}) \mid {}^t g J g = J\},$$

where as before $J = \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix}$. Then the Lie algebra $\mathfrak{g}$ of G will be the set of matrices $X \in \mathfrak{sl}_4(\mathbb{R})$ such that ${}^t X J + J X = 0$. The Cartan involution is given by $\theta(X) = -{}^t X$. Then we let $\mathfrak{k}$ and $\mathfrak{p}$ be the $+1$ and -1 eigen-spaces of θ, respectively. We have

$$\mathfrak{k} = \{\begin{pmatrix} A & B \\ -B & A \end{pmatrix} \mid A + iB \in U(2)\},$$

and

$$\mathfrak{p} = \{\begin{pmatrix} A & B \\ B & -A \end{pmatrix} \mid A = {}^t A, B = {}^t B\}.$$

Let K be the analytic subgroup defined by $\mathfrak{k}$.

2.2. From classical Siegel forms to GSp(4)**.** In this section we review the relation between classical Siegel modular forms and automorphic forms on the group GSp(4). We will follow the exposition of [**AS01**]. Let $\mathscr{H}_n$ be the complex manifold consisting of complex symmetric $n \times n$ matrices with positive definite imaginary part. Let $\Gamma_2 = \mathrm{Sp}(4, \mathbb{Z})$, and let f be a Siegel modular form with respect to Γ_2 of weight k. By definition, f is a holormophic function on $\mathscr{H}_2$ satisfying

(1) $$f|_k \gamma = f$$

for each $\gamma \in \Gamma_2$. Here

(2) $$f|_k h = \nu(h)^{nk/2} j(h, Z)^{-k} f(h < Z >)$$

for $h \in \mathrm{GSp}_4(\mathbb{R})^+$ and $Z \in \mathscr{H}_n$. In this equation, ν is the multiplier, $j(h,Z) = \det(CZ+D)$ for $h = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, and $h(Z) = (AZ+B)(CZ+D)^{-1}$. We now associate to f an adelic function Φ_f as follows. By strong approximation

$$\mathrm{GSp}(4,\mathbb{A}) = \mathrm{GSp}(4,\mathbb{Q})\mathrm{GSp}(4,\mathbb{R})^+ \prod_{v<\infty} \mathrm{GSp}(4,\mathbb{Z}_p). \tag{3}$$

According to this decomposition, write an element $g \in \mathrm{GSp}(4,\mathbb{A})$ as $g = g_{\mathbb{Q}} g_\infty k_0$ with the obvious notation. Define

$$\Phi_f(g) = (f|_k g_\infty)(I), \tag{4}$$

where $I = \mathrm{diag}(i, i, \dots, i) \in \mathscr{H}_n$. The function $\Phi = \Phi_f$ has the following properties:

1. $\Phi(\gamma g) = \Phi(g)$ for $\gamma \in \mathrm{GSp}(4,\mathbb{Q})$;
2. $\Phi(g k_0) = \Phi(g)$ for $k_0 \in K_0$;
3. $\Phi(g k_\infty = \Phi(g) j(k_\infty, I)^{-k}$ for $k_\infty \in K_\infty$;
4. $\Phi(zg) = \Phi(g)$ for $g \in Z(\mathbb{A})$.

The map $f \to \Phi_f$ sends $S_k(\Gamma_n)$ to the space of cusp forms. This map is also a Hecke-equivariant isometry between the L^2-spaces for appropriately normalized invariant measures.

Let f be a cuspidal Hecke eigenform for the full Hecke algebra. Denote by V_f the subspace of $L_0^2(\mathrm{GSp}(4,\mathbb{Q})Z(\mathbb{A})\backslash \mathrm{GSp}(4,\mathbb{A}))$ spanned by all the right translates of Φ_f. We let π_f be the irreducible automorphic cuspidal representation obtained from the right action of $\mathrm{GSp}(4,\mathbb{A})$ on V_f. Notice that here we are ignoring the issue of L-indistinguishability. As f is modular for Γ_2, the representation π_f will be unramified at all finite places. Fix a finite place v. By general theory, the v-component of π_f will be a representation of the form $\chi_1 \times \chi_2 \rtimes \chi_3$. As usual we call the complex numbers $\chi_0(\varpi)$, $\chi_1(\varpi)$, and $\chi_2(\varpi)$ the v-Satake parameters of the local v-component of the representation π_f. There is a simple relation relating these Satake parameters to the classical Satake parameters which will then induce a shift in the functional equation of the L-functions. In fact, the new Satake parameters are equal to $p^{\frac{3}{2}-k} a_0$, a_1, and a_2 with a_0, a_1, a_2 the classical ones.

Let us also say a word about the archimedean component of the representation π_f. Let

$$K_\infty = \left\{ \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \in \mathrm{GL}_{2n}(\mathbb{R}) | A\,{}^tA + B + {}^tB = I, A\,{}^tB = B\,{}^tA \right\}; \tag{5}$$

the Lie algebra of K_∞, denoted by $\mathfrak{k}$, is the collection of matrices of the same shape with A anti-symmetric and B symmetric. This is the $+1$-eigen-space of the Cartan involution $\theta X = -{}^tX$. We let $\mathfrak{p}$ be the (-1)-eigenspace of the Cartan involution.

Clearly $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. One can see that $\mathfrak{p}_{\mathbb{C}} = \mathfrak{p}_{\mathbb{C}}^{+} \oplus \mathfrak{p}_{\mathbb{C}}^{-}$ with

$$\mathfrak{p}_{\mathbb{C}}^{\pm} = \left\{ \begin{pmatrix} A & \pm iA \\ \pm iA & -A \end{pmatrix} \in \mathrm{M}_{2n}(\mathbb{R}) | A = {}^{t}A \right\}. \tag{6}$$

Let

$$T_i = -i \begin{pmatrix} 0 & D_i \\ -D_i & 0 \end{pmatrix}, \tag{7}$$

where D_i is the diagonal matrix with entry 1 at position (i,i) and zero everywhere else. Now Lemma 11 of [**AS01**] states that the representation π_∞ contains a smooth vector v_∞ with the following properties

1. $\pi(k_\infty)v_\infty = j(k_\infty, I)^{-k} v_\infty$ for all $k_\infty \in K_\infty$,
2. $T_i v_\infty = k v_\infty$ for $i = 1, 2$,
3. $\mathfrak{p}_{\infty}^{\pm} v_\infty = 0$.

This last condition is equivalent to the holomorphy of the starting function f. Because of this π_∞ is a holomorphic discrete series representation if $k > 2$ and it is a limit of discrete series when $k = 2$. There is a similar description of the automorphic form associated to a vector-valued Siegel cusp form which is nicely explained in paragraph 4.5 of [**AS01**].

In the classical setting, if for each prime p, a_0, a_1, a_2 are the Satake parameters of the form f, we set

$$L_1(s,f) = \prod_p \left((1-p^{-s}) \prod_{i=1}^{2} (1-a_i p^{-s})(1-a_i p^{-s}) \right)^{-1}, \tag{8}$$

and

$$L_2(s,f) = \prod_p \left((1-a_0 p^{-s})(1-a_0 a_1 p^{-s})(1-a_0 a_2 p^{-s})(1-a_0 a_1 a_2 p^{-s}) \right)^{-1}. \tag{9}$$

We now observe that

$$L_1(s,f) = L(s, \pi_f, \text{Standard}), \tag{10}$$

and

$$L_2(s,f) = L(s, \pi_f, \text{Spinor}) \tag{11}$$

with the right hand side of the equations being Langlands L-functions of the associated representation π_f. In this survey we will concentrate on the spinor L-functions.

3. The Spinor L-function for GSp(4)

In this section, we review the integral representation given by Novodvorsky [**Nov79**] for $G = \mathrm{GSp}(4)$. The details of the material in the following paragraphs appear in [**Bum89**], [**TB00**]. In the last paragraph, we will review a different construction due to Piatetski-Shapiro that also works for non-generic representations.

3.1. Whittaker models. As we will primarily be dealing with representations which have Whittaker models, we take a moment to review basic definition and properties of such models.

Let π be an automorphic cuspidal representation of the group G. For each $\varphi \in \pi$, we set

$$W_\varphi(g) = \int_{(\mathbb{Q}\backslash\mathbb{A})^4} \varphi\left(\begin{pmatrix} 1 & x_2 & & \\ & 1 & & \\ & & 1 & \\ & & -x_2 & 1 \end{pmatrix}\begin{pmatrix} 1 & & x_4 & x_3 \\ & 1 & x_3 & x_1 \\ & & 1 & \\ & & & 1 \end{pmatrix} g\right) \times \psi^{-1}(x_1 + x_2)\, dx_1\, dx_2\, dx_3\, dx_4.$$

Let N be the unipotent radical of the Borel subgroup. For each place v of $\mathbb{Q}$, the restriction of θ to $N(\mathbb{Q}_v)$ is denoted by θ_v. Consider the representation of G induced from the character θ_v of $N(\mathbb{Q}_v)$:

$$C^\infty_{\theta_v}(N(\mathbb{Q}_v)\backslash G(\mathbb{Q}_v)) := \left\{ W : G(\mathbb{Q}_v) \to \mathbb{C} \mid \begin{array}{c} \text{smooth,} \\ {\scriptstyle W(ng)=\theta_v(n)W(g),} \\ {\scriptstyle n\in N(\mathbb{Q}_v), g\in G(\mathbb{Q}_v)} \end{array} \right\}. \tag{12}$$

The action of $G(\mathbb{Q}_v)$ on $C^\infty_{\theta_v}(N(\mathbb{Q}_v)\backslash G(\mathbb{Q}_v))$ is by right translation.

If v is a finite place of $\mathbb{Q}$, then for any irreducible admissible representation π_v of $G(\mathbb{Q}_v)$, the intertwining space

$$\mathrm{Hom}_{G(\mathbb{Q}_v)}(\pi_v, C^\infty_{\theta_v}(N(\mathbb{Q}_v)\backslash G(\mathbb{Q}_v)))$$

is at most one-dimensional ([**Rod73**], Theorem 3). If there is a non-zero intertwining operator

$$\Psi \in \mathrm{Hom}_{G(\mathbb{Q}_v)}(\pi_v, C^\infty_{\theta_v}(N(\mathbb{Q}_v)\backslash G(\mathbb{Q}_v))) \tag{13}$$

then we say that π_v is generic, and call the image $W_u := \Psi(u)$ of $u \in \pi_v$ the *local Whittaker function corresponding to $u \in \pi_v$*. The space of all W_u ($u \in \pi_v$) is called the *Whittaker model of π_v with respect to θ_v*.

Now let $v = \infty$ be the archimedean place. We say that a $\mathbb{C}$-valued function W on $G(\mathbb{R})$ is of moderate growth if there exists $C > 0$ and $M > 0$ such that $|W(g)| \leqslant C\|g\|^M$ for all $g \in G(\mathbb{R})$. The form $\|g\|$ of $g = (g_{ij})$ is defined by $\|g\| := \max\{|g_{ij}|, |(g^{-1})_{ij}\}$. The space of functions $W \in C^\infty_{\theta_v}(N(\mathbb{Q}_v)\backslash G(\mathbb{Q}_v))$ of moderate growth is denoted by

$\mathscr{A}_{\theta_\infty}(N(\mathbb{R})\backslash G(\mathbb{R}))$. Improving Shalika's local multiplicity one theorem ([**Sha74**], Theorem 3.1), Wallach ([**Wal83**], Theorem 8.8 (1)) showed that for an arbitrary $(\mathfrak{g}, K)$-module π_∞ the intertwining space

$$\mathrm{Hom}_{(\mathfrak{g},K)}(\pi_\infty, \mathscr{A}_{\theta_\infty}(N(\mathbb{R})\backslash G(\mathbb{R})))$$

is at most one-dimensional. Again, if there is a non-zero intertwining operator

$$\Psi \in \mathrm{Hom}_{(\mathfrak{g},K)}(\pi_\infty, \mathscr{A}_{\theta_\infty}(N(\mathbb{R})\backslash G(\mathbb{R}))), \tag{14}$$

then we say π_∞ is generic and call the image $W_u := \Psi(u)$ of $u \in \pi_\infty$ the *local Whittaker function corresponding to u*.

3.2. Let φ be a cusp form on $\mathrm{GSp}(4, \mathbb{A})$, belonging to the space of an irreducible cuspidal automorphic representation π. Consider the integral

$$Z_N(s, \varphi, \mu) = \int_{\mathbb{A}^\times/\mathbb{Q}^\times} \int_{(\mathbb{A}/\mathbb{Q})^3} \varphi\left(\begin{pmatrix} 1 & x_2 & x_4 & \\ & 1 & & \\ & & 1 & \\ & z & -x_2 & 1 \end{pmatrix} \begin{pmatrix} y & & & \\ & y & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \right) \times \psi(-x_2)\mu(y)|y|^{s-\frac{1}{2}}\, dz\, dx_2\, dx_4\, d^\times y.$$

Since φ is left invariant under the matrix

$$\begin{pmatrix} & & & 1 \\ & & 1 & \\ & -1 & & \\ -1 & & & \end{pmatrix},$$

this integral has a functional equation $s \to 1 - s$. A usual unfolding process as sketched in [**Bum89**] then shows that

$$Z_N(s, \varphi, \mu) = \int_{\mathbb{A}^\times} \int_{\mathbb{A}} W_\varphi \begin{pmatrix} y & & & \\ & y & & \\ & & 1 & \\ & x & & 1 \end{pmatrix} \mu(y)|y|^{s-\frac{3}{2}}\, dx\, d^\times y. \tag{15}$$

Here the Whittaker function W_φ is given by

$$W_\varphi(g) = \int_{(\mathbb{A}/\mathbb{Q})^4} \varphi\left(\begin{pmatrix} 1 & x_2 & & \\ & 1 & & \\ & & 1 & \\ & & -x_2 & 1 \end{pmatrix} \begin{pmatrix} 1 & & x_4 & x_3 \\ & 1 & x_3 & x_1 \\ & & 1 & \\ & & & 1 \end{pmatrix} g \right) \times \psi^{-1}(x_1 + x_2)\, dx_1\, dx_2\, dx_3\, dx_4.$$

The basic idea in establishing the above identity is to show that

$$\int_{(F\backslash\mathbb{A})^3} \varphi\left(\begin{pmatrix}1 & x_2 & & x_4\\ & 1 & & \\ & & 1 & -x_2\\ & & & 1\end{pmatrix} g\right)\psi(-x_2)\,dx_2\,dx_4$$

(16)

$$= \sum_{\substack{\alpha\in F^\times\\ \beta\in F}} W\left(\begin{pmatrix}\alpha & & & \\ & \alpha & & \\ & \alpha\beta & 1 & \\ & & & 1\end{pmatrix} g\right).$$

For this one uses the following identity from Fourier analysis

(17)
$$f(0) = \sum_{\alpha\in F}\int_{F\backslash\mathbb{A}} f(x_1)\psi(-\alpha x_1)\,dx_1,$$

for any reasonable F-invariant function f.

Equation (15) implies that, in order for $Z_N(\varphi, s)$ to be non-zero, we need to assume that W_φ is not identically equal to zero. A representation satisfying this condition is called “generic." Every irreducible cuspidal representation of GL(2) is generic. On other groups, however, there may exist non-generic cuspidal representations. In fact, those representations of GSp(4) which correspond to holomorphic cuspidal Siegel modular forms are not generic.

If φ is chosen correctly, the Whittaker function may be assumed to decompose locally as $W(g) = \prod_v W_v(g_v)$, a product of local Whittaker functions. Hence, for $\Re(s)$ large, we obtain

(18)
$$Z_N(\varphi, s) = \prod_v Z_N(W_v, s),$$

where

(19)
$$Z_N(W_v, s) = \int_{F_v^\times}\int_{F_v} W_v\left(\begin{pmatrix}y & & & \\ & y & & \\ & & 1 & \\ & x & & 1\end{pmatrix}\right)|y|^{s-\frac{3}{2}}\,dx\,d^\times y.$$

As usual, we have a functional equation: There exists a meromorphic function $\gamma(\pi_v, \psi_v, s)$ (rational function in $\mathbb{N}v^{-s}$ when $v < \infty$) such that

(20)
$$Z_N(W_v, s) = \gamma(\pi_v, \psi_v, s)\tilde{Z}_N(W_v^w, 1-s),$$

with w as above,

$$\tilde{Z}_N(W_v,s)=\int_{F_v^\times}\int_{F_v} W_v\left(\begin{pmatrix} y & & & \\ & y & & \\ & & 1 & \\ & x & & 1\end{pmatrix}\right)\chi_v^{-1}(y)|y|^{s-\frac{3}{2}}\,dx\,d^\times y,$$

and χ_v the central character of π_v.

We also consider the unramified calculations. Suppose v is any nonarchimedean place of F such that W_v is right invariant by $\mathrm{GSp}(4,\mathscr{O}_v)$ and such that the largest fractional ideal on which ψ_v is trivial on $\mathscr{O}$. Then the Casselman-Shalika formula [**CS80**] allows us to calculate the last integral (cf. [**Bum89**]). Let us review the computation following [**Bum89**]. For this, we need to recall what the Casselman-Shalika formula says in this context. Set

$$u(\xi_1,\xi_2,\xi_3,\xi_4)=\begin{pmatrix}\xi_1\xi_2 & & & \\ & \xi_1\xi_3 & & \\ & & \xi_2\xi_4 & \\ & & & \xi_3\xi_4\end{pmatrix}. \tag{21}$$

Then the Weyl group acts on the polynomial ring $\mathbb{C}[x_1^{\pm1},x_2^{\pm1},x_3^{\pm1},x_4^{\pm1}]$. The action is in such a way that the two generators

$$w_1=\begin{pmatrix}1 & & & \\ & & -1 & \\ & 1 & & \\ & & & 1\end{pmatrix},\quad w_2=\begin{pmatrix} & 1 & & \\ 1 & & & \\ & & & 1\\ & & 1 & \end{pmatrix} \tag{22}$$

act via $w_1: x_1\to x_2\to x_1, x_3\to x_4\to x_3$ and $w_2: x_1\to x_1, x_2\to x_3\to x_2, x_4\to x_4$. Now define a group algebra element

$$\mathscr{B}=\sum_{w\in W}(-1)^{l(w)}w, \tag{23}$$

where $l(w)$ is the length function on the Weyl group. Let

$$T_{k_1,k_2}(x_1,x_2,x_3,x_4)=\frac{\mathscr{B}(x_1^{k_1+k_2+3}x_2^{k_1+2}x_3x_4^{-k_2})}{\mathscr{B}(x_1^3x_2^2x_3)}. \tag{24}$$

The Casselman-Shalika formula states the following: Let $X_\pi = u(\xi_1, \xi_2, \xi_3, \xi_4)$ be the semi-simple conjugacy class in $\mathrm{GSp}_4(\mathbb{C})$ associated to the unramified representation π, and W the normalized Whittaker function for π. Then if $ord(y_i) = k_i$, then

$$W\left(\begin{pmatrix} y_0y_1y_2 & & & \\ & y_0y_1 & & \\ & & y_0 & \\ & & & y_0y_2^{-1} \end{pmatrix}\right) = q^{-\frac{3}{2}k_1-2k_2}(\xi_1\xi_2\xi_3\xi_4)^{k_0}T_{k_1,k_2}(\xi_1,\xi_2,\xi_3,\xi_4) \tag{25}$$

if $k_1, k_2 \geqslant 0$, and zero otherwise.

The result is the following:

$$Z_N(W_v, s) = L(s, \pi_v, \mathrm{Spin}). \tag{26}$$

Let us explain the notation. The connected L-group ${}^LG^0$ is $\mathrm{GSp}_4(\mathbb{C})$. Let ${}^LT$ be the maximal torus of elements of the form

$$t(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = \begin{pmatrix} \alpha_1 & & & \\ & \alpha_2 & & \\ & & \alpha_3 & \\ & & & \alpha_4 \end{pmatrix},$$

where $\alpha_1\alpha_4 = \alpha_2\alpha_3$. The fundamental dominant weights of the torus are λ_1 and λ_2 where

$$\lambda_1 t(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = \alpha_1,$$

and

$$\lambda_2 t(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = \alpha_1\alpha_3^{-1}.$$

The dimensions of the representation spaces associated with these dominant weights are four and five, respectively. In our notation, Spin is the representation of $\mathrm{GSp}(4,\mathbb{C})$ associated with the dominant weight λ_1, i.e. the standard representation of $\mathrm{GSp}(4,\mathbb{C})$ on $\mathbb{C}^4$. The L-function $L(s, \pi, \mathrm{Spin})$ is called the Spinor, or simply the Spin, L-function of $\mathrm{GSp}(4)$.

Next step is to use the integral introduced above to extend the definition of the Spinor L-function to ramified non-archimedean and archimedean places.

Remark 3.1. There are constructions for the degree 8 (resp. 12) L-function for generic representations of $\mathrm{GSp}(4) \times \mathrm{GL}_2$ (resp. $\mathrm{GSp}(4) \times \mathrm{GL}_3$). For a review see [**Bum89**].

3.3. Local Euler factors of the spinor L-function. We now sketch the computation of the local non-archimedean Euler factors of the Spin L-function of generic representations given by the integral representation introduced above. In order for this to make sense, we need the following lemma:

Lemma 3.2 (Theorem 2.1 of [TB00]). *Suppose Π is a generic representation of* GSp(4) *over a non-archimedean local field K, q order of the residue field. For each $W \in \mathscr{W}(\Pi,\psi)$, the function $Z_N(W,s)$ is a rational function of q^{-s}, and the ideal $\{Z_N(W,s)\}$ is principal.*

Sketch of proof. For $W \in \mathscr{W}(\Pi,\psi)$, we set

$$Z(W,s) = \int_K W\left(\begin{pmatrix} y & & & \\ & y & & \\ & & 1 & \\ & & & 1 \end{pmatrix}\right) |y|^{s-\frac{3}{2}}\, d^\times y. \tag{27}$$

The first step of the proof is to show that the vector space $\{Z(W,s)\}$ is the same as $\{Z_N(W,s)\}$ (cf. Proposition 3.2 of [**TB00**]). Next, we use the asymptotic expansions of the Whittaker functions along the torus to prove the existence of the g.c.d. for the ideal $\{Z(W,s)\}$. Indeed, Proposition 3.5 of [**TB00**] (originally a theorem in [**CS80**]) states that there is a finite set of finite functions S_Π, depending only on Π, with the following property: for any $W \in \mathscr{W}(\Pi,\psi)$, and $c \in S_\Pi$, there is a Schwartz-Bruhat function $\Phi_{c,W}$ on K such that

$$W(\begin{pmatrix} y & & & \\ & y & & \\ & & 1 & \\ & & & 1 \end{pmatrix}) = \sum_{c\in S_\Pi} \Phi_{c,W}(y) c(y) |y|^{\frac{3}{2}}.$$

The lemma is now immediate. □

We have the following theorem:

Theorem 3.3. *Suppose Π is a generic representation of the group* GSp(4) *over a non-archimedean local field K. Then*

1. *If Π is supercuspidal, or a sub-quotient of a representation induced from a supercuspidal representation of the Klingen parabolic subgroup, then* $L(s,\pi,\mathrm{Spin}) = 1$.
2. *If π is a supercuspidal representation of* GL(2) *and χ a quasi-character of $K^\times$, and $\Pi = \pi \rtimes \chi$ is irreducible, we have*

$$L(s,\Pi,\mathrm{Spin}) = L(s,\chi)\cdot L(s,\chi.\omega_\pi).$$

3. *If χ_1, χ_2, and χ_3 are quasi-characters of $K^\times$, and $\Pi = \chi_1 \times \chi_2 \rtimes \chi_3$ is irreducible, we have*

$$L(s,\Pi,\mathrm{Spin}) = L(s,\chi_3)\cdot L(s,\chi_1\chi_3)\cdot L(s,\chi_2\chi_3)\cdot L(s,\chi_1\chi_2\chi_3).$$

4. *When Π is not irreducible, one can prove similar statements for the generic subquotients of $\Pi = \pi \rtimes \chi$ (resp. $\Pi = \chi_1 \times \chi_2 \rtimes \chi_3$) according to the classification*

theorems of Sally-Tadic [**ST93**] *and Shahidi* [**Sha90**] *(cf. Theorems 4.1 and 5.1 of* [**TB00**]*).*

Remark 3.4. Sally and Tadic [**ST93**] and Shahidi [**Sha90**] have completed the classification of representations supported in the Borel and Siegel parabolic subgroups. In particular, they have determined for which representations the parabolic induction is reducible. From their result, one can immediately establish a classification for all the generic representations supported in the Borel or Siegel parabolic subgroups.

Sketch of proof. By the proof of the lemma, we need to determine the asymptotic expansion of the Whittaker functions in each case. The argument consists of several steps:

Step 1. *Bound the size of* S_Π. Fix $c \in S_\Pi$, and define a functional Λ_c on $\mathscr{W}(\Pi,\psi)$ by

$$\Lambda_c(W) = \Phi_{c,W}(0). \tag{28}$$

If $c, c' \in S_\Pi$, and $c \neq c'$, the two functionals Λ_c and $\Lambda_{c'}$ are linearly independent. Furthermore, the functionals Λ_c belong to the dual of a certain twisted Jacquet module $\Pi_{N,\bar{\theta}}$ (notation from [**TB00**], page 1095). Hence $\#S_\Pi = \dim \Pi_{N,\bar{\theta}}$. Then one uses an argument similar to those of [**Sha74**], distribution theory on p-adic manifolds, to bound the dimension of the Jacquet module. The result (Proposition 3.9 of [**TB00**]) is that if Π is supercuspidal or supported in the Klingen parabolic subgroup (resp. Siegel parabolic, resp. Borel parabolic), then $\#S_\Pi = 0$ (resp. $\leqslant 2$, resp. $\leqslant 4$). Note that this already implies the first part of the theorem.

From this point on, we concentrate on the Siegel parabolic subgroup, the Borel subgroup case being similar. We fix some notation. Suppose $\Pi = \pi \rtimes \chi$, with π supercuspidal of GL(2). Let λ_Π (resp. λ_π) be the Whittaker functional of Π (resp. π) from [**Sha81**]. It follows from the proof of the Lemma 3.2 that, for $f \in \Pi$, there is a positive number $\delta(f)$, such that

$$\lambda_\Pi(\Pi(\begin{pmatrix} y & & & \\ & y & & \\ & & 1 & \\ & & & 1 \end{pmatrix})f) = \sum_{c\in S_\Pi} \Lambda_c(f)c(y)|y|^{\frac{3}{2}},$$

for $|y| < \delta(f)$. Here, Λ_c is the obvious functional on the space of Π.

Step 2. *Uniformity.* For $f \in Ind(\pi \times \chi | P \cap K, K)$, and $\tau \in \mathbb{C}$, define f_τ on G by

$$f_\tau(pk) = \delta_P(p)^{\tau+\frac{1}{2}} \pi \otimes \chi(p) f(k).$$

It is clear that f_τ is a well-defined function on G, and that it belongs to the space of a certain induced representation Π_τ. The *Uniformity Theorem* (Proposition 3.9 of [**TB00**]) asserts that one can take $\delta(f_\tau) = \delta(f)$.

Step 3. *Regular representations.* This is the case where $\omega_\pi \neq 1$. In this situation, we have

$$\begin{aligned}\lambda_\Pi(\Pi(\begin{pmatrix} y &&& \\ & y && \\ && 1 & \\ &&& 1\end{pmatrix})f) = \\ \lambda_\pi(A(w,\Pi)(f)(e))\chi(y)|y|^{\frac{3}{2}} + C(w\Pi, w^{-1})^{-1}\lambda_\pi(f(e))\chi(y)\omega_\pi(y)|y|^{\frac{3}{2}},\end{aligned} \tag{29}$$

for $|y| < \delta(f)$. Here $w = \begin{pmatrix} && 1 & \\ &&& 1 \\ -1 &&& \\ & -1 && \end{pmatrix}$, $A(w,\Pi)$ is the intertwining integral of [**Sha81**], and $C(w\Pi, w^{-1})$ is the local coefficient of [**Sha81**]. The proof of this identity follows from the the above Lemma 3.2, and the *Multiplicity One Theorem* [**Sha74**]. The idea is to find one term of the asymptotic expansion using the open cell; then apply the long intertwining operator to find the other term.

Note that the identity of *Step* 3 also applies to reducible cases. For example, if $f \in \Pi$ is in the kernel of the intertwining operator $A(w,\Pi)$, the first term of the right hand side vanishes.

Step 4. *Irregular Representations.* The idea is the following: we twist everything in *Step* 3 by the complex number τ, so that the resulting representation Π_τ is regular. By *Step* 2, the identity still holds uniformly for all τ. By a theorem of Shahidi [**Sha81**] (essentially due to Casselman and Shalika [**CS80**]), we know that the left hand side of the identity is an entire function of τ. This implies that the poles of the right hand side, coming from the intertwining operator and the local coefficient, must cancel out. Next, we let $\tau \to 0$. An easy argument (l'Hopital's rule!) shows the appearance of $\chi(y)|y|^{\frac{3}{2}}$ and $\chi(y)|y|^{\frac{3}{2}}\log_q|y|$ in the asymptotic expansion.

This finishes the sketch of proof of the theorem. □

Corollary 3.5. *Let π be an irreducible generic representation of* GSp(4) *over a non-archimedean local field K. Let μ be a quasi-character of $K^\times$. If μ is highly ramified, we have*

$$L(s, \pi \otimes \mu) = 1.$$

3.4. Moriyama's results at the archimedean place. In this paragraph we review Moriyama's computation of the archimedean Euler factors of Novodvorsky's integral for generic (limits of) discrete series. In order to state his results, however, we need to set up some notation.

Here let $G = \mathrm{GSp}(4)$ and $G_0 = \mathrm{Sp}(4,\mathbb{R})$. A maximal compact subgroup K (resp. K_0) of $G(\mathbb{R})$ (resp. G_0) is given by $K := G(\mathbb{R}) \cap \mathrm{O}(4)$ (resp. $K_0 := G_0 \cap \mathrm{O}(4)$). The group K_0 is isomorphic to the unitary group $U(2) := \{g \in \mathrm{GL}(2,\mathbb{C}) |\, {}^t\bar{g}g = I_2\}$. Define an isomorphism $\kappa : U(2) \cong K_0$ by

$$A + \sqrt{-1}B \mapsto \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \in K_0 \tag{30}$$

if $A + \sqrt{-1}B \in U(2)$. We write the Lie algebra of $G(\mathbb{R})$, G_0, and K_0 by $\mathfrak{g}$, $\mathfrak{g}_0$ and $\mathfrak{k}$ respectively. For an arbitrary Lie algebra $\mathfrak{l}$, we denote its complexification by $\mathfrak{l}_{\mathbb{C}}$. The differential κ_* of κ defines an isomorphism of complex Lie algebras $\mathfrak{gl}(2,\mathbb{C}) \cong \mathfrak{k}_{\mathbb{C}}$. The simple Lie algebra $\mathfrak{g}_0$ has a compact Cartan subalgebra $\mathfrak{h} := \mathbb{R}T_1 \oplus \mathbb{R}T_2$, where

$$T_1 := \kappa_* \begin{pmatrix} \sqrt{-1} & 0 \\ 0 & 0 \end{pmatrix}, \quad T_2 = \kappa_* \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{-1} \end{pmatrix}. \tag{31}$$

Define a $\mathbb{C}$-basis $\{\beta_1, \beta_2\}$ of $\mathfrak{h}_{\mathbb{C}}^*$ by $\beta_i(T_j) = \sqrt{-1}\delta_{ij}$, and fix an inner product $<,>$ on $\mathfrak{h}_{\mathbb{C}}^*$ by $<\beta_i, \beta_j>= \delta_{ij}$. Then the root system $\Delta = \Delta((\mathfrak{g}_0)_{\mathbb{C}}, \mathfrak{h}_C)$ is given by $\{\pm 2\beta_1, \pm 2\beta_2, \pm\beta_1 \pm \beta_2\}$. Denote by Δ_c the set of compact root $\{\pm(\beta_1 - \beta_2\}$ and take as a positive system of compact roots the set $\Delta_c^+ = \{\beta_1 - \beta_2\}$.

The irreducible finite dimensional representations of K_0 are parametrized by the set of their highest weights relative to Δ_c^+:

$$\{q = q_1\beta_1 + q_2\beta_2 = (q_1, q_2) \in \mathfrak{h}_C^*, q_i \in \mathbb{Z}, q_1 \geqslant q_2\}. \tag{32}$$

For each dominant weight $q = (q_1, q_2)$, we set $d_q = q_1 - q_2 \geqslant 0$. Then the degree of the representation $(\tau_{(q_1,q_2)}, V_{(q_1,q_2)})$ with highest weight (q_1, q_2) is $d_q + 1$. We can take a basis $\{v_k | 0 \leqslant k \leqslant d\}$ of $V_{(q_1,q_2)}$ so that the representation of $\mathfrak{k}_{\mathbb{C}}$ associated to (q_1, q_2) is given by

$$\begin{aligned} \tau(\kappa_* \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}) v_k &= (q_2 + k) v_k; \\ \tau(\kappa_* \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}) v_k &= (q_1 - k) v_k; \\ \tau(\kappa_* \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}) v_k &= (k+1) v_{k+1}; \\ \tau(\kappa_* \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}) v_k &= (d+1-k) v_{k-1}. \end{aligned} \tag{33}$$

We call $\{v_k\}$ the standard basis of $V_{(q_1,q_2)}$.

There are four positive systems of Δ containing Δ_c^+:

$$\begin{aligned} \Delta_I^+ &= \{(1,-1),(2,0),(1,1),(0,2)\};\\ \Delta_{II}^+ &= \{(1,-1),(0,-2),(2,0),(1,1)\};\\ \Delta_{III}^+ &= \{(1,-1),(-1,-1),(0,-2),(2,0)\};\\ \Delta_{IV}^+ &= \{(1,-1),(-2,0),(-1,-1),(0,-2)\}. \end{aligned} \tag{34}$$

Let $J \in \{I, II, III, IV\}$. Set $\Delta_{J,n}^+ = \Delta_J^+ \backslash \Delta_c^+$. For each index J, we denote by Ξ_J the set of integral weight $\Lambda = (\Lambda_1, \Lambda_2) \in \mathfrak{h}_C^*$, $\Lambda_i \in \mathbb{Z}$, satisfying (i) $< \Lambda, \beta >\geqslant 0$ for all $\beta \in \Delta_{J,n}^+$ and (ii) $< \Lambda, \beta >> 0$ for all $\beta \in \Delta_c^+$ that is a simple root in Δ_J^+. Then the set $\{(J,\Lambda) | \Lambda \in \Xi_J\}$ gives the collection of Harish-Chandra parameters of the (limits of) discrete series representations for G_0. We denote by $\pi(\Lambda, \Delta_J^+)$ the representation of G_0 associated to the parameter. If $< \Lambda, \beta >> 0$ for all $\beta \in \Delta_J^+$, then $\pi = \pi(\Lambda, \Delta_J^+)$ is a discrete series representations; otherwise a limit of discrete series. The Blattner parameter $\lambda_{min} \in \mathfrak{h}_{\mathbb{C}}^*$ of π is given by

$$\lambda_{min} := \Lambda + \frac{1}{2} \sum_{\alpha \in \Delta_J^+} \alpha - \sum_{\beta \in \Delta_c^+} \beta. \tag{35}$$

The highest weights of the K_0-types of π are of the form $\lambda_{min} + \sum_{\alpha \in \Delta_J^+} m_\alpha \alpha$ with m_α integral and non-negative. Furthermore, $\tau_{\lambda_{min}}$ occurs in π with multiplicity one, and we call it the minimal K_0 type of π. We denote by $D_{(q_1,q_2)}$ the representation of this form with minimal K_0-type equal to $\tau_{(q_1,q_2)}$. A (limit of) discrete series representation $\pi(\Lambda, \Delta_J^+)$ is called *large* if $J = II$ or III.

Now suppose $\Pi_{\mathbb{R}}$ is a representation of $\mathrm{GSp}(4,\mathbb{R})$ whose restriction to $\mathrm{Sp}(4,\mathbb{R})$ is the direct sum of two (limits of) discrete series representations $D_{(\lambda_1,\lambda_2)}$ and $D_{(-\lambda_2,-\lambda_1)}$. Let $\omega_{\Pi_{\mathbb{R}}}$ be the central character of $\Pi_{\mathbb{R}}$ and define a complex number ω_∞ by $\omega_{\Pi_{\mathbb{R}}}(t) = t^{\omega_\infty}$ ($t \in \mathbb{R}, t > 0$). Assume that the representation $\Pi_{\mathbb{R}}$ has a local Whittaker model. Hence by a theorem of Kostant on the existence of local Whittaker models, the representation of G_0 occurring in Π_∞ must be large. This means that we must have $1 - \lambda_1 \leqslant \lambda_2 \leqslant 0$ or $1 + \lambda_2 \leqslant -\lambda_1 \leqslant 0$. Without loss of generality we will assume that $1 - \lambda_1 \leqslant \lambda_2 \leqslant 0$. Set

$$L(s, \Pi_{\mathbb{R}}) = \Gamma_{\mathbb{C}}(s + \frac{\omega_\infty + \lambda_1 + \lambda_2 - 1}{2})\Gamma_{\mathbb{C}}(s + \frac{\omega_\infty + \lambda_1 - \lambda_2 - 1}{2}). \tag{36}$$

Here $\Gamma_{\mathbb{C}}(s) = 2(2\pi)^{-s}\Gamma(s)$.

Define the ε-factor by $\varepsilon(s, \Pi_{\mathbb{R}}, \psi_\infty) := (-1)^{\lambda_1}$.

Let $\{v_k | 0 \leqslant k \leqslant d = \lambda_1 - \lambda_2\}$ be the standard basis of the minimal K_0-type $\tau_{(-\lambda_2,-\lambda_1)}$ of $D_{(-\lambda_2,-\lambda_1)}$. We denote by $W_k \in \mathrm{W}(\Pi_{\mathbb{R}}, \psi_\infty)$ the local Whittaker function corresponding to $v_k \in \Pi_{\mathbb{R}}$. Define a vector subgroup A of $\mathrm{GSp}(4,\mathbb{R})$ by

$$A := \{\begin{pmatrix} a_1 & & & \\ & a_2 & & \\ & & a_1^{-1} & \\ & & & a_2^{-1} \end{pmatrix} | a_i > 0\}. \tag{37}$$

Moriyama [**Mor02**] proves the following explicit formula for the value of the function W_k on A:

Theorem 3.6. *For each $0 \leqslant k \leqslant d$, the support of W_k is contained in the identity component of* $\mathrm{GSp}(4,\mathbb{R})$. *Furthermore, if (σ_1, σ_2) is a pair of real numbers satisfying*

$$\sigma_1 + \sigma_2 + 1 > 0, \quad \sigma_1 > 0 > \sigma_2, \tag{38}$$

then there is a non-zero constant $C \in \mathbb{C}^\times$ independent of $0 \leqslant k \leqslant d$ such that

$$\begin{aligned} W_k \begin{pmatrix} a_1 & & & \\ & a_2 & & \\ & & a_1^{-1} & \\ & & & a_2^{-1} \end{pmatrix} &= C \binom{d}{k} (s\sqrt{-1})^{-k} \exp(-2\pi a_2^2) \\ \times \int_{L(\sigma_1)} \frac{ds_1}{2\pi\sqrt{-1}} \frac{\Gamma(s_1+k)}{\Gamma(s_1)} (4\pi^3 a_1^2)^{\frac{-s_1+\lambda_1+1-k}{2}} & \int_{L(\sigma_2)} \frac{ds_2}{2\pi\sqrt{-1}} (4\pi a_2^2)^{\frac{-s_2+\lambda_2+k}{2}} \\ \Gamma(\frac{s_1+s_2-2\lambda_2+1}{2}) \Gamma(\frac{s_1+s_2+1}{2}) \Gamma(\frac{s_1}{2}) \Gamma(\frac{-s_2}{2}). & \end{aligned} \tag{39}$$

Here for a real number σ, $L(\sigma)$ denotes the path $\sigma - \sqrt{-1}\infty$ to $\sigma + \sqrt{-1}\infty$.

For an outline of the proof, which involves solving partial differential equations, see 3.2 in [**Mor04**].

Moriyama then introduces the following slightly more general integral

$$Z_N^\infty(s, y_1, W) = \int_{\mathbb{R}^\times} \int_{\mathbb{R}} W \begin{pmatrix} yy_1 & & & \\ & y & & \\ & & y_1^{-1} & \\ & x & & 1 \end{pmatrix} |y|^{s-\frac{3}{2}} \, dx \, dy^\times. \tag{40}$$

Clearly we are interested in the value of this integral when $y_1 = 1$.

Theorem 3.7 (Proposition 8 of [Mor04]). *The integrals* $Z_N^\infty(s, y_1, W_k)$, $0 \leqslant k \leqslant d$, *converge absolutely for* $\Re s > (-\lambda_1 - \lambda_2 + 1)/2$ *and*

(41)
$$\frac{Z_N^\infty(s, y_1, W_k)}{L(s, \Pi_{\mathbb{R}})} = C_1 \binom{d}{k} (\sqrt{-1})^{-k} (4\pi)^s \int_{L(\sigma_1)} \frac{ds_1}{2\pi\sqrt{-1}} (4\pi y_1)^{-s_1 - s + \lambda_1 + 1} \frac{\Gamma(s_1 + \frac{-\lambda_1 - \lambda_2 + 1}{2})\Gamma(s_1 + \frac{-\lambda_1 + \lambda_2 + 1}{2})}{\Gamma(\frac{s_1 - s - d + k + 2}{2})\Gamma(\frac{s_1 + s - k + 1}{2})}$$

with a non-zero constant $C_1 \in \mathbb{C}$ *independent of* $0 \leqslant k \leqslant d$. *Here* $\sigma \in \mathbb{R}$ *is taken so that* $\sigma_1 > (\lambda_1 - \lambda_2 - 1)/2$. *Moreover, for each fixed* $y_1 > 0$, *the integral in the above expression extends to an entire function of* s.

The proof of this theorem involves a very clever application of Barnes' first lemma. Barnes' first lemma says

$$\frac{1}{2\pi\sqrt{-1}} \int_{L(0)} \Gamma(a+s)\Gamma(b+s)\Gamma(c-s)\Gamma(d-s)\, ds = \frac{\Gamma(a+c)\Gamma(a+d)\Gamma(b+c)\Gamma(b+d)}{\Gamma(a+b+c+d)} \tag{42}$$

for all complex numbers a, b, c, d, provided that neither of the numbers $a + c$, $a + b$, $b + c$, $b + d$ is a non-positive integer.

When $y_1 = 1$, one can simplify this expression in the theorem even further. For $\lambda_2 = 0$, the expression turns out to be zero for all k (!). For $\lambda_2 < 0$, we have

$$\frac{Z_N^\infty(s, W_k)}{L(s, \Pi_{\mathbb{R}})} = C_1 \binom{d}{k} (\sqrt{-1})^{-k} \sum_{l=1}^{-\lambda_2} (-4\pi)^{-l} \frac{(l-1)!}{(-\lambda_2 - l)!} \Gamma\Big(\frac{s - k + l + \frac{\lambda_1 + \lambda_2 + 1}{2}}{2}\Big)^{-1} \Gamma\Big(\frac{-s + k + l + \frac{-\lambda_1 + 3\lambda_2 + 3}{2}}{2}\Big)^{-1}. \tag{43}$$

This is about the best one could have hoped for. Clearly, the expression on the right hand side has a very strong arithmetic flavor. It should now be possible to obtain arithmeticity results for generic automorphic forms with archimedean components in the generic limits of discrete series.

[**Mor04**] contains the treatment of some other representations which are induced from (limits of) discrete series representations of the Levi factor of the Klingen parabolic subgroups. The main result of [**Mor04**] is the statement that if Π is a generic irreducible automorphic cuspidal representation of GSp(4) over a totally real field, then $L(s, \Pi, \text{Spinor})$ is entire, and satisfies the correct functional equation. This statement now trivially follows from the general results of Asgari and Shahidi

[**AS06b**]. But what does not follow from [**AS06b**] is the explicit computation of the integral as in equation (43).

3.5. A different construction. Here we review the article [**PS97**] which contains a construction for L-functions of automorphic representations of GSp(4) which works for representations that are not necessarily generic. This is a construction of the spinor L-function that also works for non-generic representations. This is also very closely related to Bessel functionals. The genesis of this method is an idea of Andrianov for the case of holomorphic Siegel modular forms. Let the base field be $\mathbb{Q}$. Let K be a quadratic extension of $\mathbb{Q}$ such that φ, which in our case is an automorphic form coming from a Siegel cusp form, has a non-zero Fourier coefficient parametrized by a symmetric matrix having eigenvalues in K; K is then an imaginary quadratic field. Then Sp(4) contains a copy of $Res_{K/\mathbb{Q}}\mathrm{SL}(2)$ the restriction of scalars of SL from K to $\mathbb{Q}$. Andrianov then proved that restricting to this subgroup and integrating against an Eisenstein series gives the degree four L-function of φ. Piatetski-Shapiro's contribution was to interpret this adelically and apply it to cusp forms that were not necessarily associated to holomorphic Siegel cusp forms.

Let P be the Siegel parabolic subgroup, and let S be its unipotent radical. Let ℓ be a non-degenerate linear form on S, and let D be the connected component of the stabilizer of ℓ in M, the Levi component of P. Then there is a unique semisimple algebra K over the base field k of degree two, such that $D = K^\times$. It is known that either $K = k \oplus k$ or K is a quadratic extension of k. The important subgroup is $R = DS$. Let $N = \{s \in S | \ell(s) = 0\}$. Set $V = K^2$ and consider the group

$$G = \{g \in \mathrm{GL}_2(K) | \det(g) \in k^\times\}. \tag{44}$$

We will write vectors in V in row form and let G act on the right. On V we consider the skew symmetric form

$$\rho(x, y) = \mathrm{tr}_{K/k}(x_1 y_2 - x_2 y_1) \tag{45}$$

where $x = (x_1, x_2)$ and $y = (y_1, y_2)$ as elements of V. Then G preserves ρ up to a factor in $k^\times$. If we consider V as a four dimensional vector space over k then we obtain an embedding

$$G \hookrightarrow \mathrm{GSp}_\rho. \tag{46}$$

If we define the k-linear transformation ι on V by

$$(x_1, x_2) \mapsto (\bar{x}_1, \bar{x}_2) \tag{47}$$

then ι preserves ρ and gives a well-defined element of GSp_ρ.

Let ψ be a non-degenerate character on $S_{\mathbb{A}}$, and ν a character on $D_{\mathbb{A}} \cong I_K$. Let $V_{\mathbb{A}}$ be the adelic points of the vector space on which $G_{\mathbb{A}}$ acts. Take $\Phi \in \mathscr{S}(V_{\mathbb{A}})$, the

Schwartz-Bruhat functions on $V_{\mathbb{A}}$. Let μ be a Hecke character on k. Then we can associate to Φ a function on $G_{\mathbb{A}}$ defined by

$$f^{\Phi}(g,\mu,\nu,s)=\mu(\det g)|\det g|^{s+\frac{1}{2}}\int_{I_K}\Phi((0,t)g)|t\bar{t}|^{s+\frac{1}{2}}\nu(t)d^{\times}t. \tag{48}$$

Here $|\cdot|$ is the idele norm on I_K. Note that $f^{\Phi}(g,\mu,\nu,s)\in Ind_{B'_{\mathbb{A}}}^{G_{\mathbb{A}}}\chi$ where χ is a character on $B'_{\mathbb{A}}$ defined by

$$\chi\left(\begin{pmatrix}x & \\ & 1\end{pmatrix}\begin{pmatrix}\bar{t} & \\ & t\end{pmatrix}\begin{pmatrix}1 & n\\ & 1\end{pmatrix}\right)=\mu(x)|x|^{s+\frac{1}{2}}\nu^{-1}(t). \tag{49}$$

We then form the Eisenstein series

$$E^{\Phi}(g,\mu,\nu,s)=\sum_{\gamma\in B'_k\backslash G_k}f^{\Phi}(\gamma g,\mu,\nu,s). \tag{50}$$

Then E^{Φ} is a meromorphic function on $\mathbb{C}$, and satisfies a functional equation of the usual type. Furthermore, if $\mu(t\bar{t})\nu(t)\neq 1$, E^{Φ} has no poles. If $\mu(t\bar{t})\nu(t)=1$, poles are only at $s=-\frac{1}{2}$ with residue $\Phi(0)\mu(\det g)$, at $s=\frac{3}{2}$ with $\hat{\Phi}(0)\mu^{-1}(\det g)\nu_k(\det g)$, where ν_k is obtained by restricting ν to I_k.

Now let π be an automorphic cuspidal representation of GSp(4) acting on V_{π}. Suppose for ψ and ν as above we have

$$\int_{Z_{\mathbb{A}}R_k\backslash R_{\mathbb{A}}}\varphi(r)\alpha_{\nu,\psi}^{-1}(r)\,dr\neq 0 \tag{51}$$

for some $\varphi\in V_{\pi}$. Here $R=SD$, and $\alpha_{\nu,\psi}^{-1}(sd)=\psi(s)\nu(d)$ if $s\in S$ and $d\in D$. If the above integral is non-zero for some choice of φ, we may set

$$W_{\varphi}(g)=\int_{Z_{\mathbb{A}}R_k\backslash R_{\mathbb{A}}}\varphi(rg)\alpha_{\nu,\psi}^{-1}(r)\,dr. \tag{52}$$

These functions satisfy

$$W_{\varphi}(rg)=\alpha_{\nu,\psi}(r)W_{\varphi}(g), \tag{53}$$

for $r\in R_{\mathbb{A}}$. Denote the space of all such functions by $\mathscr{W}^{\nu,\psi}$. The action of $\mathrm{GSp}_4(\mathbb{A})$ on $\mathscr{W}^{\nu,\psi}$ is equivalent to π. As in the case of the Whittaker model, there will be local models for the the representations π_v, and the uniqueness of the global model follows from the uniqueness of the local models.

For $\varphi\in V_{\pi}$ set

$$Z(s,\varphi,\mu)=\int_{Z_{\mathbb{A}}G_k\backslash G_{\mathbb{A}}}\varphi(g)E^{\varphi}(g,\mu,\nu,s)\,dg. \tag{54}$$

Then $Z(s,\varphi)$ converges in some right half plane, and has a meromorphic continuation, and furthermore satisfies a functional equation dictated by that of the Eisenstein series. In fact, if $\mu(t\bar{t})\nu(t)\neq 1$, then $Z(\varphi,s,\mu)$ is entire; otherwise it has a poles at $s=-\frac{1}{2}$ with residue

$$\Phi(0)\int_{Z_{\mathbb{A}}G_k\backslash G_{\mathbb{A}}}\mu(\det g)\varphi(g)\,dg, \tag{55}$$

and at $s=\frac{3}{2}$ with residue

$$\hat{\Phi}(0)\int_{Z_{\mathbb{A}}G_k\backslash G_{\mathbb{A}}}\mu^{-1}(\det g)\nu_k(\det g)\varphi(g)\,dg. \tag{56}$$

One can also show that

$$Z(s,\varphi,\mu)=\int_{D_{\mathbb{A}}N_{\mathbb{A}}\backslash G_{\mathbb{A}}}W_{\varphi}(g)f^{\Phi}(g,\mu,\nu,s)\,dg. \tag{57}$$

If the data is chosen correctly the latter integral is an infinite product, and this gives us a definition of the local zeta integral. One then proceeds to define the local L-factors $L(s,\pi_p,\mu_p)$. It turns out that if π_p is unramified, then the resulting L-factors are the same as the Langlands L-factors obtained from tensoring the natural embedding $\mathrm{GSp}(4,\mathbb{C})\hookrightarrow \mathrm{GL}_4(\mathbb{C})$ with $\mathbb{C}^{\times}$. The global L-function $L(s,\pi,\mu)$ defined by this zeta integral then converges in some right half plane, and has a meromorphic continuation to the entire complex plane. It also satisfies the appropriate functional equation. Furthermore, if some local component of π is generic, the L-function $L(s,\pi,\mu)$ is entire.

Remark 3.8. It is likely that the construction reviewed in this paragraph is the one needed in [**Sch05**].

4. Bessel functionals

4.1. Bessel functionals. Here we study the unique models introduced in the previous section in more details. We recall the notion of Bessel model introduced by Novodvorsky and Piatetski-Shapiro [**NPS73**]. We follow the exposition of [**Fur93**]. Let $S\in M_2(\mathbb{Q})$ be such that $S={}^tS$. We define the discriminant $d=d(S)$ of S by $d(S)=-4\det S$. Let us define a subgroup $T=T_S$ of GL(2) by

$$T=\{g\in \mathrm{GL}(2)\mid {}^tgSg=\det g.S\}.$$

Then we consider T as a subgroup of GSp(4) via

$$t\mapsto\begin{pmatrix}t & \\ & \det t.{}^tt^{-1}\end{pmatrix},$$

$t\in T$.

Let us denote by U the subgroup of GSp(4) defined by

$$U=\{u(X)=\begin{pmatrix}I_2 & X\\ & I_2\end{pmatrix}\mid X={}^tX\}.$$

Finally, we define a subgroup R of GSp(4) by $R = TU$.

Let ψ be a non-trivial character of $\mathbb{Q}\backslash\mathbb{A}$. Then we define a character ψ_S on $U(\mathbb{A})$ by $\psi_S(u(X)) = \psi(\mathrm{tr}(SX))$ for $X = {}^tX \in \mathsf{M}_2(\mathbb{A})$. Usually when there is no danger of confusion, we abbreviate ψ_S to ψ. Let Λ be a character of $T(\mathbb{Q})\backslash T(\mathbb{A})$. Denote by $\Lambda\otimes\psi_S$ the character of $R(\mathbb{A})$ defined by $(\Lambda\otimes\psi)(tu) = \Lambda(t)\psi_S(u)$ for $t \in T(\mathbb{A})$ and $u \in U(\mathbb{A})$.

Let π be an automorphic cuspidal representation of $\mathrm{GSp}_4(\mathbb{A})$ and V_π its space of automorphic functions. We assume that

$$\Lambda|_{\mathbb{A}^\times} = \omega_\pi. \tag{58}$$

Then for $\varphi \in V_\pi$, we define a function B_φ on $\mathrm{GSp}_4(\mathbb{A})$ by

$$B_\varphi(g) = \int_{Z_{\mathbb{A}}R_{\mathbb{Q}}\backslash R_{\mathbb{A}}} (\Lambda\otimes\psi_S)(r)^{-1}\cdot\varphi(rh)\,dh. \tag{59}$$

We say that π has a global Bessel model of type (S,Λ,ψ) for π if for some $\varphi \in V_\pi$, the function B_φ is non-zero. In this case, the $\mathbb{C}$-vector space of functions on $\mathrm{GSp}_4(\mathbb{A})$ spanned by $\{B_\varphi \,|\, \varphi \in V_\pi\}$ is called the space of the global Bessel model of π.

Similarly, one can consider local Bessel models. Fix a local field $\mathbb{Q}_v$. Define the algebraic groups T_S, U, and R as above. Also, consider the characters Λ, ψ, ψ_S, and $\Lambda\otimes\psi_S$ of the corresponding local groups. Let (π, V_π) be an irreducible admissible representation of the group GSp(4) over $\mathbb{Q}_v$, when v is finite, or a $(\mathfrak{g}, K)$-module when v is archimedean. Then we say that the representation π has a local Bessel model of type (S,Λ,ψ) if there is a non-zero map in

$$\mathrm{Hom}(\pi_v, Ind(\Lambda\otimes\psi|R,G)). \tag{60}$$

Here the Hom space is the collection of $G(\mathbb{Q}_v)$-intertwining maps when v is finite, and the collection of all $(\mathfrak{g}, K)$-maps when v is archimedean. Also in the archimedean case, as in the Whittaker case, we work with that subspace of Ind which consists of functions of moderate growth.

Typically one is interested in two different types of Bessel models corresponding to two choices of the symmetric matrix S. The two choices of S are:

1. $S = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$,
2. $S = \begin{pmatrix} 1 & \\ & d \end{pmatrix}$, with d a positive square-free rational number.

Below, we will determine the subgroups T_S, and R, and explicitly write down the corresponding global Bessel functionals. We fix an irreducible automorphic cuspidal representation π of $\mathrm{GSp}_4(\mathbb{A})$ and a unitary character ψ of $\mathbb{A}$ throughout.

(1) $S = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$. This is the case of interest for us. In this case, the subgroup T_S is equal to the subgroup consisting of diagonal matrices. A straightforward analysis then shows that for every character Λ of $T_S(\mathbb{Q})\backslash T_S(\mathbb{A})$ subject to (58), there is a Hecke character of $\mathbb{A}^\times$ such that the global Bessel functional (59) is given by

$$B_\chi^{\text{split}}(g;\varphi) = \int_{F^\times\backslash\mathbb{A}^\times} \varphi^U(\begin{pmatrix} y & & & \\ & 1 & & \\ & & 1 & \\ & & & y \end{pmatrix})\chi(y)\,d^\times y.$$

Here when φ is a cusp form on GSp(4), we have set

$$\varphi^U(g) = \int_{(F\backslash\mathbb{A})^3} \varphi(\begin{pmatrix} 1 & & u & w \\ & 1 & w & v \\ & & 1 & \\ & & & 1 \end{pmatrix} g)\psi^{-1}(w)\,du\,dv\,dw.$$

(2) $S = \begin{pmatrix} 1 & \\ & d \end{pmatrix}$. In this case, the subgroup T_S is equal to a non-split torus. Then there is a Hecke character of the torus T_S, say χ, in such a way that

$$B_\chi(g;\varphi) = \int_{T_S(F)\mathbb{A}^\times\backslash T_S(\mathbb{A})} \varphi^U(\begin{pmatrix} \alpha & \\ & \det\alpha.{}^t\alpha^{-1} \end{pmatrix})\chi(\alpha)\,d\alpha,$$

with φ^U defined as before. The case of immediate interest is the case where $d = 1$, in which case,

$$\begin{aligned} T_S &= \{g \in \mathrm{GL}_2 \mid {}^t g \cdot g = \det g\} \\ &= \{\begin{pmatrix} a & b \\ -b & a \end{pmatrix} \mid a^2 + b^2 \in \mathrm{GL}_1\}. \end{aligned}$$

The problems of existence of Bessel functionals for this choice of the matrix S seem to be more delicate.

4.2. An interesting relevant pair. In this paragraph, we explain how Bessel functionals are related to the setup of [**GP94**]. Here we follow the conventions of [**PS83**]. Let k be a field of characteristic not equal to two. The space

$$V = \left\{T \in \mathsf{M}_4(k) \mid TJ_2 \text{ is skew-symmeric and } \operatorname{tr} T = 0\right\}$$

is a five dimensional vector space over k. Here $J_n = \begin{pmatrix} & I_n \\ -I_n & \end{pmatrix}$. The group $\mathrm{GSp}(4,k)$ acts on V by

$$g : T \mapsto g^{-1}Tg.$$

A symmetric non-degenerate form on V is given by

$$(T_1, T_2) = \frac{1}{4}\operatorname{tr} T_1 T_2.$$

If we set $Q(T) = (T, T)$, then we have $Q(g \cdot T) = Q(T)$ for all $g \in \mathrm{GSp}(4)$ and $T \in V$. In fact, we have an isomorphism $\mathrm{PGSp}(4) \simeq \mathrm{SO}(5)$.

More explicitly, the vector space V can be given in the following way:

$$V = \left\{ \begin{pmatrix} M & xJ_1 \\ yJ_1 & {}^T M \end{pmatrix} \mid M \in \mathsf{M}_2(k), \operatorname{tr} M = 0, x, y \in k \right\}.$$

Also, the quadratic form Q is given by

$$Q(T) = -\det M - xy,$$

for $T = \begin{pmatrix} M & xJ_1 \\ yJ_1 & {}^T M \end{pmatrix}$. We will denote the element T by $[M, x, y]$. The action of the group $\mathrm{GSp}(4)$ on the space V is explicitly given by the following relations:

$$\begin{pmatrix} A & \\ & \lambda\, {}^T A^{-1} \end{pmatrix} \cdot T = [A^{-1} M A, x\lambda(\det A)^{-1}, y\lambda^{-1} \det A],$$

$$\begin{pmatrix} I_2 & S \\ & I_2 \end{pmatrix} \cdot T = [M - ySJ_1, (x - y \det S) J_1 + MS - {}^T(MS), y]$$

where $S = {}^T S$, and

$$\begin{pmatrix} & -I_2 \\ I_2 & \end{pmatrix} \cdot T = [{}^T M, -y, -x].$$

We will also need the following even more explicit realization. The space is given by

$$V = \left\{ T = \begin{pmatrix} t & v+w & 0 & x \\ v-w & -t & -x & 0 \\ 0 & y & t & v-w \\ -y & 0 & v+w & -t \end{pmatrix} \right\}$$

equipped with the quadratic form

$$Q(T) = t^2 + v^2 - w^2 - xy.$$

Then V has a two dimensional quadratic subspace

$$W = \left\{ T' = \begin{pmatrix} t & v & & \\ v & -t & & \\ & & t & v \\ & & v & -t \end{pmatrix} \right\}$$

equipped with

$$Q'(T') = t^2 + v^2.$$

The pair (V,W) is relevant, as $\dim V - \dim W = 3$ is odd, and

$$W^{\perp} = \left\{ T'' = \begin{pmatrix} 0 & w & 0 & x \\ -w & 0 & -x & 0 \\ 0 & y & 0 & -w \\ -y & 0 & w & 0 \end{pmatrix} \right\}$$

equipped with $Q''(T'') = -w^2 - xy$ is split. We set

$$X = \{[0_2, x, 0], x \in k\},$$
$$X' = \left\{[0_2, 0, y], y \in k\right\}.$$

Then X and X' are isotropic dual spaces. The stabilizer of X is the Siegel parabolic subgroup given by

$$P = \left\{ \begin{pmatrix} A & \\ & \lambda\, {}^T A^{-1} \end{pmatrix} \begin{pmatrix} I_2 & S \\ & I_2 \end{pmatrix} \right\}.$$

The group $\mathrm{SO}(W)$ has the following realization

$$\left\{ \begin{pmatrix} A & \\ & {}^T A^{-1} \end{pmatrix}, A = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \text{ with } a^2 + b^2 = 1 \right\}.$$

We note that $\mathrm{SO}(W) \subset P$. Then we consider a character of the unipotent radical N of the parabolic subgroup P. The subgroup N is abelian. The stabilizer of this character will just be a subgroup isomorphic to $\mathrm{SO}(W)$ in M. So in this case, the spherical subgroup is simply $\mathrm{SO}(W) \rtimes N_P$.

It remains to say something about the pure inner forms. We know from the discussion following Corollary 8.10 of [**GP94**] that the only relevant inner form of the above $G = \mathrm{SO}(V) \times \mathrm{SO}(W)$, which is $\mathrm{SO}(3,2) \times \mathrm{SO}(2,0)$, is $G' = \mathrm{SO}(1,4) \times \mathrm{SO}(0,2)$. Here we give a realization for this group, following [**FS03**].

Let D be the division algebra of Hamiltonian quaternions over $\mathbb{R}$, and let us denote the canonical involution of D by $x \mapsto \bar{x}$. Then we define G_D, the quaternion similitude unitary group of degree two over D, by

$$G_D = \left\{ g \in \mathrm{GL}_2(D) \mid g^* \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \mu(g) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \mu(g) \in D^\times \right\},$$

where

$$g^* = \begin{pmatrix} \bar{a} & \bar{c} \\ \bar{b} & \bar{d} \end{pmatrix} \quad \text{where} \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

We regard G_D as an algebraic group over $\mathbb{R}$. The group G_D is an inner form of GSp(4). In fact, if $D = \mathsf{M}_2(\mathbb{R})$, then in $\mathrm{GL}_4(\mathbb{R})$ we have

$$\xi G_D \xi^{-1} = G \quad \text{where} \quad \xi = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

We refer the reader to Lemma 1.2 of [**FS03**] for the proof of the last statement.

4.3. Existence. The existence of Bessel functionals of various types poses an interesting problem. It is a theorem of Li [**Li92**] that any automorphic cuspidal representation of GSp(4) either has a Whittaker model, or some Bessel model. When $S = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$, it follows from [**TB00**] that the existence of the the local Bessel functional, at least at the non-archimedean places, is equivalent to the existence of Whittaker models. At the archimedean place, the existence of the Bessel functional certainly implies the genericity. We certainly expect the converse too as suggested by the results of [**Mor04, TB05**]. If this is indeed the case, we will have the following global result:

Theorem 4.1. *Let (π, V_π) be an irreducible generic automorphic cuspidal representation. Let $S = I_2$, and χ a finite order Hecke character with χ_∞ trivial. Then π has a non-zero global (S,χ)-Bessel model if and only if $L(\frac{1}{2}, \pi \otimes \chi) \neq 0$.*

This is of course not a hard theorem, and follows from simple observations.

It is well-known that automorphic representations associated to holomorphic Siegel modular forms are not generic; that is, they fail to have Whittaker models. It is also known that the genericity of such representations certainly fails at the archimedean place. For this reason it is desirable to determine when holomorphic discrete series representations posses Bessel models which seem to be the next best thing in applications to L-functions [**Fur93, FS03**]. We now turn our attention to the case where $S = \begin{pmatrix} 1 & \\ & d \end{pmatrix}$. For simplicity, assume $d = 1$. The conjecture of Gross and Prasad (Conjecture 6.9 of [**GP94**]) predicts that the existence of Bessel models for holomorphic discrete series is intertwined with the existence of such models for other members of the Vogan L-packet of the given discrete series representation, in particular the generic discrete series.

Let Π be a generic discrete series representation of $\mathrm{GSp}(4,\mathbb{R})$, with trivial central character. Then there is a pair (D_k, D_l) of discrete series representations of $\mathrm{GL}(2,\mathbb{R})$ with trivial central character such that Π is obtained by a theta lift from $\mathrm{GO}(2,2)$ by the representation that the pair (D_k, D_l) defines. In order to land in generic discrete series, we need to assume that $k, l \geqslant 2$ satisfy $k \neq l$ and they have the same parity. Let n be an integer with $n > \max(k, l)$, and with different parity from k (or l). We set $\chi_n\left(\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\right) = e^{in\theta}$. With these notations, we prove in [**TB05**] that

Theorem 4.2. Π *has a* $(\begin{pmatrix} 1 & \\ & 1 \end{pmatrix}, \chi_n, \psi)$*-Bessel model.*

A few remarks are in order. It is clear from the setup that the proof of the theorem uses theta correspondence; in fact, we will use the pull-back of the Bessel functional via the global theta correspondence, along with various substantial local and global results from the theory of automorphic forms [**Har93, Mic, Rob01, Wal85**]. It may be desirable to find a direct local proof of the existence theorem as in [**Wal03**]. Our attempts in this direction, however, have not been successful. Inspired by [**Sha80, Sha85**], one is tempted to write down an integral and try to prove that the integral does not vanish for the correct choice of the data. There are convergence issues that one needs to deal with. In the Whittaker situation, what saves the day is the fact that one can do the analysis of the integrals "one root at a time"; we have not been able to successfully follow such an approach for the Bessel integrals. In order to establish the conjecture of Gross-Prasad for the pair $(\mathrm{SO}(5), \mathrm{SO}(2))$ for discrete series packets, one needs to study generic discrete series representations of $\mathrm{PGSp}(4)$, holomorphic discrete series representations of $\mathrm{PGSp}(4)$, and related representations of $\mathrm{SO}(4,1)$. The case of $\mathrm{SO}(4,1)$ is simpler as the group in question has rank one. Here we have considered the representations of the group $\mathrm{PGSp}(4)$. Thanks to Wallach's recent paper [**Wal03**], the case of holomorphic representations is much better understood. This is the reason why we concentrated our efforts on the generic case. Shalika has informed the author that he can prove the converse statement of the theorem using local methods based on [**KW76**]. Consequently, the "if" in the theorem may be replaced by "if and only if." Perhaps, it should also be pointed out here that, in light of Theorem 3.4 of [**Wal88**], our results automatically extend to generic limits of discrete series. In [**TB05**] we have used the same idea of pulling back global Bessel functional via the theta correspondence to prove the analytic continuation of the spinor L-function of a large class of automorphic cusp forms. The class of cuspidal representation to which our result applies is restricted by a condition at the archimedean place, and for this reason our theorem is weaker than those obtained by Asgari and Shahidi. At a crucial point in our argument we have to use twisting with highly ramified characters and a density result from [**Ven**].

5. Saito-Kurokawa and CAP representations

5.1. The failure of multiplicity one. One of the features that distinguishes the symplectic group from the general linear groups is the failure of the *multiplicity one phenomena.* We know by the results of Jacquet and Shalika that if $\Pi_1 = \otimes_v \Pi_{1v}$ and $\Pi_2 = \otimes_v \Pi_{2v}$ are two automorphic cuspidal representations of GL_n such that $\Pi_{1v} \simeq \Pi_{2v}$ for almost all v, then $\Pi_1 = \Pi_2$. This means that first of all $\Pi_{1v} \simeq \Pi_{2v}$ for *all* v, and that the representations Π_1 and Π_2 correspond to the *same* irreducible subspace of L^2. Let us construct two irreducible cuspidal representations of GSp(4) that agree at almost all places, but are not isomorphic globally. Let D be a quaternion algebra over $\mathbb{Q}$ and compact at infinity. Let π_1, π_2 be two cuspidal representations of $D^\times$ with, say, trivial central character. Also assume that π_1 and π_2 are not equal. Then we know that the pair (π_1, π_2) defines a cuspidal representation of GSO(4); extend this representation trivially to GO(4). Then if we consider the dual pair (GO(4), GSp(4)), we can transfer the resulting representation to GSp(4) and obtain a non-zero automorphic cuspidal representation Π_1 of GSp(4) which is globally non-generic. On the other hand, let π_1^{JL} and π_2^{JL} be the representations of GL(2) which are the respective images of π_1, π_2 under the Jacquet-Langlands correspondence. Then we can construct an automorphic cuspidal representation of GO(2,2), and by considering the pair (GO(2,2), GSp(4)), a non-zero automorphic cuspidal representation of GSp(4) which is globally generic. Then it follows that the the representations Π_1 and Π_2 agree at all those places where D is split. For details of these constructions, see [**HPS83**].

5.2. CAP representations. Another feature of sympletic groups that puts them in sharp contrast with the general linear groups is the existence of representations that are *Cuspidal Associated to a Parabolic.* In general, let G be a quasi-split reductive group, and let $P = MU$ be a parabolic subgroup of G. Also let τ be an automorphic representation of M. It is well-known that any irreducible constituent of $Ind_{P(\mathbb{A})}^{G(\mathbb{A})}$ is an automorphic representation, and that if $\Pi_1 = \otimes_v \Pi_{1v}$ and $\Pi_2 = \otimes_v \Pi_{2v}$ are two irreducible constituents then $\Pi_{1v} = \Pi_{2v}$ for almost all v.

Definition 5.1. Let $\pi = \otimes_v \pi_v$ be an irreducible automorphic cuspidal representation of G. We say π is *Cuspidal Associated to a Parabolic*, or simply CAP, if there is a parabolic subgroup $P = MU$ and an automorphic representation τ of M, and an irreducible constituent $\Pi = \otimes_v \Pi_v$ of $\mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})}$ such that $\pi_v \simeq \Pi_v$ for almost all v.

When G is not quasi-split over the number field F, then it has an inner form G' that is quasi-split over F. Note that for almost all v, we have $G(F_v) = G'(F_v)$. In that situation, we call a cuspidal representation π of G *Cuspidal Associated to a Parabolic* if there is a parabolic subgroup $P = MU$ of G' and an automorphic representation τ

of M such that π agrees locally at almost all places with an irreducible constituent of $Ind_{P(\mathbb{A})}^{G'(\mathbb{A})}$. This definition is suggested in [**Gan**].

In the case of GSp(4) we have three conjugacy classes of parabolic subgroups: the Borel subgroup B, the Siegel parabolic subgroup P, and the Klingen parabolic subgroup Q. So we may have representations that are CAP with respect to either of these parabolics. Note that if a representation is CAP with respect to B then it is CAP with respect to P and Q as $B \subset P$ and $B \subset Q$. For this reason, we introduce the notion of *Strongly Cuspidal Associated to a Parabolic.* This means that the representation τ in the definition is cuspidal. In the sequel, unless otherwise noted, CAP with respect to a parabolic subgroup is always to mean strongly CAP with respect to that parabolic.

5.3. CAP representations for GSp(4) **and its inner forms.** The following theorem is due to Piatetski-Shapiro [**PS83**]:

Theorem 5.2. *If π is CAP with respect to B or P with central character ω_π, then there is a representation π' of* GSp(4)*, CAP with respect to B or P and with trivial central character, such that for all $g \in \mathrm{GSp}(4,\mathbb{A})$ we have*

$$\pi(g) = \pi'(g)\omega(\nu(g)). \tag{61}$$

A representation of GSp(4) with trivial central character is nothing but a representation of PGSp(4). We have identified the latter group with SO(3,2). We may then consider the dual reductive pair $(\widetilde{\mathrm{SL}}_2, \mathrm{PGSp}(4))$. The following theorem is one of the main results of [**PS83**]:

Theorem 5.3. *Let π be an irreducible cuspidal representation of* PGSp(4)*. Then the following are equivalent:*

1. *π is CAP with respect to B or P;*
2. *$L(s,\pi,Spinor)$ has poles;*
3. *π is the theta lift of an irreducible cuspidal automorphic representation σ of $\widetilde{\mathrm{SL}}_2(\mathbb{A})$.*

Observe that implicit in the theorem is the statement that the theta lift of any representation σ as in the last part of the theorem is always a non-trivial representation; this is Theorem 5.1 of [**PS83**]. This follows from the Rallis inner product formula and the fact that the pair $(\widetilde{\mathrm{SL}}_2, \mathrm{PGSp}(4))$ is in the stable range. For details see Theorem 2.9, 2.16, and 2.17 of [**Gan**]. The proof in [**PS83**] is more direct.

E. Sayag has proved the analogous statement for rank one inner form of the group PGSp(4); see [**Gan**] for the description of the results. Sayag's argument does not work in the anisotropic situation. [**Gan**] contains an interesting result for all inner forms, but Gan's result falls short of the complete characterization of the collection of CAP representations.

6. Further developments

6.1. Roberts Packets. Here we have used the lecture notes of a talk by Roberts at Princeton seminar in February of 2002. Let F be a totally real number field and let E be either a totally real quadratic extension of F or $F \times F$. Let τ be a tempered cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_E)$ such that τ is not Galois invariant, and ω_τ is Galois invariant. Then there must exist a Hecke character χ of F such that $\omega = \chi \circ N_{E/F}$. Then for each place v of F Roberts [**Rob01**] defined a local L-packet of tempered irreducible representations $\Pi(\chi_v, \tau_v)$ of $\mathrm{GSp}_4(F_v)$ associated conjecturally to an L-parameter $\varphi(\chi_v, \tau_v) : L_{F_v} \to \mathrm{GSp}_4(\mathbb{C})$. It is shown that these packets have many of the properties that one expects of Langlands packets. For example, $|\Pi(\chi_v, \tau_v)| = 1$ or 2, and equal to 1 for almost all v. One can also define global L-packets via

$$\Pi(\chi, \tau) = \otimes_v \Pi(\chi_v, \tau_v). \tag{62}$$

Roberts further proves the following multiplicity statement which is compatible with Arthur's conjectures:

Theorem 6.1. 1. *If E is a field, then every element of $\Pi(\chi, \tau)$ occurs with multiplicity one in the cusp forms of $\mathrm{GSp}(4)$ of central character χ;*

2. *Assume $E = F \times F$, and $\Pi = \otimes_v \Pi_v \in \Pi(\chi, \tau)$. Let N be the number of times Π_v is non-generic. Then Π occurs with multiplicity $(1 + (-1)^N)/2$ in the cusp forms on $\mathrm{GSp}(4)$.*

Note that one still does not know that the packets only depend on the equivalent class of $\varphi(\chi_v, \tau_v)$, i.e. we do not know whether $\varphi(\chi_v, \tau_v) \cong \varphi(\chi'_v, \tau'_v)$ implies that $\Pi(\chi_v, \tau_v) = \Pi(\chi'_v, \tau'_v)$. Also, we do not have a purely local construction of the packets i.e. one that does not require global objects τ and χ. It would also be interesting to know exactly which local or global packets are obtained this way. Locally, if v is non-archimedean place not dividing 2, then $\varphi(\chi_v, \tau_v)$ include all but one parameter $\mathrm{sp}(2) \otimes \rho$ with $\dim \rho = 2$ irreducible orthogonal. When $v|2$, there a finite number of parameters not included. So most parameters are covered. The situation is similar to GL(2) in that there are in fact representations that cannot be obtained as theta lifts, namely the special representation.

6.2. Asgari and Shahidi's results. As explained earlier the connected component of the identity of the L-group of $\mathrm{GSp}(4)$ is the group $\mathrm{GSp}_4(\mathbb{C})$. This group has an obvious embedding into the group $\mathrm{GL}_4(\mathbb{C})$. Then Langlands' theory of functoriality predicts the existence of a map φ from the collection of automorphic forms on GSp(4) over a global field F to those on GL(4) over the same field F in such a way that for any automorphic form π on GSp(4) we have

$$L(s,\pi,\mathrm{Spinor}) = L(s,\varphi(\pi)). \tag{63}$$

The right hand side of this equation is the standard L-function of the automorphic form $\varphi(\pi)$ on GL(4). In a series of two papers [**AS06a, AS06b**], Asgari and Shahidi have established this transfer for the case where the representation in question is generic. In [**AS06a**] they establish the weak transfer for split general or groups. This means that they can show that given a globally generic representation π on a general spinor group, there is an automorphic representation on the corresponding general linear group such that the local representation match up via the local Langlands' at almost all places. Currently they cannot show the matching at all places. Observing that GSp(4) is nothing but the split general spinor group of order five, they get weak transfer in the case of GSp(4). Then they use what is know about the theory of L-function a la [**PS97**] and Piatetski-Shapiro and Soudry to get more precise information about the transfer. Let us describe their result more carefully. By Langlands' theory of Eisenstein series we just need to do the transfer for cuspidal unitary automorphic representations.

Theorem 6.2 (Asgari and Shahidi). *Let π be a unitary cuspidal representation of* $\mathrm{GSp}(4,\mathbb{A})$ *which is globally generic. Then π has a unique transfer to an automorphic representation Π of* $\mathrm{GL}(4,\mathbb{A})$. *The transfer is globally generic (hence locally generic). If ω_π, ω_Π are the central characters of π and Π, respectively, then $\omega_\Pi = \omega_\pi^2$ and $\Pi \simeq \tilde{\Pi} \otimes \omega_\pi$.*

Asgari and Shahidi can also determine when the resulting representation on GL(4) is cuspidal. They show that it is cuspidal unless when the starting representation π is in a Roberts packet, in which case the resulting representation is an isobaric sum of two GL(2) cuspidal representations. In [**AS06b**] various applications of these important results are listed including bounds towards Ramanujan, and the entireness of the global spinor L-function for generic representations. It should be noted that if the automorphic cuspidal representation π on GSp(4) is not generic, then the L-function may not be entire.

Remark 6.3. The transfer of a general automorphic form on GSp(4) to GL(4) should follow from the trace formula. This is explained in the paper of Arthur in the Shalikafest [**Art04**]. There is the recent thesis of David Whitehouse under Dinakar Ramakrishnan which establishes the *fundamental lemma* for the situation under consideration. Here we should also point that there is an unpublished manuscript due by Flicker addrerring the transfer from GSp(4) to GL(4).

6.3. Roberts and Schmidt's theory of new forms. The theory of new forms in the context of classical modular forms is well-understood. There is an interpretation and extension of this theory to the representation theory of the group GL(2) due to Casselman [**Cas73**] who also found interesting connections to Jacquet-Langlands theory. Casselman's theory of new vectors was then generalized to GL(n) by Jacquet, Piatetski-Shapiro, and Shalika in an amazingly beautiful work [**JPSS81**] who also showed the connection to the theory of L-functions. Unfortunately, in general we do not have a theory of new forms for a general reductive group, even conjectural, except for a few isolated cases. One of the rare cases in which there now seems to exist a nice theory of new forms is the case of PGSp(4) where the theory is due to Roberts and Schmidt [**RS**] where a fairly precise conjecture has been formulated; they have recently claimed to have proved their conjecture and are now preparing a manuscript. Let us now explain their conjecture.

We work over a non-archimedean local field F with ring of integers denoted by $\mathcal{O}$, and suppose $\mathfrak{p}$ be the prime idea, and ϖ a local uniformizer. We first define a compact open subgroup of the Klingen parabolic subgroup. For a non-negative integer n, let $Kl(\mathfrak{p}^n)$ be the collection of matrices k in the subgroup

$$\begin{pmatrix} \mathcal{O} & \mathcal{O} & \mathcal{O} & \mathcal{O} \\ \mathfrak{p}^n & \mathcal{O} & \mathcal{O} & \mathcal{O} \\ \mathfrak{p}^n & \mathfrak{p}^n & \mathcal{O} & \mathfrak{p}^n \\ \mathfrak{p}^n & \mathcal{O} & \mathcal{O} & \mathcal{O} \end{pmatrix} \tag{64}$$

subject to the extra condition that $\nu(k) \in \mathcal{O}^\times$. Define the Atkin-Lehner element of level $\mathfrak{p}^n$ by

$$u_n = \begin{pmatrix} & & & 1 \\ & & -1 & \\ & -\varpi^n & & \\ \varpi^n & & & \end{pmatrix} \tag{65}$$

We finally define the paramodular group $K(\mathfrak{p}^n)$ of level $\mathfrak{p}^n$ to the subgroup of the group GSp$(4, F)$ generated by $Kl(\mathfrak{p}^n)$ and $u_n Kl(\mathfrak{p}^n) u_n^{-1}$. An equivalent description is to say that $K(\mathfrak{p}^n)$ is the collection of matrices in the subgroup

$$\begin{pmatrix} \mathcal{O} & \mathcal{O} & \mathfrak{p}^{-n} & \mathcal{O} \\ \mathfrak{p}^n & \mathcal{O} & \mathcal{O} & \mathcal{O} \\ \mathfrak{p}^n & \mathfrak{p}^n & \mathcal{O} & \mathfrak{p}^n \\ \mathfrak{p}^n & \mathcal{O} & \mathcal{O} & \mathcal{O} \end{pmatrix} \tag{66}$$

which have their similitude norm in the group $\mathcal{O}^\times$.

Conjecture 6.4 (Roberts, Schmidt). *Let π be a generic irreducible admissible representation of* $\mathrm{GSp}(4,F)$ *with trivial central character. For each non-negative integer n, let $\pi(\mathfrak{p}^n)$ be the subspace of π of vectors fixed by $K(\mathfrak{p}^n)$. Then*

1. *For some non-negative integer n, the space $\pi(\mathfrak{p}^n)$ is non-zero.*
2. *If N_π is the smallest n such that $\pi(\mathfrak{p}^n)$ is non-zero, then*

$$\dim \pi(\mathfrak{p}^{N_\pi}) = 1.$$

3. *There is a W_π in the Whittaker model of π coming from a non-zero vector in $\pi(\mathfrak{p}^{N_\pi})$ such that*

$$Z_N(s, W_\pi) = L(s,\pi).$$

As in the $\mathrm{GL}(n)$ theory there is a nice connection to ε-factors. For example it is shown in [**RS**] that if the conjecture is true, then

$$\pi(u_{N_\pi})W_\pi = \varepsilon_\pi W_\pi \tag{67}$$

for some $\varepsilon_\pi \in \{\pm 1\}$. Furthermore,

$$\varepsilon(s,\pi,\psi) = \varepsilon_\pi q^{-N_\pi(s-\frac{1}{2})}. \tag{68}$$

Observe that the subgroups $K(\mathfrak{p}^n)$ do NOT form a descending chain of subgroups. This is an indication of how strange a theory of new forms for a general reductive group might seem at first. Recall that in the case of $\mathrm{GL}(k)$, the role of the subgroups $K(\mathfrak{p}^n)$ is played by the subgroup

$$\left\{ \begin{pmatrix} & & & u \\ & g & & \vdots \\ & & & v \\ r & \dots & s & m \end{pmatrix}; \begin{array}{c} g \in \mathrm{GL}_{k-1}(\mathcal{O}), u, \dots, v \in \mathcal{O}, \\ r, \dots, s \equiv 0 \mod \mathfrak{p}^n, m \equiv 1 \mod \mathfrak{p}^n \end{array} \right\}. \tag{69}$$

and in this case, the subgroups clearly form a descending chain.

In an interesting recent paper [**Sch05**], Schmidt considers the problem for the case of non-generic representations and offers a theory for the square-free case. Schmidt considers various congruence subgroups, and studies each case carefully. What is needed to complete the theory is a theory of L functions of degree four for representations with Iwahori-fixed vectors similar to the one proposed by Novodvorsky, and worked out by Bump, Takloo-Bighash, and Moriyama for the generic situation.

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